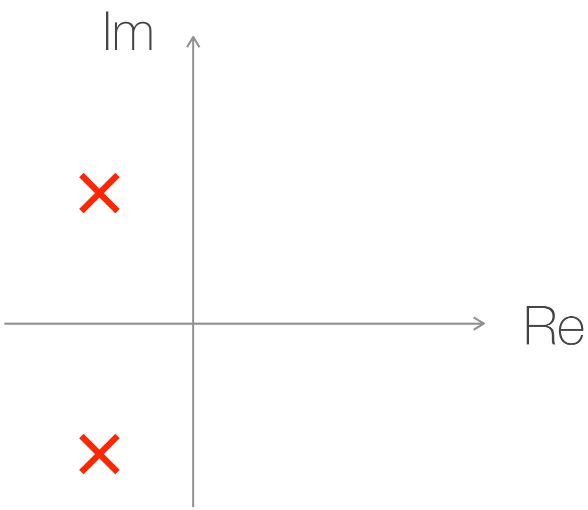
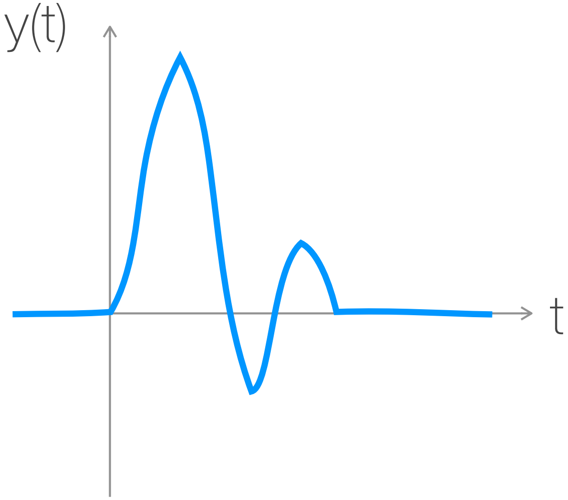
458.308 Process Control & Design

Lecture 4: Models for Control (part 2) - Complex Dynamics





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General Form of Transfer Function

$$\frac{Y(s)}{U(s)} = G(s) = \frac{N(s)}{D(s)}e^{-\theta s} = \gamma\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}e^{-\theta s}$$

Poles: Roots of the denominator polynomial D(s)

related to stability, oscillation, speed of response …

Zeros: Roots of the numerator polynomial N(s)

related to inverse response and overshoot

$$G(s) = \frac{2s+1}{s^2+4s+3} \qquad p_1 = -3,\ p_2 = -1, z_1 = -1/2$$

$$G(s) = \frac{5}{s^2-s+1} \qquad p_1,\ p_2 = \frac{1 \pm \sqrt{3}j}{2}$$

Quick Analysis

Given a transfer function G(s), what can you say quickly (without doing a lot of calculation) about the dynamics that the transfer function represents?

Stability: input returning to the original equilibrium value (after some excursion) ⟶ output eventually returning to the original equilibrium value?

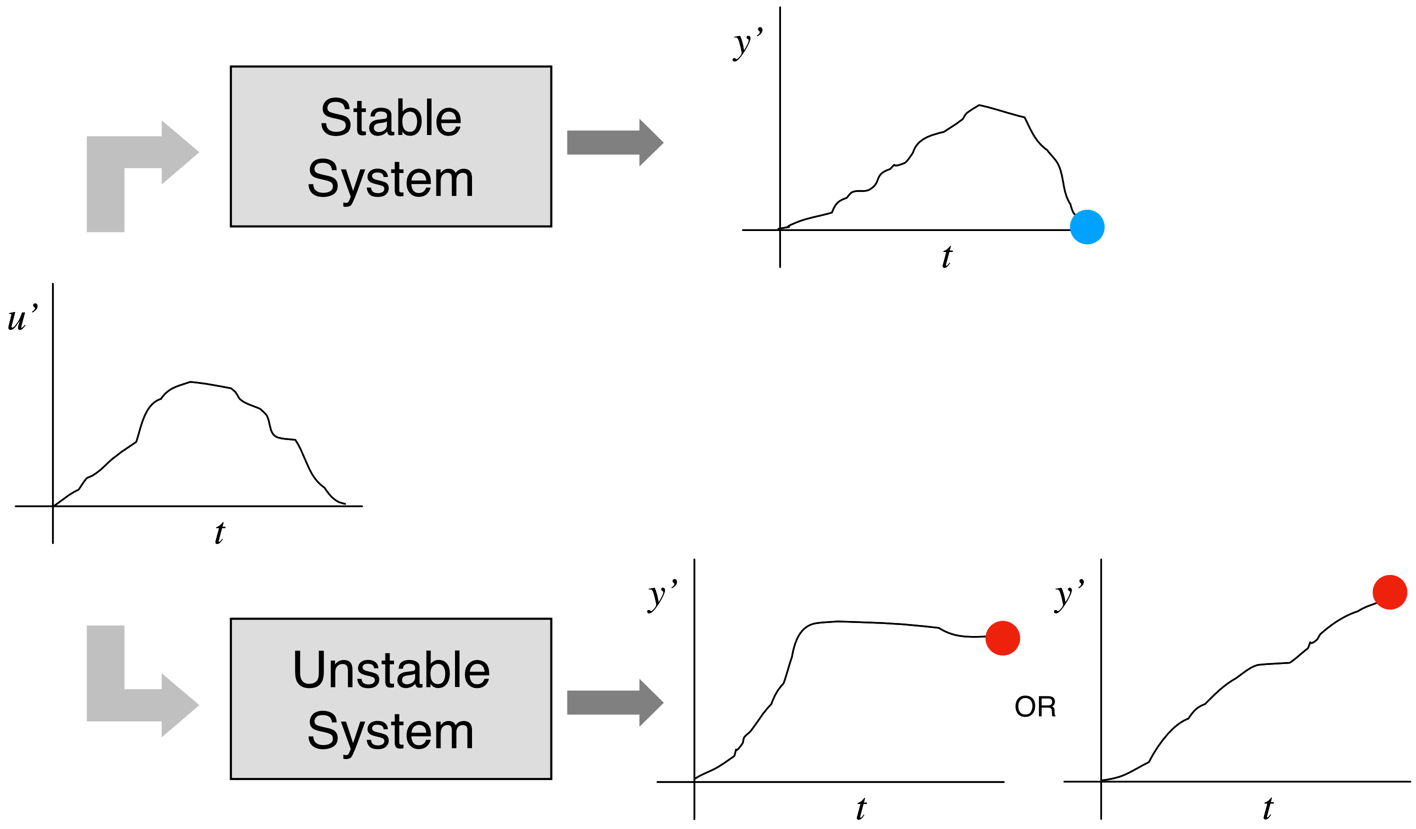
Gain: output change / input change

Overdamped or underdamped: If underdamped, frequency of oscillation?

Any inverse response or overshoot?

General speed of response (e.g., settling time)

Definition of Stability



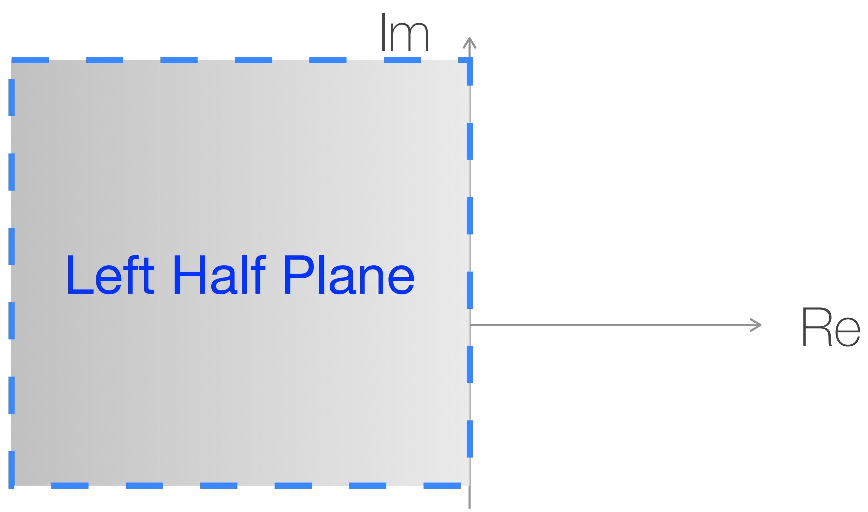
For linear systems, same as BIBO (Bounded-Input/Bounded-Output) stability

Stability of Linear (Linearized) Systems

If all poles have (   ) real part, the dynamics is stable

$$\forall i, \;\; Re(p_i) < 0$$

If any of the poles have positive or zero real part, the dynamics is unstable



Pole's Location

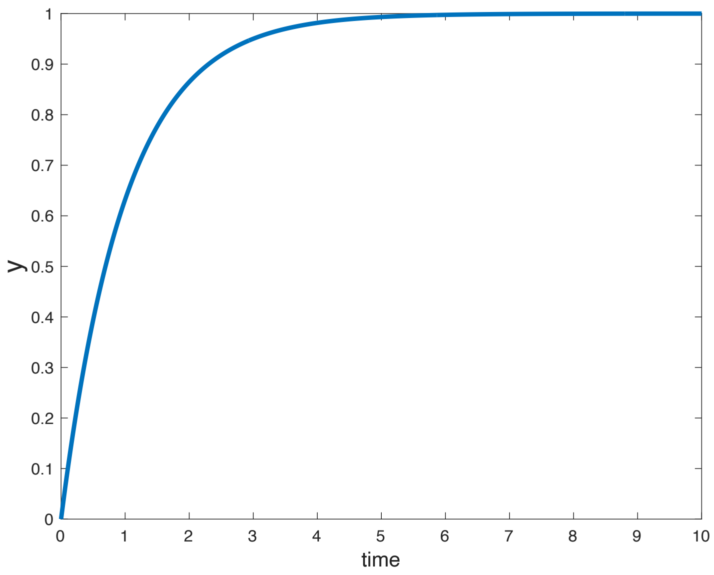
Ex) Unit step input $G_1(s) = \frac{1}{s+1}$ $G_2(s) = \frac{1}{s-1}$

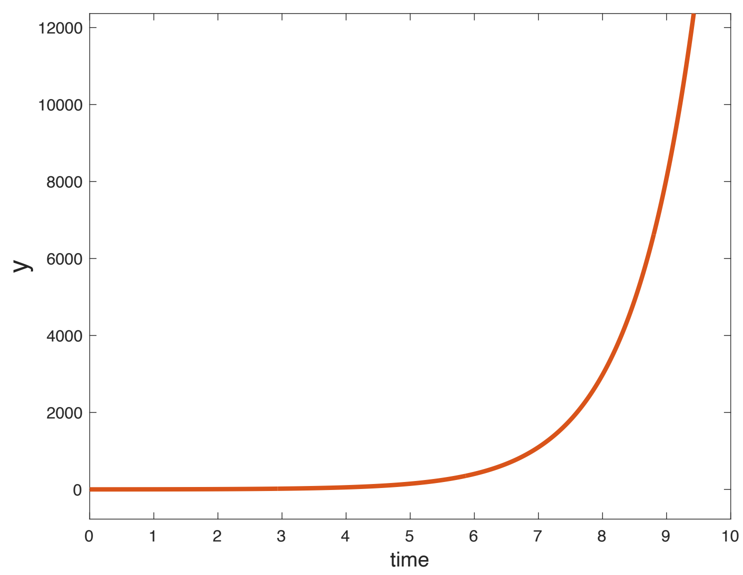
$$Y_1(s) = \frac{1}{s+1} \cdot \frac{1}{s} = -\frac{1}{s+1} + \frac{1}{s}$$

$$Y_2(s) = \frac{1}{s-1} \cdot \frac{1}{s} = \frac{1}{s-1} - \frac{1}{s}$$

$$y_1(t) = 1 - e^{-t}$$

$$y_2(t) = e^{t} - 1$$





Impulse response

$$y_1(t) = e^{-t}$$

$$y_2(t) = e^{t}$$

Match the transfer functions with impulse responses and determine their stability

TFs	Impulse Response	
$\frac{1}{(s-1)(s+5)}$	$a + be^{-5t}$	Stable
$\frac{1}{s(s+5)}$	$ae^{-2t} + be^{-5t}$	Unstable
$\frac{1}{(s+2)(s+5)}$	$ae^{t} + be^{-5t}$	Can't tell

System Gain

$$\text{Gain} = \frac{\text{Output Change}}{\text{Input Change}} = \frac{y'(\infty)}{u'(\infty)}$$

Step change in the input of size M $\longrightarrow$ Ultimate response in y?

$$y'(\infty) = \lim_{s\to 0} s\left(G(s)\frac{M}{s}\right) = \lim_{s\to 0} G(s)M$$

Hence, we get

$$\text{Gain} = \frac{y'(\infty)}{u'(\infty)} = \frac{\lim_{s\to 0} G(s)M}{M} = \lim_{s\to 0} G(s)$$

(   ) is the gain; this works only when the dynamics is stable.

Determine the process gain of the TFs below

$$G(s) = \frac{1}{(s+2)(s+5)}$$

$$G(s) = \frac{5s+2}{(6s+7)(7s^2+2s+5)}$$

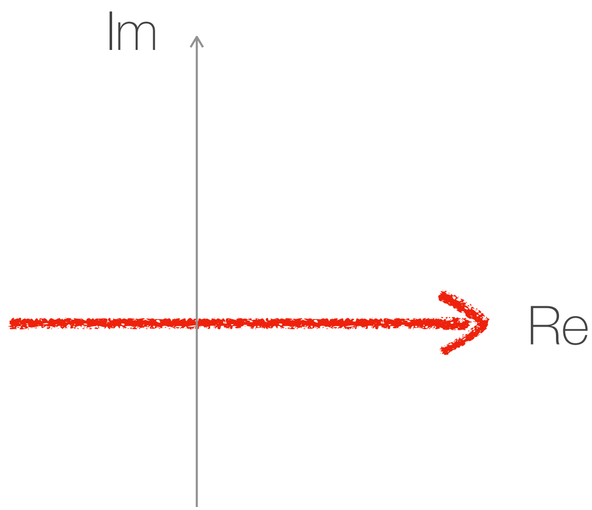
$$G(s) = \frac{1}{(s-2)(s+5)}$$

Oscillation (Underdamped)

Nonoscillatory input ⟶ Oscillatory response

If the poles are (   ) numbers (w/ nonzero imaginary parts), the dynamics is underdamped

$$\exists_i \quad Im(P_i) \neq 0$$



Pole's Location

$$G_1(s) = \frac{1}{s^2 + 2s + 2}$$ oscill. stable

$p = -1 \pm j$

$$G_1(s) = \frac{1}{s^2 - 2s + 2}$$ oscill. unstable

$p = +1 \pm j$

Unit step input

$$Y_1(s) = \frac{1}{(s^2 + 2s + 2) \cdot s} = \frac{-0.25 + 0.25j}{s - (-1 + j)} + \frac{-0.25 - 0.25j}{s - (-1 - j)} + \frac{0.5}{s}$$

$$y_1(t) = (-0.25 + 0.25j)e^{(-1+j)t} + (-0.25 - 0.25j)e^{(-1-j)t} + 0.5$$

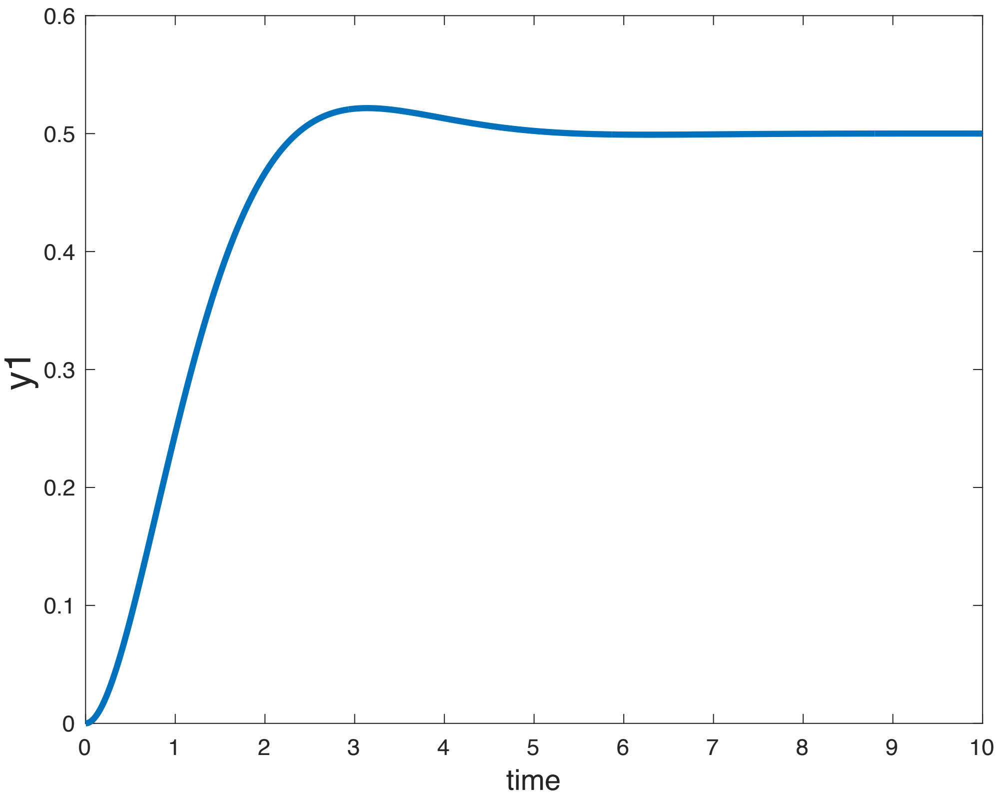
$$e^{(-1+j)t} = e^{-t}\,(\cos t + j \sin t)$$

Leonhard Euler (1707-1783)

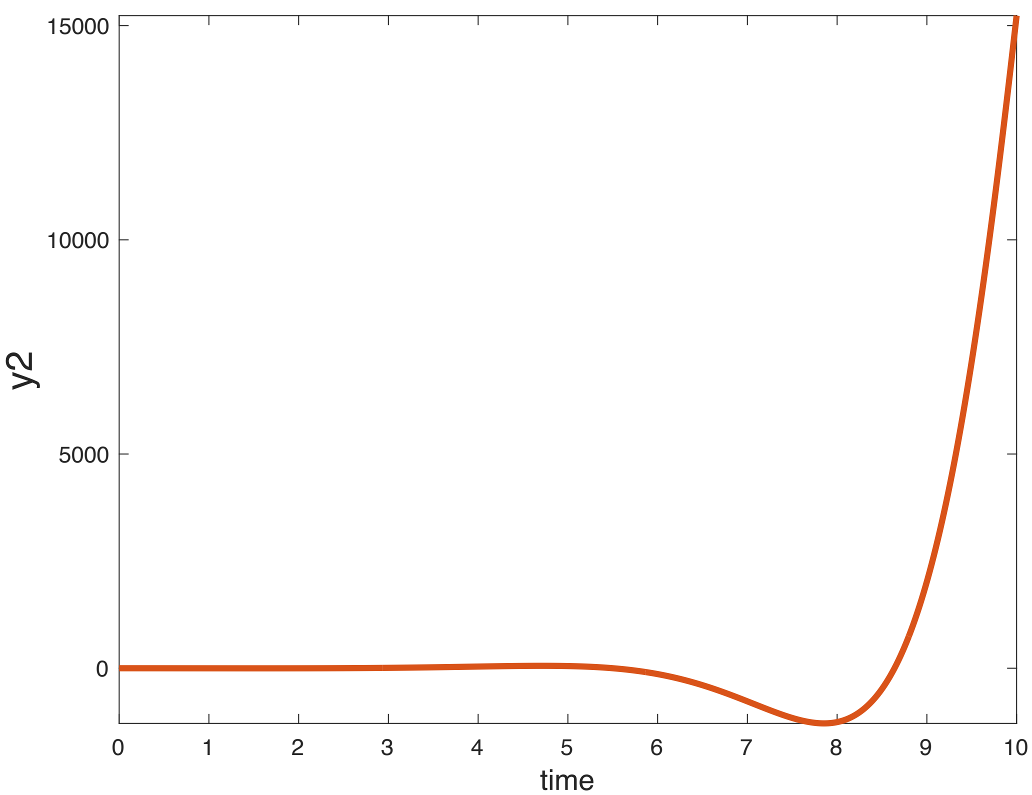
$$= 0.5 - 0.5e^{-t}(\cos t + \sin t)$$

$$Y_2(s) = \frac{1}{(s^2 - 2s + 2) \cdot s} = \frac{-0.25 - 0.25j}{s - (1 + j)} + \frac{-0.25 + 0.25j}{s - (1 - j)} + \frac{0.5}{s}$$

$$y_2(t) = 0.5 - 0.5e^{t}(\cos t + \sin t)$$



0.6
0.5
0.4
y1
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
8
9
10
time



15000
10000
y2
5000
0
0
1
2
3
4
5
6
7
8
9
10
time

Overshoot and Inverse Response

Existence of overshoot or inverse response can be determined from zeros of the TF

Overshoot: "a" real Left-Half-Plane (negative) zero closer to the origin than the dominant pole (the pole that's closest to the origin)

Inverse response: "a" real Right-Half-Plane (positive) zero

The closer the RHP zero to the origin, the more pronounced the inverse response

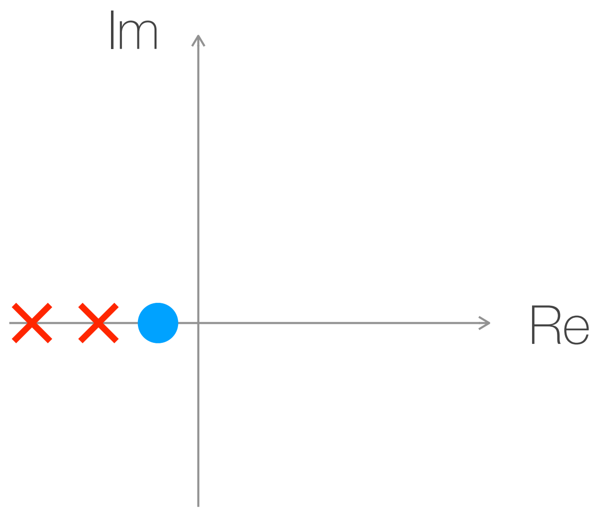
Draw the step response to each TF

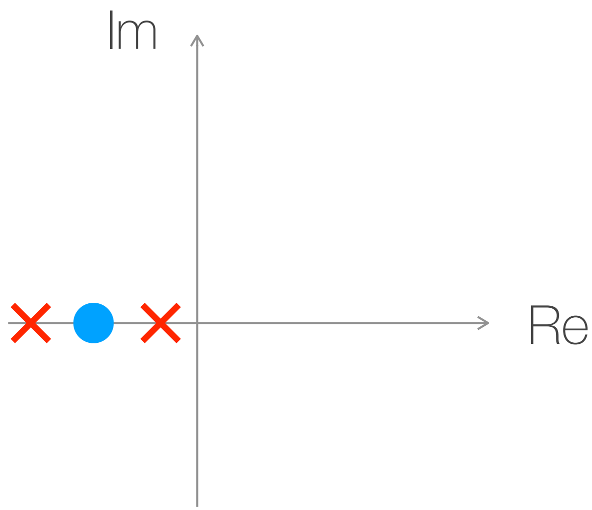
$$G_1(s) = \frac{(10s+1)}{(3s+1)(2s+1)}$$

$$G_2(s) = \frac{(2.5s+1)}{(3s+1)(2s+1)}$$

$$p_1 = -\frac{1}{3},\ p_2 = -\frac{1}{2},\ z_1 = -\frac{1}{10}$$

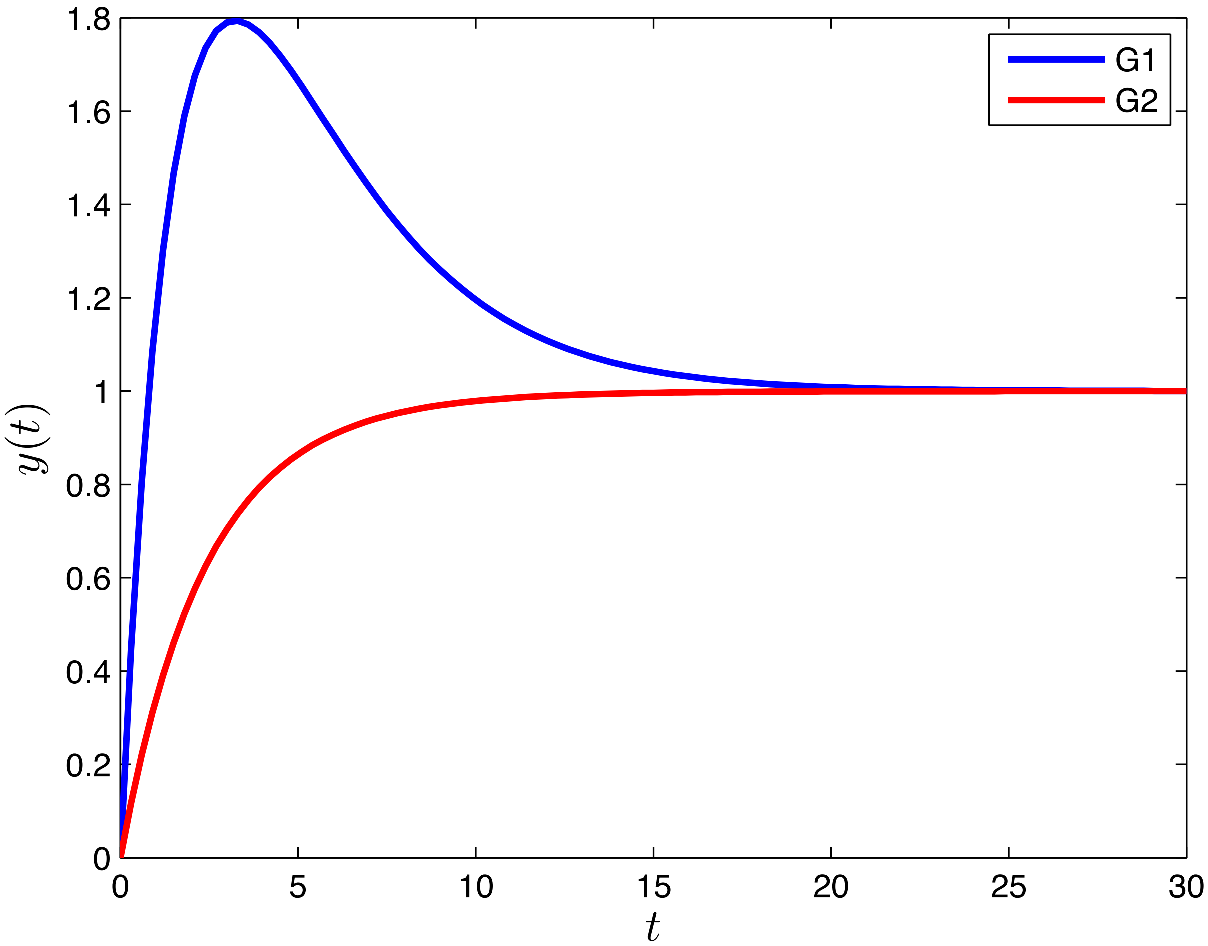
$$p_1 = -\frac{1}{3},\ p_2 = -\frac{1}{2},\ z_1 = -\frac{1}{2.5}$$





LHP zero, (   )

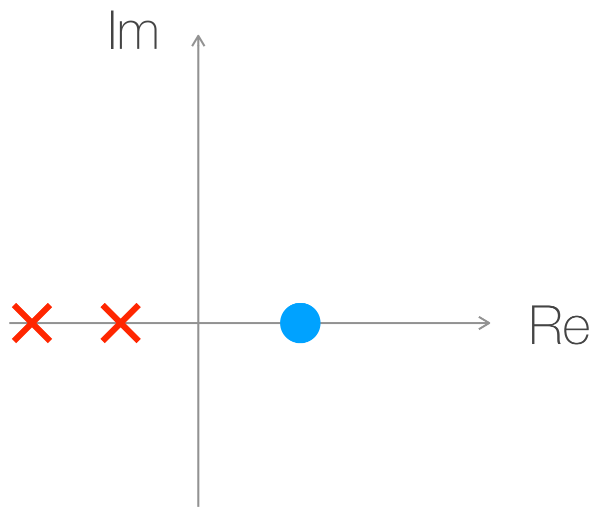
LHP zero, (   )


1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$y(t)$
0
5
10
15
20
25
30
t
G1
G2

Draw the step response to each TF

$$G_3(s) = \frac{(-2.5s+1)}{(3s+1)(2s+1)}$$

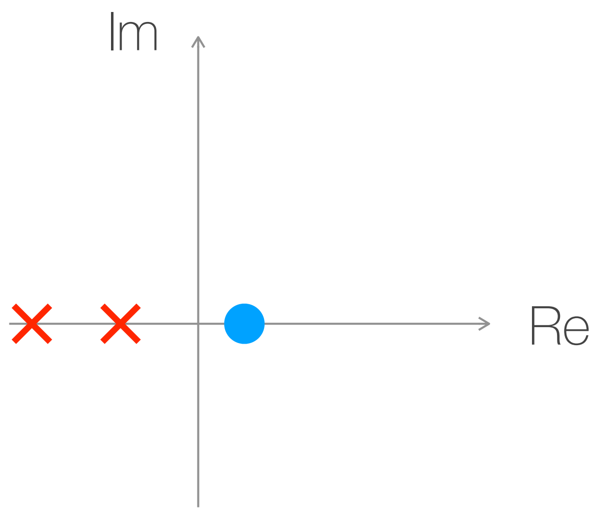
$$p_1 = -\frac{1}{3},\ p_2 = -\frac{1}{2},\ z_1 = \frac{1}{2.5}$$



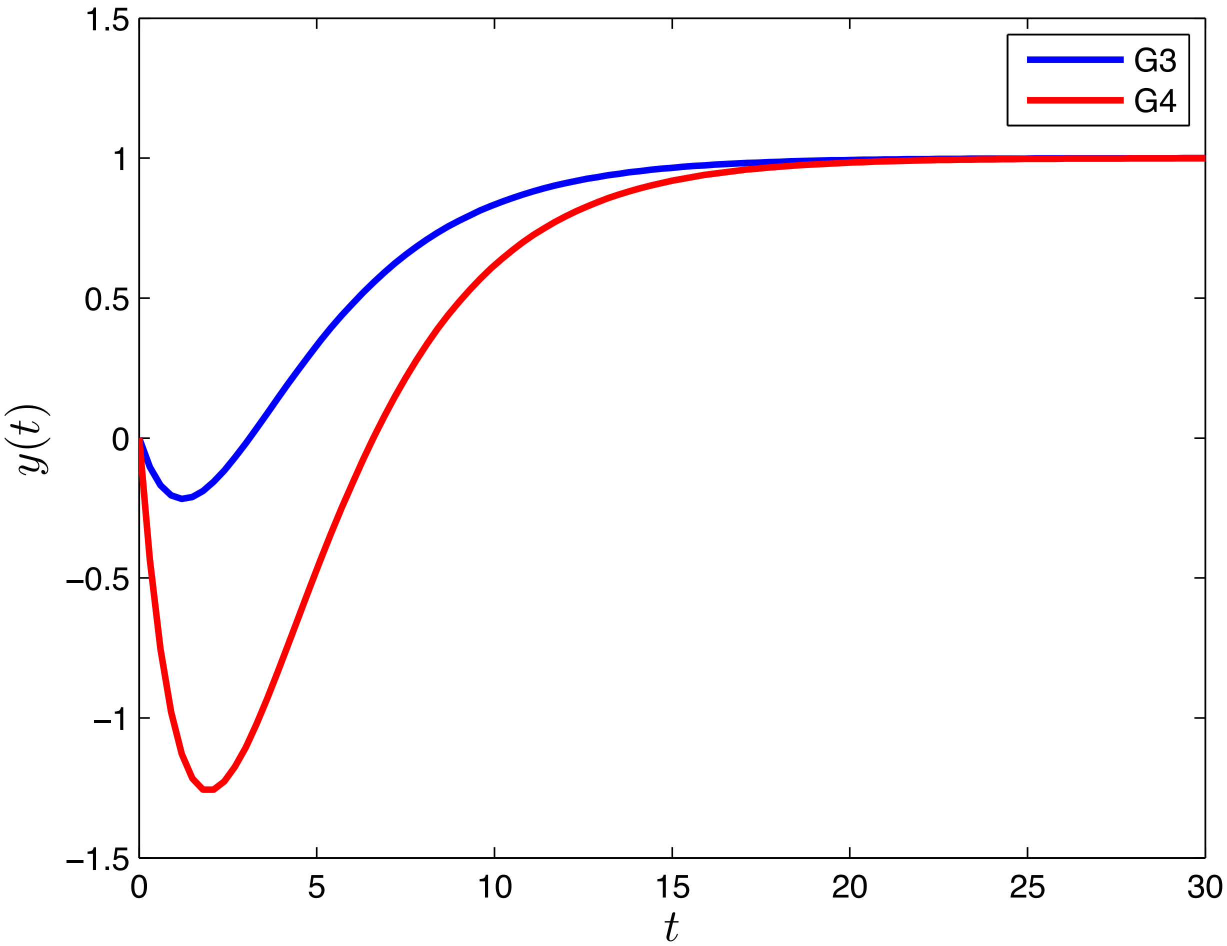
RHP zero, Inverse Response

$$G_4(s) = \frac{(-10s+1)}{(3s+1)(2s+1)}$$

$$p_1 = -\frac{1}{3},\ p_2 = -\frac{1}{2},\ z_1 = \frac{1}{10}$$

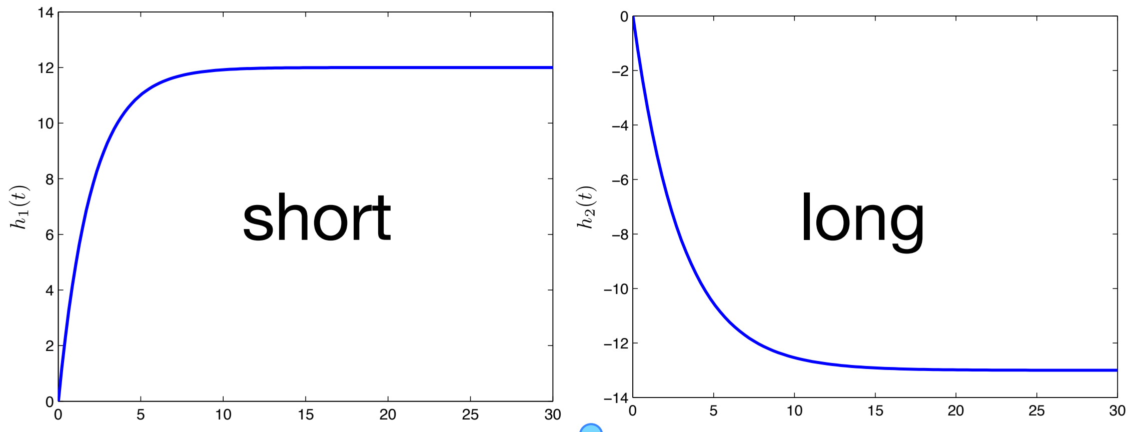


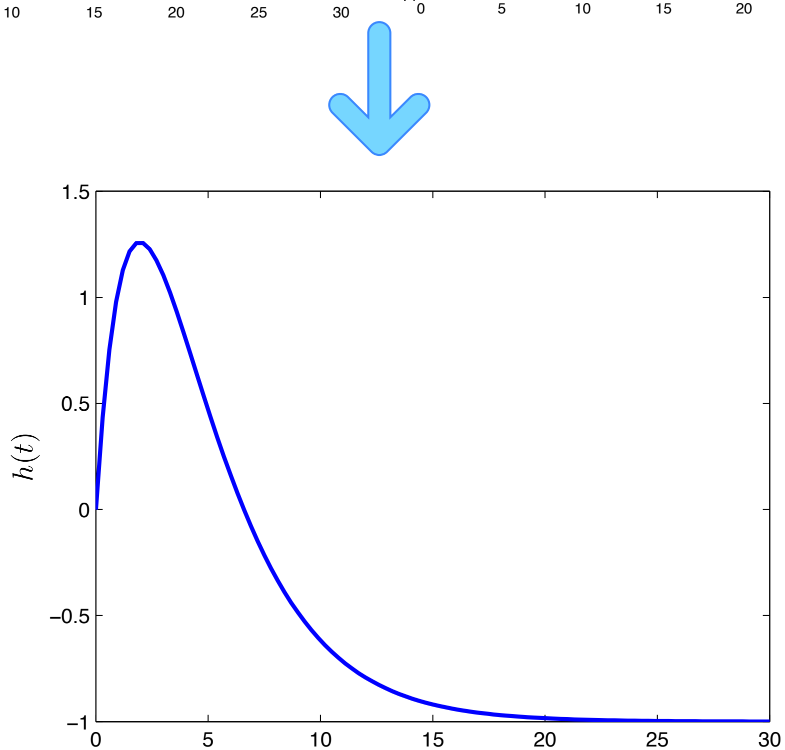
RHP zero, ( )



1.5
1
0.5
0
−0.5
−1
−1.5
$y(t)$
0
5
10
15
20
25
30
t
G3
G4

Inverse Response Example





Reboiler level response to an increase in the steam flow rate

Short-term (fast), smaller effect: An increase in the boiler rate intensifies frothing in the tray above and causes a larger spill-over, increasing the level

Long-term (slow), dominant effect: More liquid is boiled-off, decreasing the level

Some Specific Results

- A real positive zero is a sufficient condition for inverse response to be exhibited
- An odd number of zeros with positive real part results in the initial slope of a step response being in the "wrong" direction (inverse response)
- Only complex (i.e. non-real) zeros with positive real parts are not sufficient to cause inverse response - but "may" occur depending on the definition of inverse response. (cannot generalize)

Varied Definition of Inverse Response

Textbook Definition

The initial response to a step input is in one direction but the final steady state is in the opposite direction

$$\left.\frac{dy'}{dt}\right|_{t=0^+} \times y'(t=\infty) < 0$$

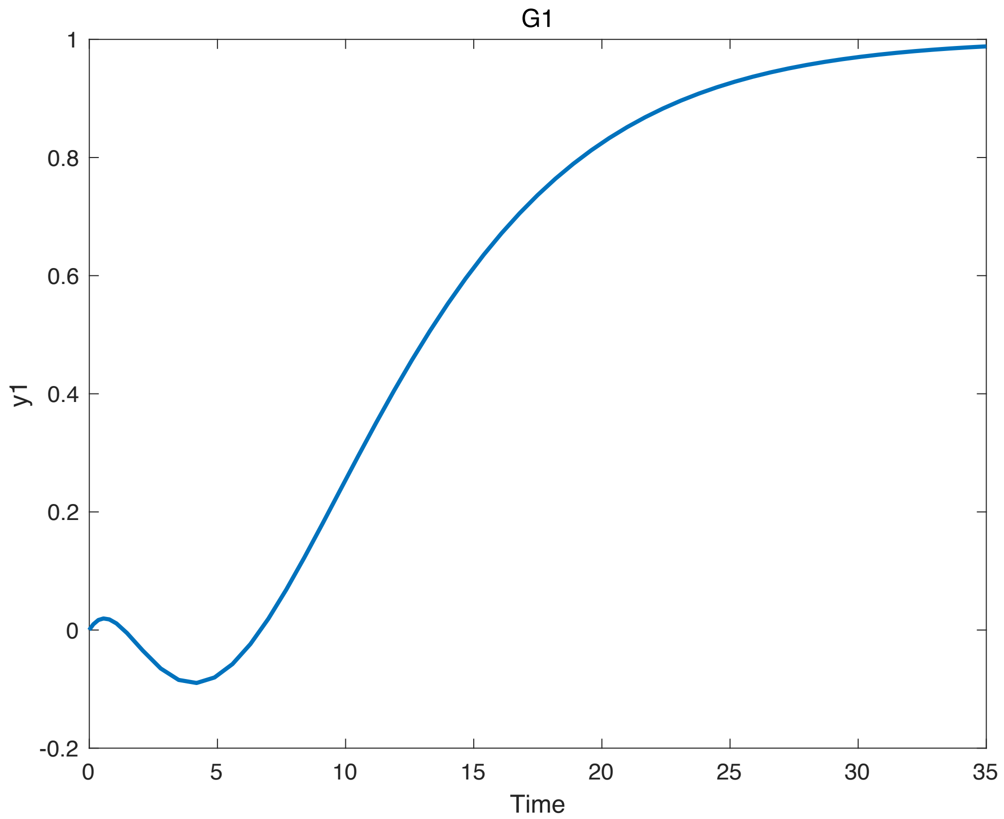
Another Definition (Not Well-Known)

$$\frac{|y'(t=\infty)|}{\max|y'(t)|} \rightarrow 1 \quad \text{AND} \quad y'(t_0) = 0 \text{ for some } t_0$$

Multiple RHP Zeros

$$G_1(s) = \frac{(-3s+1)(-s+1)}{(2s+1)(5s+1)(4s+1)}$$

The system with an **odd** number of RHP zeros exhibits **true** inverse response in the sense that the **initial** direction of the step response will always be **opposite** to the direction of the final steady state, regardless of the number of inversions involved in this response.

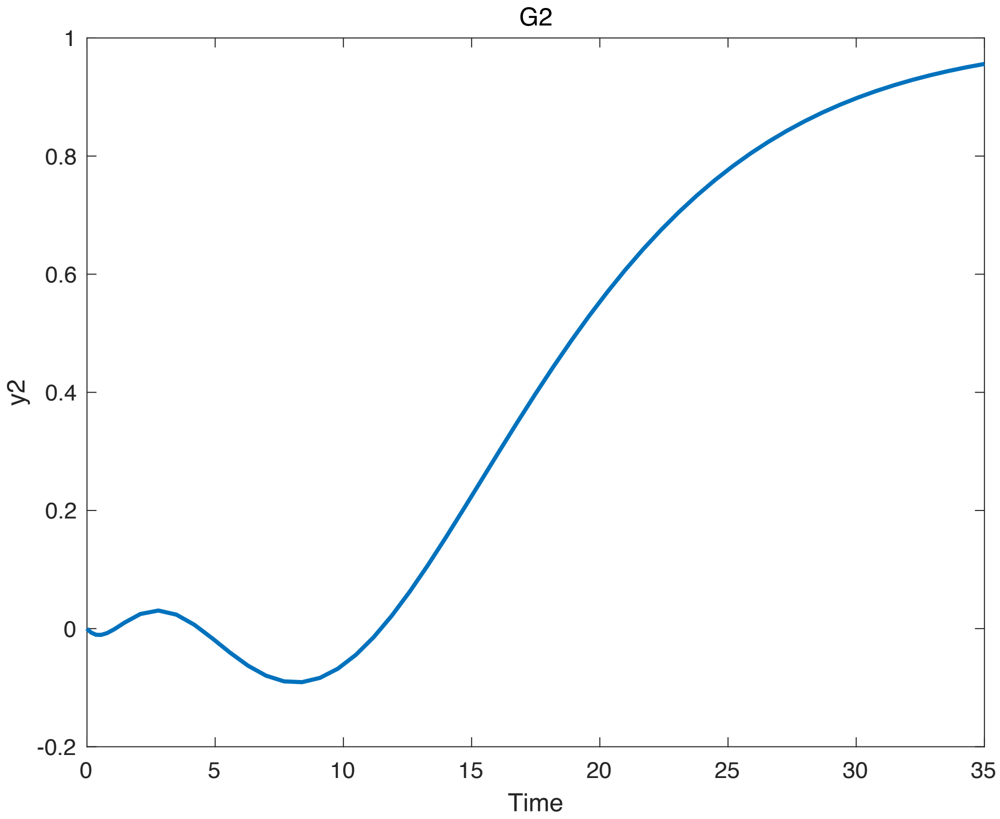


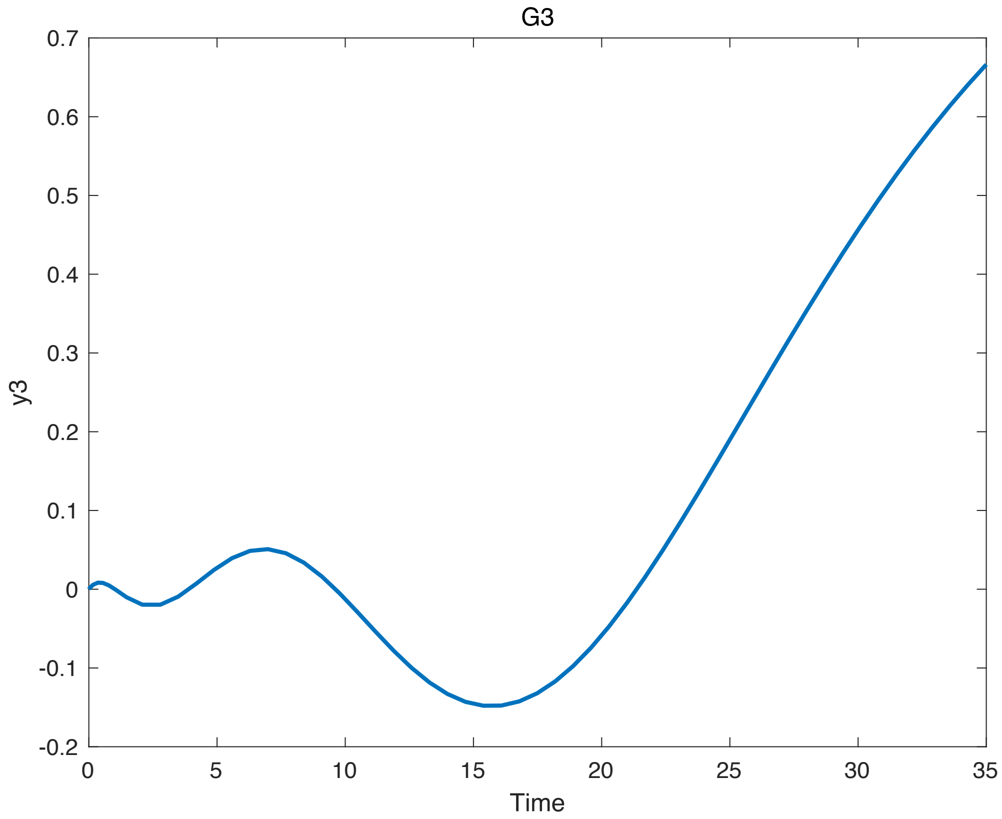
On the other hand, the initial portion of the step response of a system with an **even** number of RHP zeros exhibits the same even number of inversions before heading in the direction of the final steady state, but the initial direction is always the same as the direction of the final steady state.

Multiple RHP Zeros

$$G_2(s) = \frac{(-3s+1)(-s+1)(-2.5s+1)}{(2s+1)(5s+1)(4s+1)(3.5s+1)}$$

$$G_3(s) = \frac{(-3s+1)(-s+1)(-2.5s+1)(-6s+1)}{(2s+1)(5s+1)(4s+1)(3.5s+1)(7s+1)}$$

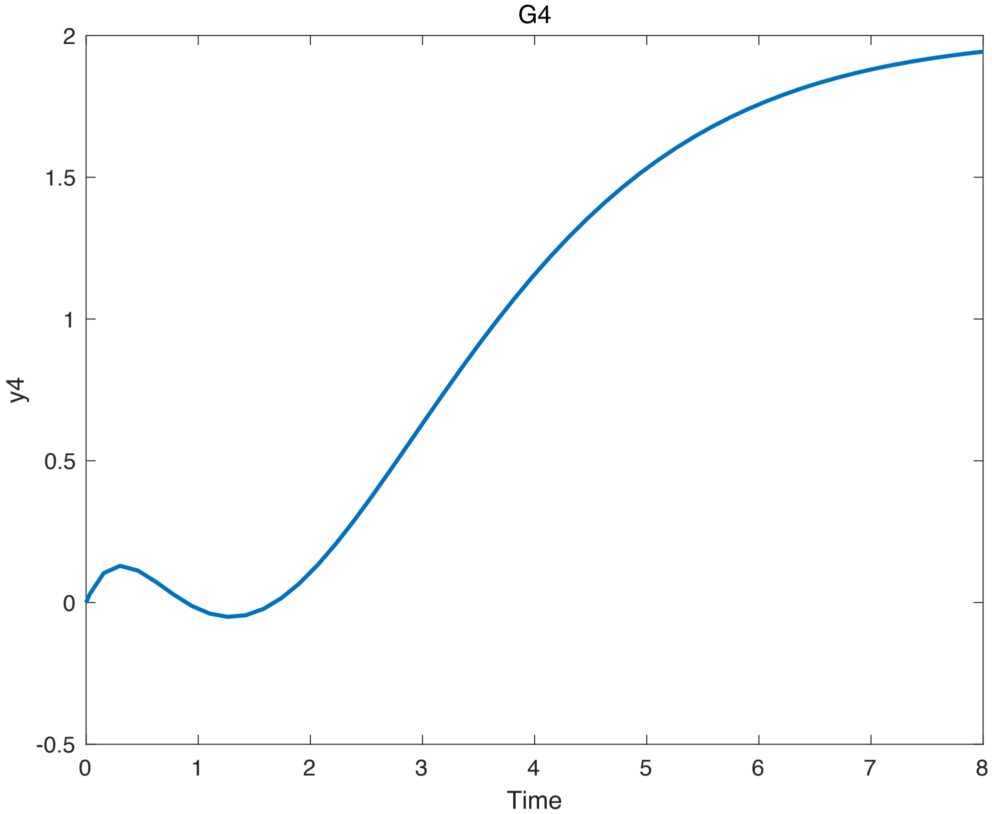




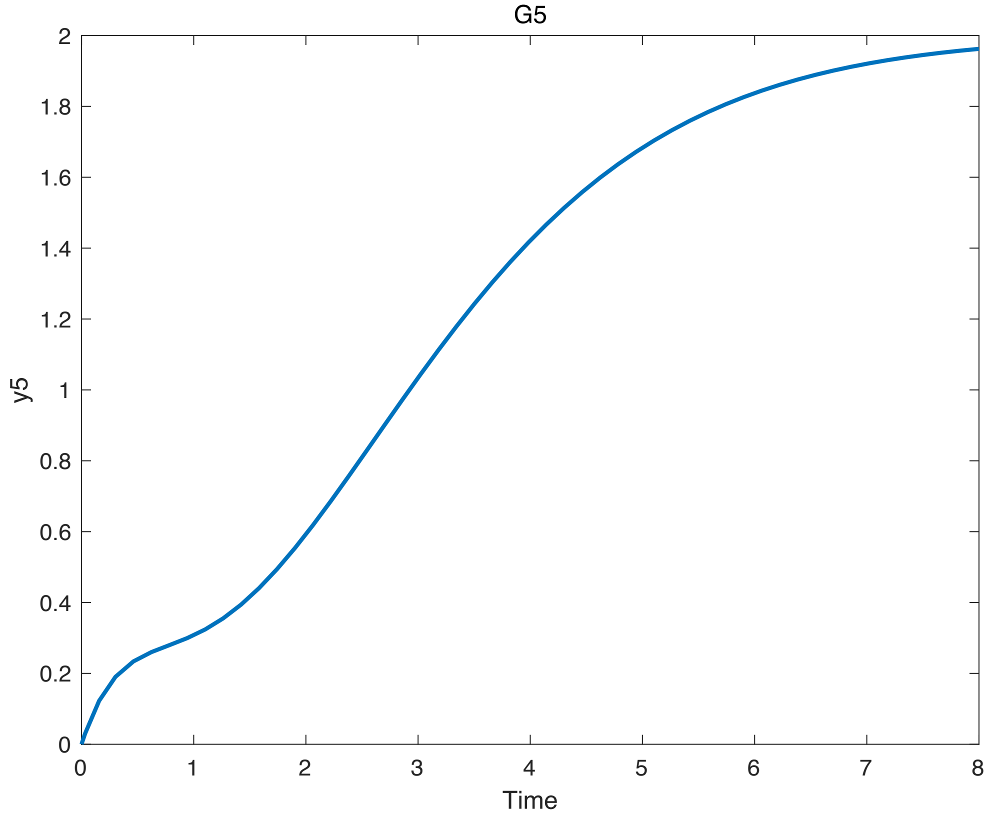
Complex RHP Zeros; even number

$$G_4(s) = \frac{s^2 - 2s + 2}{(s+1)^3}$$

$$G_5(s) = \frac{s^2 - 0.2s + 2}{(s+1)^3}$$



Two complex RHPs



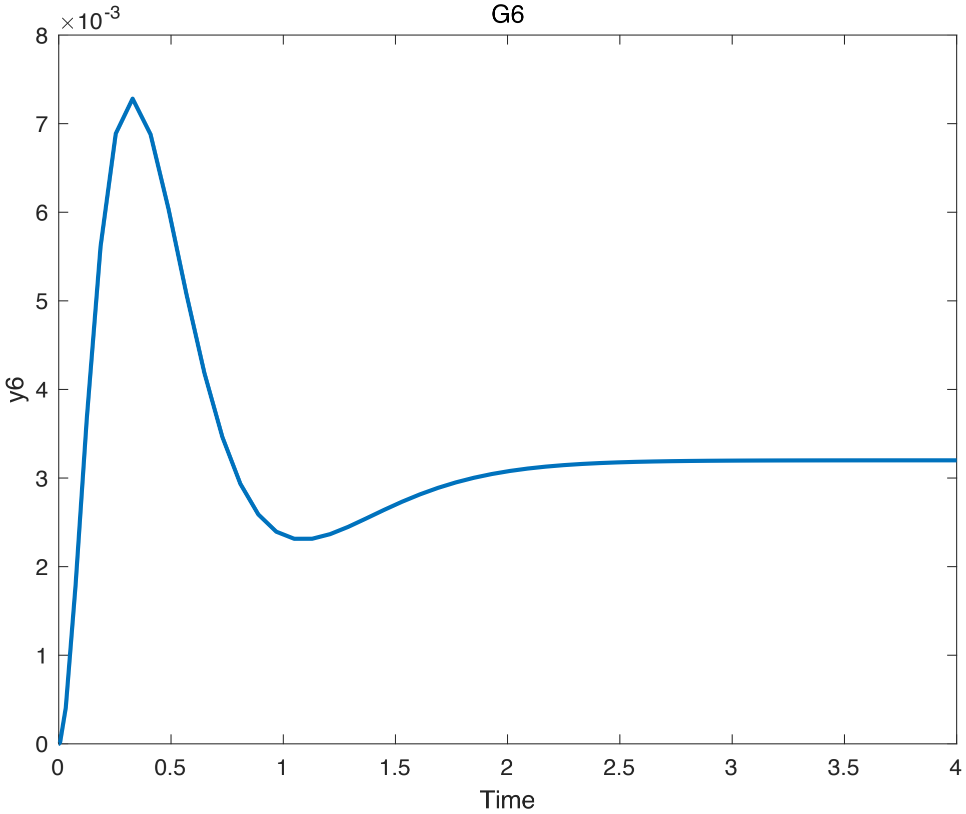
Two complex RHPs

(No inverse response at all)

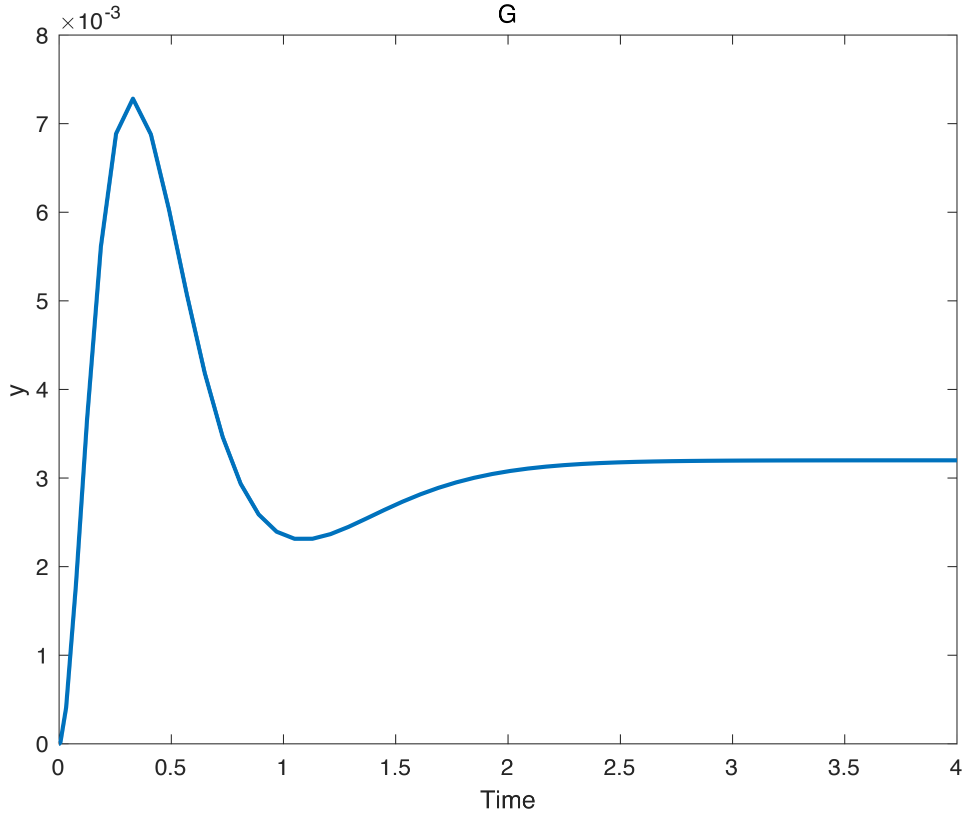
Complex LHP Zeros

$$G_6(s) = \frac{s^2 + 2s + 2}{(s+5)^4}$$

$$G_7(s) = \frac{(s^2 + 2s + 2)(s+3)}{(s+5)^4}$$



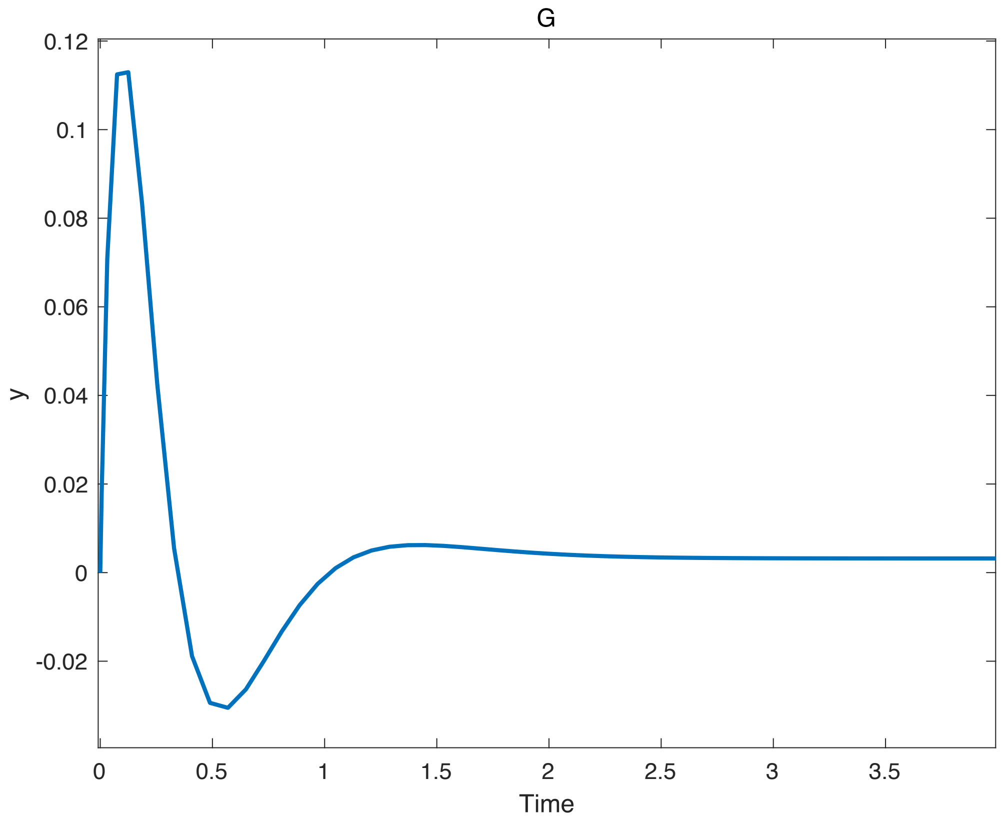
Two complex LHP zeros
(overshoot + undershoot)



Two “dominant” complex LHP zeros
(overshoot + undershoot)

Complex LHP Zeros

$$G_8(s) = \frac{(s^2+2s+2)(3s+1)}{(s+5)^4}$$



One "dominant" real LHP zero (overshoot + undershoot)

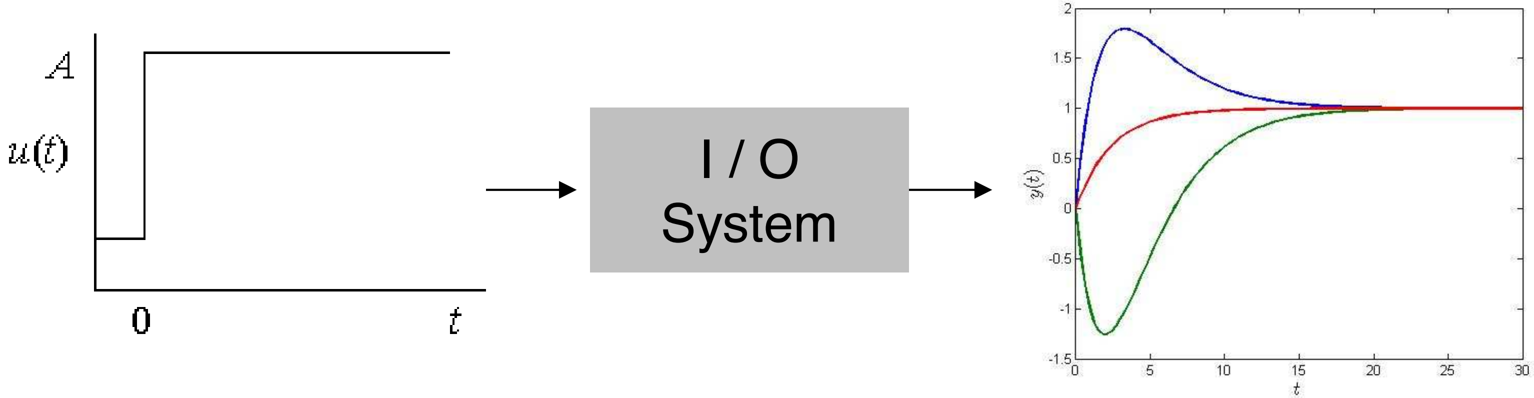
Does not show a true overshoot — cannot generalize

Speed of Response

Speed of response is determined roughly by the dominant pole (the pole that's close to the origin), which corresponds to the slowest time constant.

$$\text{Settling time} \cong 3 \sim 5 \times \frac{1}{\text{dominant pole}}$$

2^{nd} Order System Plus a Zero



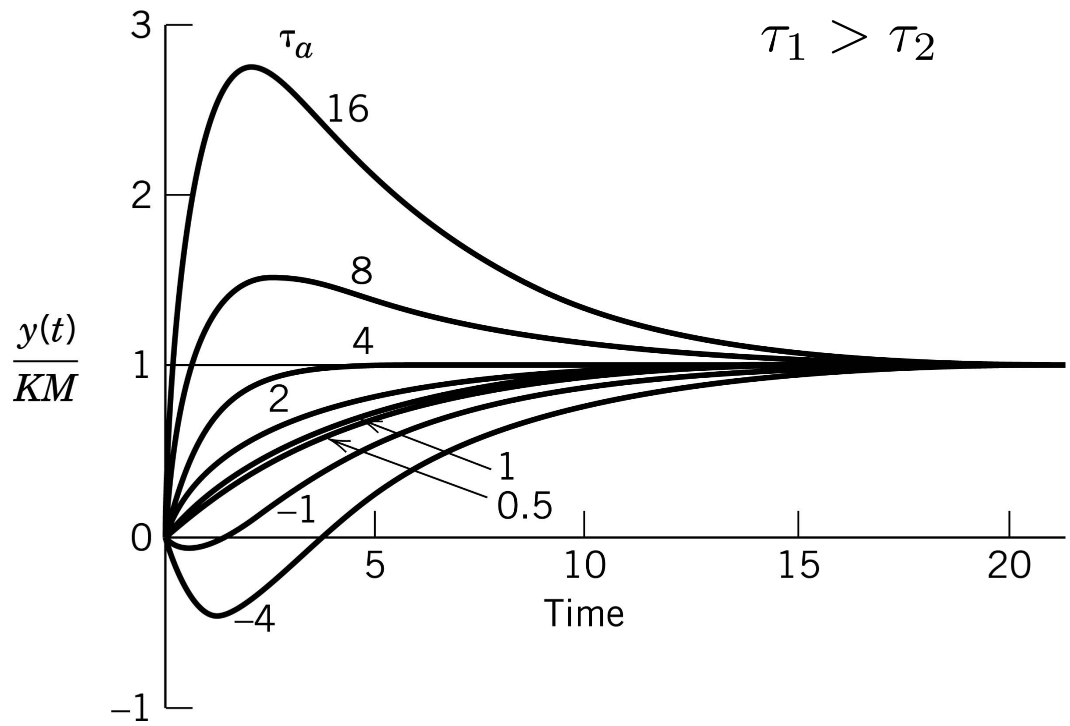
$$Y(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} U(s)$$

$$\Longleftrightarrow \quad \tau_1 \tau_2 \frac{d^2 y}{dt^2} + 2(\tau_1 + \tau_2)\frac{dy}{dt} + y = K\left(\tau_a \frac{du}{dt} + u\right)$$

Possible responses

- Monotonic response (like the over damped 2^{nd} order system)
- Overshoot
- Inverse response

Effect of τ_a



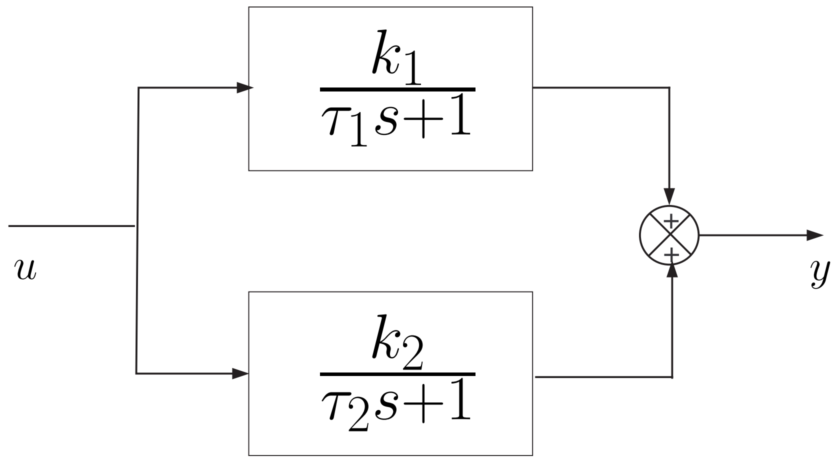
$\tau_a > \tau_1$: Overshoot

$\tau_a \leq \tau_1$: Overdamped response with no overshoot

$\tau_a < 0$: Inverse response (the initial response is the opposite direction to the final response

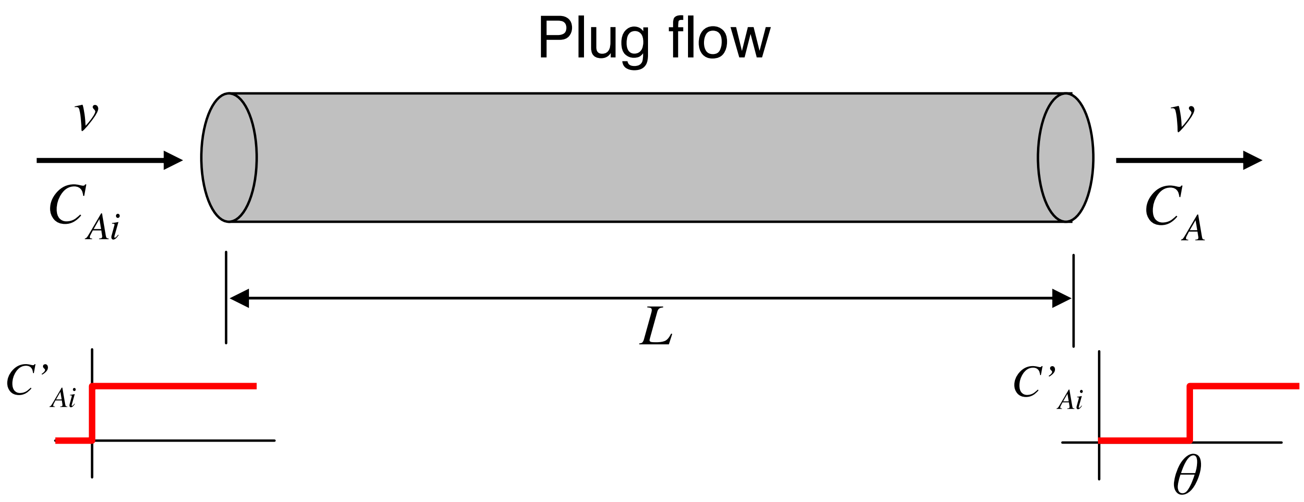
Example Scenario?

Two first-order effects in parallel:



$$\frac{Y(s)}{U(s)} = \frac{k_1}{\tau_1 s + 1} + \frac{k_2}{\tau_2 s + 1} = \frac{(k_1 + k_2)\left(\frac{\tau_2 k_1 + \tau_1 k_2}{k_1 + k_2} s + 1\right)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

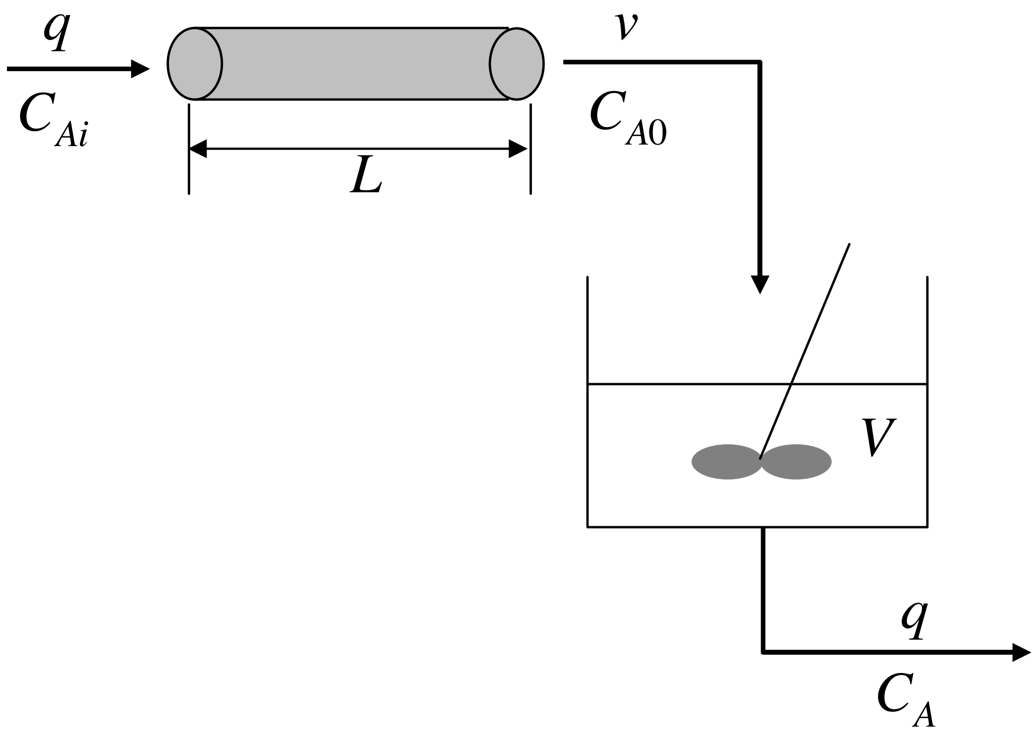
Transport Delays



$$C'_A(t) = C'_{Ai}(t-\theta) \overset{\mathscr{L}}{\to} C'_A(s) = e^{-\theta s} C'_{Ai}(s)$$

$\theta = \dfrac{L}{v}$: dead time or transport delay

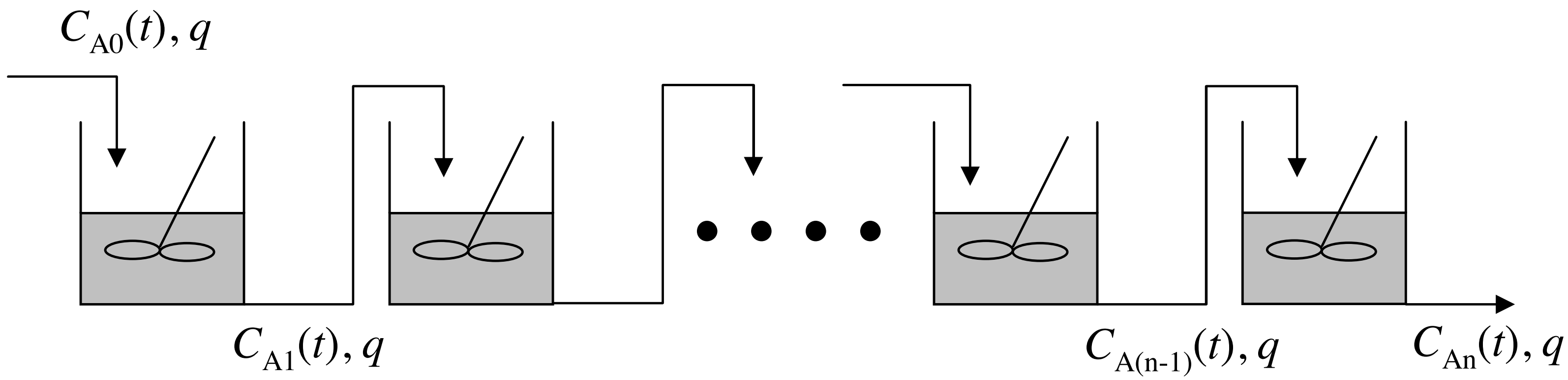
FOPTD System



$$C'_A(s) = \frac{K}{\tau s + 1} C'_{A0}(s), \quad C'_{A0}(s) = e^{-\theta s} C'_{Ai}(s)$$

$$C'_A(s) = \frac{K}{\tau s + 1} e^{-\theta s} C'_{Ai}(s), \quad K = 1, \ \tau = \frac{V}{q}, \ \theta = \frac{A \cdot L}{q}$$

Approximating a High-Order System with a Delay



$$C'_{A(i+1)}(s) = \frac{1}{\tau s + 1} C'_{Ai}(s) \Rightarrow C'_{An}(s) = \frac{1}{(\tau s + 1)^n} C'_{A0}(s)$$

$$\approx \boxed{} \text{(for large } n\text{)}$$

Note: $e^x \approx 1 + x, \quad \tau = V/q$

Approximating High-Order Systems with 1st or 2nd Order Plus Delay

Most chemical processes are of very high-order dynamics due to imperfect mixing, wall effect, flow dynamics, etc.

$$G(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)\cdots(\tau_n s + 1)}$$

Suppose τ_1 is a single dominant time constant: $\tau_1 >> \tau_2, \cdots, \tau_n$

$$G(s) \approx \frac{K}{\tau_1 s + 1} e^{-\theta s},$$

Suppose a case of two dominant time constants:

$\tau_1, \tau_2 >> \tau_3, \cdots, \tau_n$

$$G(s) \approx \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} e^{-\theta s},$$

Development of Empirical Models from Process Data

- In some situations, it is not feasible to develop a theoretical (first-principles) model due to:
  - Lack of information
  - Model complexity
  - Engineering effort required
- An attractive alternative: Develop an empirical dynamic model from input-output data
  - Advantage: Less effort is required
  - Disadvantage: The model is only valid (at best) for the range of data used in its development
  - Empirical models usually don't extrapolate very well

Fitting First-Order/Second-Order Model using Step Tests

- Simple TF models can be obtained graphically from step response data
- Process reaction curve: a plot of the output response of a process to a step change input
- If the process of interest can be approximated by a first- or second-order linear model, the model parameters can be obtained by inspection of the process reaction curve

First-Order Plus Time Delay Model

$$G(s) = \frac{Ke^{-\theta s}}{\tau s + 1}$$

- For this FOPTD model we note the following characteristics of its step response:
  - The response attains 63.2% of its final response at time, t = $\tau + \theta$
  - The line drawn tangent to the response at maximum slope (t = θ) intersects the y/KM = 1 line at t = $\tau + \theta$
  - The step response is essentially completed at t = 5τ. In other words, the settling time is $t_s = 5\tau$