Laplace's Equation & Poisson's Equation

Introduction to Electromagnetism with Practice

Theory & Applications

Sunkyu Yu

Dept. of Electrical and Computer Engineering

Seoul National University

Laplace Equations – Spherical Coordinates

Starting from SoV for Spherical Coordinates

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]V=0$$

Separation of Variables: $V=R(r)Y(\theta,\varphi)$

$$\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}=0$$

$\times\frac{r^2}{RY}$

$$\underbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)}_{l(l+1)}+\underbrace{\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}}_{-l(l+1)}=0$$

$l(l+1)$

$-l(l+1)$

The Radial Equation

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1)$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-l(l+1)R=0$$

$$\boxed{R=c_l r^l + d_l r^{-l-1}}$$

The Angular Equation

$$\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}=-l(l+1)$$

$\times Y\sin^2\theta$

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{\partial^2 Y}{\partial\varphi^2}=-\left[l(l+1)\sin^2\theta\right]Y$$

Separation of Variables: $Y(\theta,\varphi)=P(\theta)Q(\varphi)$ $\quad \times\frac{1}{PQ}$

$$\underbrace{\frac{1}{P}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[l(l+1)\sin^2\theta\right]}_{m^2}+\underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{-m^2}=0$$

The Angular Equation

$$\frac{d^2 Q}{d\varphi^2} + m^2 Q = 0 \quad \Rightarrow \quad Q = e^{im\varphi} \qquad (m = 0, \ \pm 1, \ \pm 2, \ \ldots)$$

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \left[l(l+1)\sin^2\theta - m^2\right]P = 0$$

$$\frac{d}{d\theta} = -\sin\theta \frac{d}{dx} \qquad x = \cos\theta$$

$$(1-x^2)\frac{d^2 P}{dx^2} - 2x\frac{dP}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0$$

$$P = P(x) = P(\cos\theta), \quad -1 \le x \le 1 \text{ from } 0 \le \theta \le \pi$$

P(*x*): The Associated Legendre Functions

Interim Summary 1

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]V=0$$

$$\boxed{V(r,\theta,\varphi)=R(r)Y(\theta,\varphi)=R(r)P(\cos\theta)Q(\varphi)}$$

The Radial Equation

$$\boxed{R(r)=c_l r^l + d_l r^{-l-1}}$$

$$\boxed{Q(\varphi)=e^{im\varphi}\quad (m=0,\ \pm1,\ \pm2,\ \ldots)}$$

P(*x*): The Associated Legendre Functions

$$\boxed{(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0}$$

$$P=P(x)=P(\cos\theta),\ \ -1\le x\le 1 \text{ from } 0\le\theta\le\pi$$

The Associated Legendre Functions

P(*x*): The Associated Legendre Functions

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0$$

$$(m=0,\ \pm1,\ \pm2,\ \ldots)$$

We'd like to remove this denominator

$$P=P(x)=P(\cos\theta),\ \ -1\le x\le 1 \text{ from } 0\le\theta\le\pi$$

$$P(x)=\left(1-x^2\right)^n v(x)$$

$$P(x)=\left(1-x^2\right)^{\frac{m}{2}} v(x)$$

$$(1-x^2)v''-2(m+1)xv'+\left[l(l+1)-m(m+1)\right]v=0$$

when *n* = *m*/2

Differentiating

v -> v', m -> m+1

$$(1-x^2)\left(v'\right)''-2\left[(m+1)+1\right]x\left(v'\right)'+\left[l(l+1)-(m+1)(m+1+1)\right]\left(v'\right)=0$$

Differentiating

v -> v'', m -> m+2

$$(1-x^2)\left(v''\right)''-2\left[(m+2)+1\right]x\left(v''\right)'+\left[l(l+1)-(m+2)(m+2+1)\right]\left(v''\right)=0$$

The Associated Legendre Functions

$$(1-x^2)v''-2(m+1)xv'+\left[l(l+1)-m(m+1)\right]v=0$$ when $n = m/2$

Differentiating

$v \to v'$, $m \to m+1$

$$(1-x^2)(v')''-2\left[(m+1)+1\right]x(v')'+\left[l(l+1)-(m+1)(m+1+1)\right](v')=0$$

Differentiating

$v \to v''$, $m \to m+2$

$$(1-x^2)(v'')''-2\left[(m+2)+1\right]x(v'')'+\left[l(l+1)-(m+2)(m+2+1)\right](v'')=0$$

If we know the solution $v = P_l$ for $m = 0$

$$(1-x^2)P_l''-2xP_l'+l(l+1)P_l=0$$

Differentiating

$$v\big|_{m=1}=\frac{dP_l}{dx}$$

Differentiating

$$v\big|_{m=2}=\frac{d^2P_l}{dx^2}$$

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0$$

$$v\big|_{m}=\frac{d^mP_l}{dx^m}$$

$$P(x)=P_l^m(x)=\left(1-x^2\right)^{\frac{m}{2}}\frac{d^mP_l}{dx^m}$$

$$(m=0,\ \pm1,\ \pm2,\ \ldots)$$

Interim Summary 2

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]V=0$$

$$V(r,\theta,\varphi)=R(r)Y(\theta,\varphi)=R(r)P_l^m(\cos\theta)Q_m(\varphi)$$

The Radial Equation

$$R(r)=c_l r^l+d_l r^{-l-1}$$

$$Q_m(\varphi)=e^{im\varphi}$$

$$(m=0,\ \pm1,\ \pm2,\ \ldots)$$

$$P_l^m(x)=\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m P_l}{dx^m}$$

$P_l^m(x)$: The Associated Legendre Functions

$$(1-x^2)\frac{d^2P_l^m}{dx^2}-2x\frac{dP_l^m}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P_l^m=0$$

$$P_l^m=P_l^m(x)=P_l^m(\cos\theta),\ \ -1\le x\le 1 \text{ from } 0\le\theta\le\pi$$

Legendre's Equation

$$(1-x^2)\frac{d^2P_l}{dx^2}-2x\frac{dP_l}{dx}+l(l+1)P_l=0$$

The Legendre Functions

Legendre's Equation

$$(1-x^2)\frac{d^2P_l}{dx^2}-2x\frac{dP_l}{dx}+l(l+1)P_l=0$$

$$P_l=\sum_{k=0}^{\infty}a_kx^k$$

Homework or Exam

$$a_{k+2}=-\frac{(l+k+1)(l-k)}{(k+1)(k+2)}a_k$$

Convergence for **|*x*| < 1**

Remember that ...

$$\psi=R(r)Y(\theta,\varphi)=\frac{u(r)}{r}P_l^m(\cos\theta)Q_m(\varphi)$$

$$P_l^m(x)=P_l^m(\cos\theta)=\left(1-x^2\right)^{\frac{m}{2}}\frac{d^mP_l}{dx^m},$$

$$-1\le x\le 1 \text{ from } 0\le\theta\le\pi$$

We need to obtain the convergence for **|*x*| ≤ 1**

$$P_l=a_0\left[1-\frac{l(l+1)}{2!}x^2+\frac{l(l+1)(l-2)(l+3)}{4!}x^4-\frac{l(l+1)(l-2)(l+3)(l-4)(l+5)}{6!}x^6+\cdots\right]$$
$$+a_1\left[x-\frac{(l-1)(l+2)}{3!}x^3+\frac{(l-1)(l+2)(l-3)(l+4)}{5!}x^5-\frac{(l-1)(l+2)(l-3)(l+4)(l-5)(l+6)}{7!}x^7+\cdots\right]$$

To obtain the convergence at **|*x*| = 1:** Integer value of l

$l=0$	$l=1$	$l=2$	$l=3$	...
$P_0=1$	$P_1=x$	$P_2=\frac{1}{2}\left(3x^2-1\right)$	$P_3=\frac{1}{2}\left(5x^3-3x\right)$	

Legendre Polynomial

Restrictions on *l* and *m*

$$P_l = a_0\left[1-\frac{l(l+1)}{2!}x^2+\frac{l(l+1)(l-2)(l+3)}{4!}x^4-\frac{l(l+1)(l-2)(l+3)(l-4)(l+5)}{6!}x^6+\cdots\right]$$
$$+a_1\left[x-\frac{(l-1)(l+2)}{3!}x^3+\frac{(l-1)(l+2)(l-3)(l+4)}{5!}x^5-\frac{(l-1)(l+2)(l-3)(l+4)(l-5)(l+6)}{7!}x^7+\cdots\right]$$

l **is set to be a nonnegative integer**

$l=3$	$P_3=\frac{1}{2}\left(5x^3-3x\right)$
$l=2$	$P_2=\frac{1}{2}\left(3x^2-1\right)$
$l=1$	$P_1=x$
$l=0$	$P_0=1$
$l=-1$	$P_{-1}=1=P_0$
$l=-2$	$P_{-2}=x=P_1$
$l=-3$	$P_{-3}=\frac{1}{2}\left(3x^2-1\right)=P_2$

$\boldsymbol{m = 0, \pm 1, \pm 2, \ldots, \pm l}$

$$(m=0,\ \pm 1,\ \pm 2,\ \ldots)$$

$$P_l^m(x)=P_l^m(\cos\theta)=\left(1-x^2\right)^{\frac{m}{2}}\frac{d^mP_l}{dx^m},$$

$$-1\le x\le 1 \text{ from } 0\le\theta\le\pi$$

$$P_l^m(x)=0 \text{ for } |m|>l$$

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0 \qquad P_l^m(x)\propto P_l^{-m}(x)$$

Solution Summary

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]V=0$$

$$V(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}R_l^m(r)P_l^m(\cos\theta)Q_m(\varphi)$$

The Radial Equation

$$R_l^m(r)=c_l^m r^l+d_l^m r^{-l-1}$$

$$Q_m(\varphi)=e^{im\varphi}$$

$l=0,\ 1,\ 2,\ \dots$

$m=0,\ \pm1,\ \pm2,\ \dots,\ \pm l$

$P_0=1,\ P_1=x,\ P_2=\frac{1}{2}\left(3x^2-1\right),\dots$

$P_l^m(x)$: The Associated Legendre Functions

$$(1-x^2)\frac{d^2P_l^m}{dx^2}-2x\frac{dP_l^m}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P_l^m=0$$

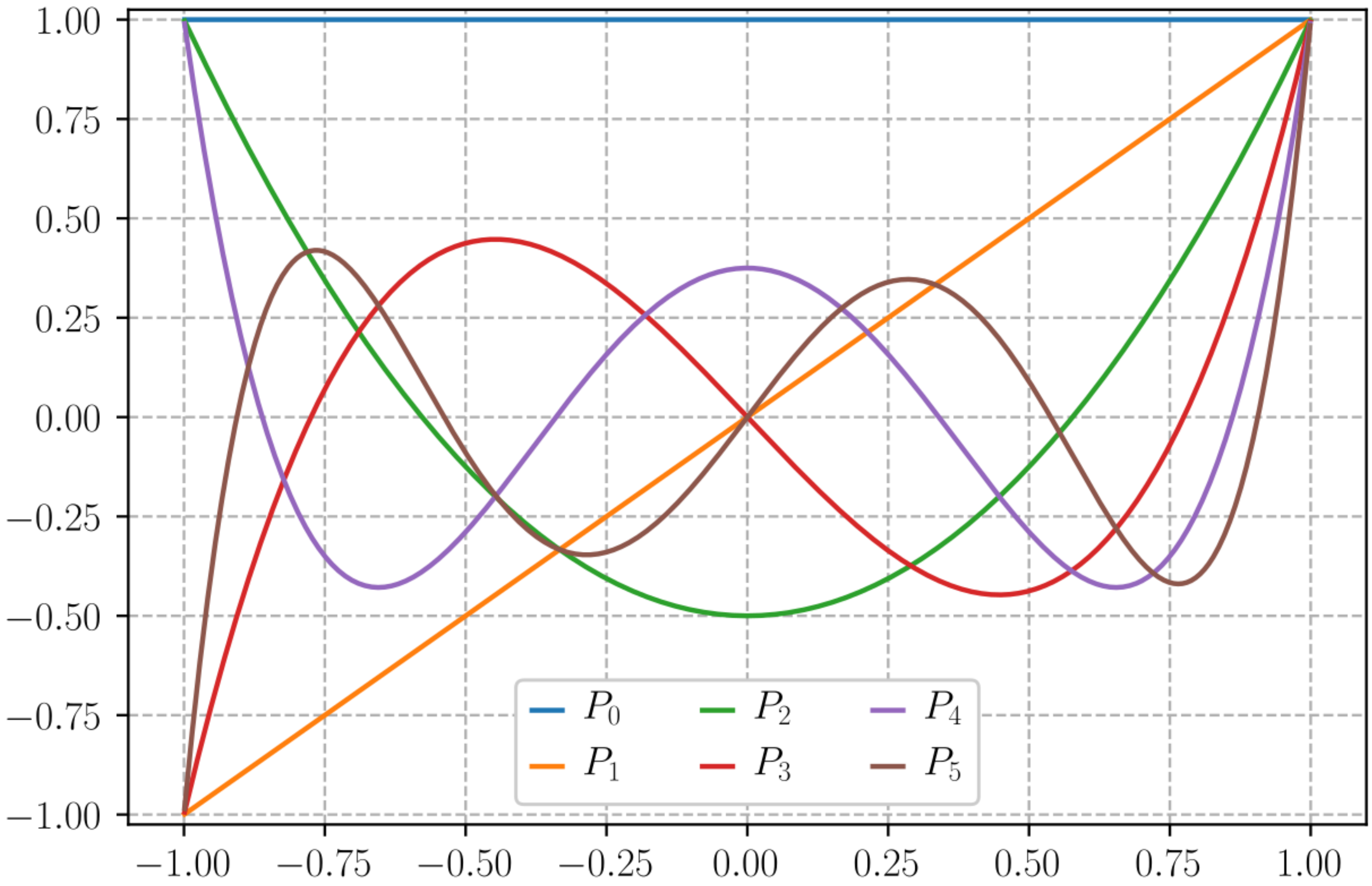
$P_l^m=P_l^m(x)=P_l^m(\cos\theta),\ \ -1\le x\le 1$ from $0\le\theta\le\pi$

$$P_l^m(x)=\left(1-x^2\right)^{\frac{m}{2}}\frac{d^mP_l}{dx^m}$$

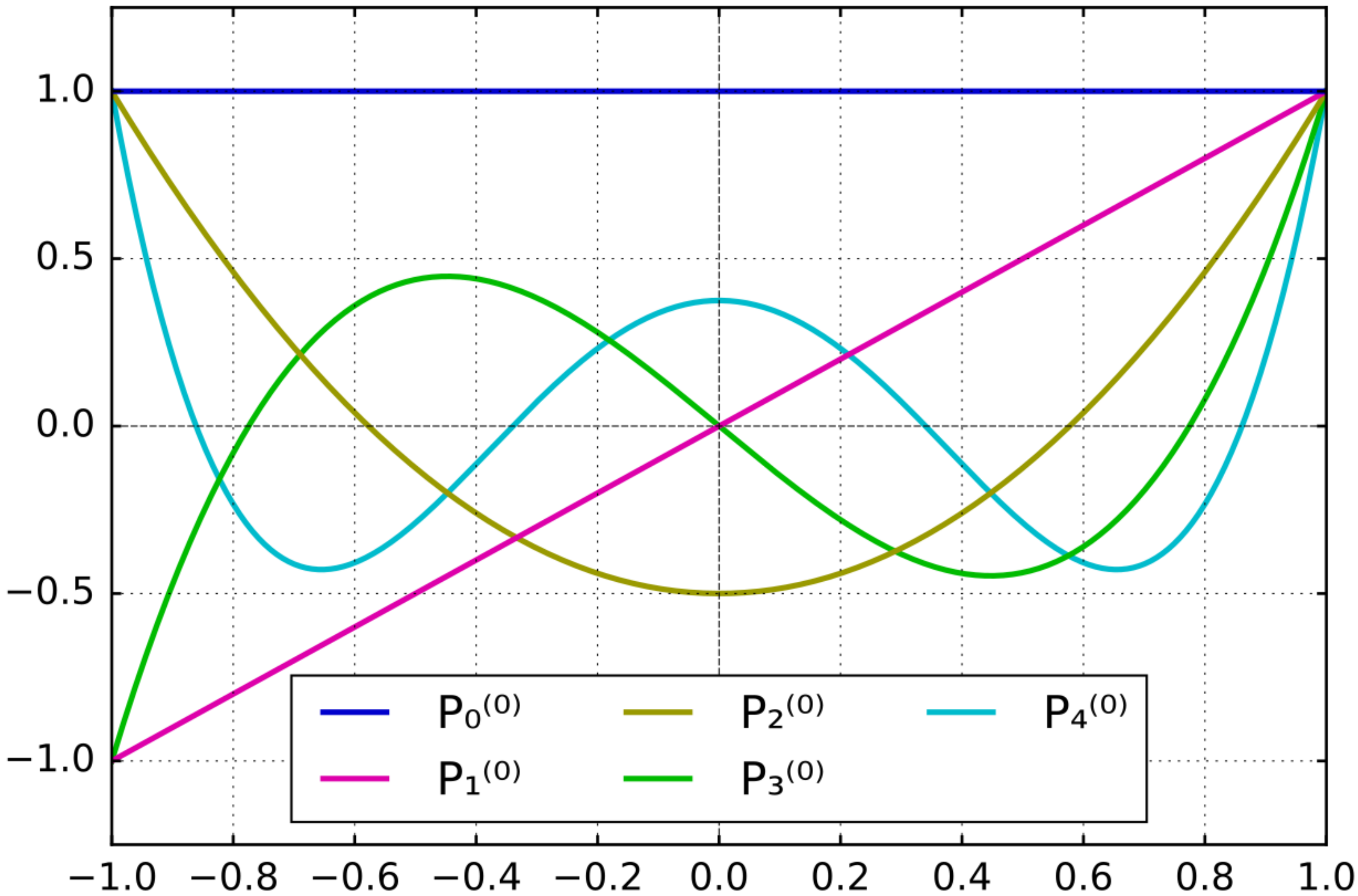
Legendre's Equation

$$(1-x^2)\frac{d^2P_l}{dx^2}-2x\frac{dP_l}{dx}+l(l+1)P_l=0$$

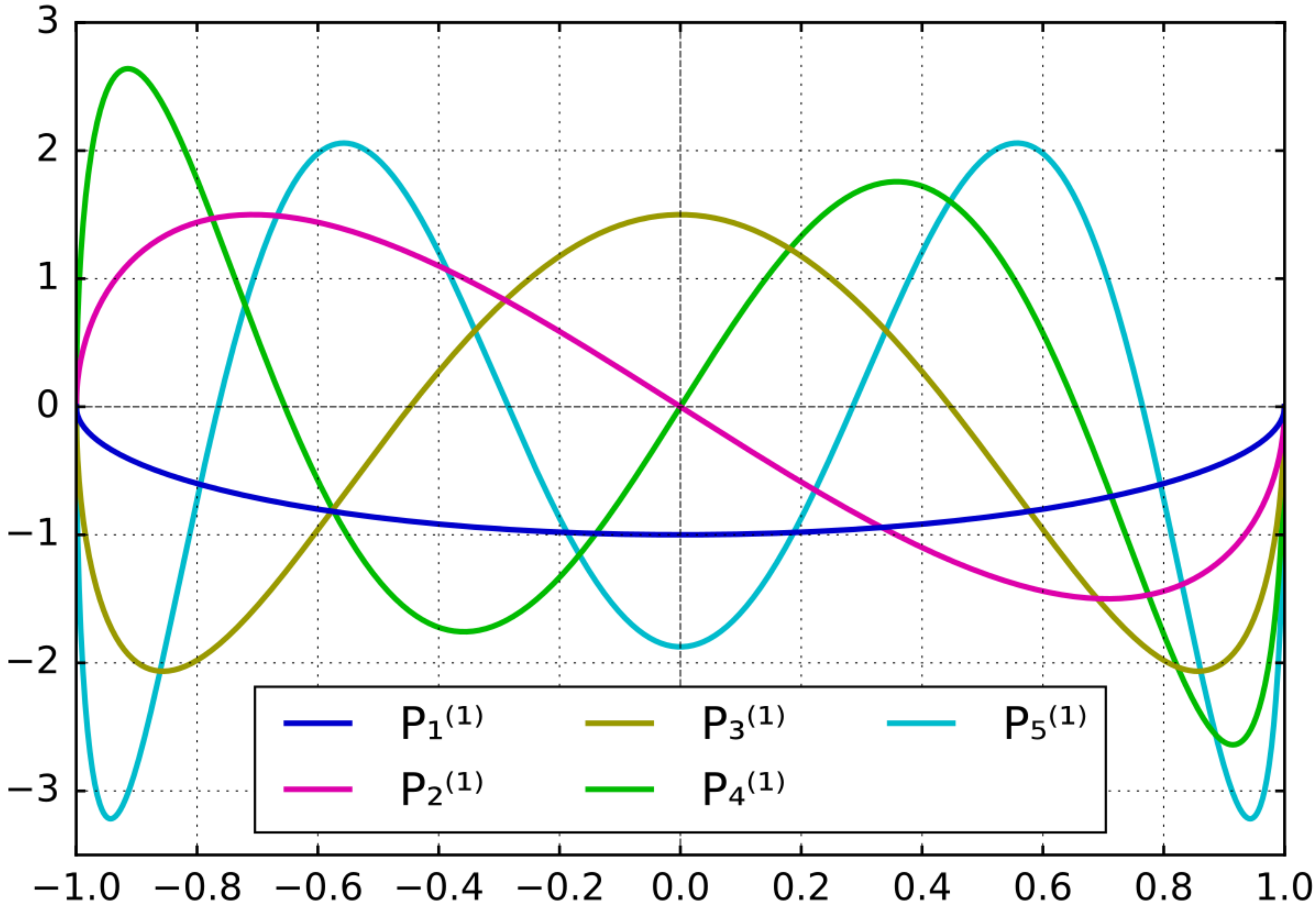
Legendre Polynomials



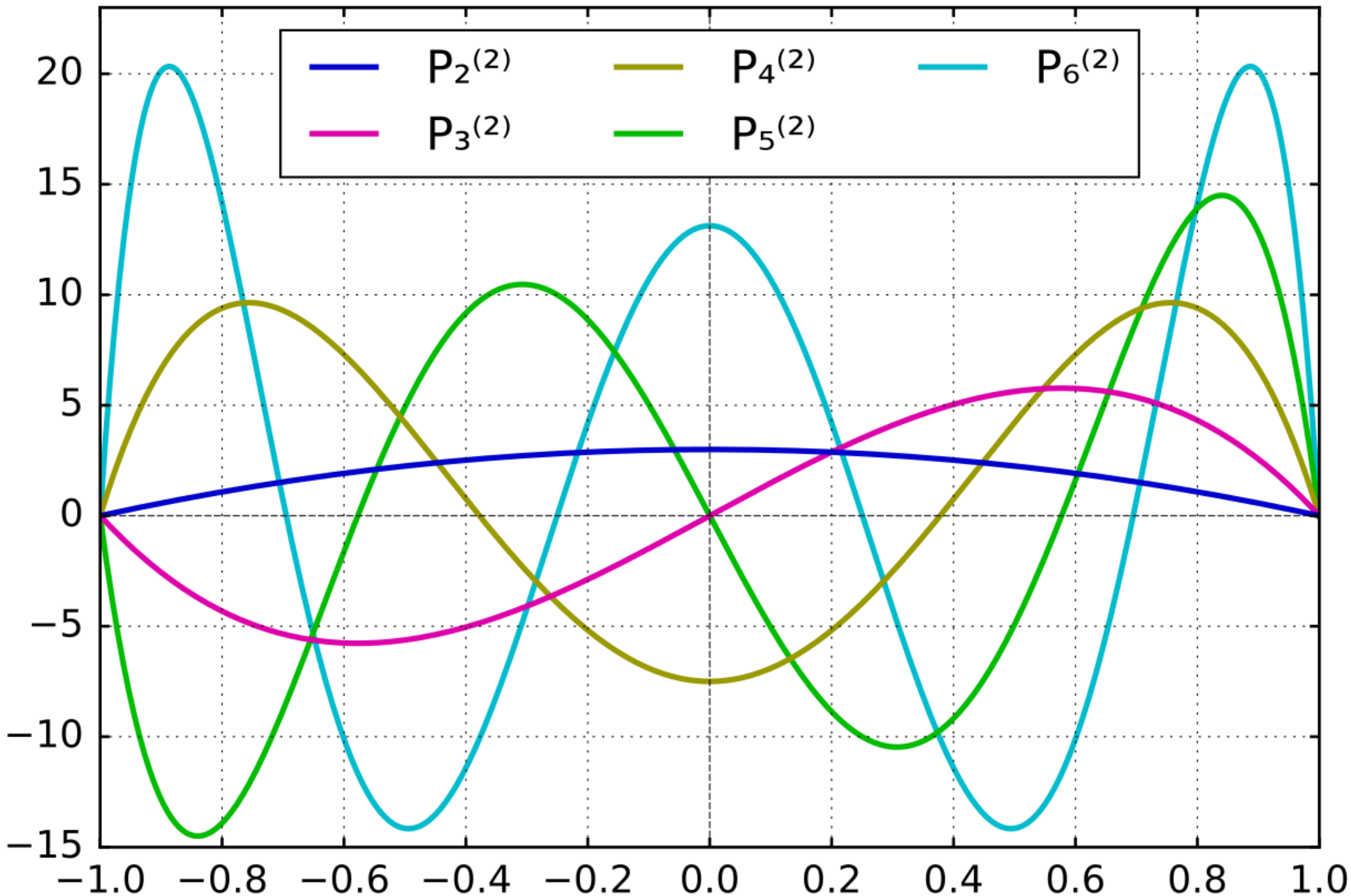
Associated Legendre Polynomials: *m* = 0



Associated Legendre Polynomials: m = 1



Associated Legendre Polynomials: *m* = 2



Spherical Harmonics – Basis for Angular Responses

$$Y_l^m(\theta,\varphi) = c_{lm} P_l^m(\cos\theta) Q_m(\varphi) = c_{lm} e^{im\varphi} P_l^m(\cos\theta)$$

Let's consider the integral:

$$\int_0^{2\pi}\int_0^{\pi}\left[Y_{l'}^{m'}(\theta,\varphi)\right]^*\left[Y_l^m(\theta,\varphi)\right]\sin\theta\, d\theta\, d\varphi$$

$$\int_0^{2\pi}\int_0^{\pi}\left[c_{l'm'} e^{im'\varphi} P_{l'}^{m'}(\cos\theta)\right]^*\left[c_{lm} e^{im\varphi} P_l^m(\cos\theta)\right]\sin\theta\, d\theta\, d\varphi$$

$$= c_{l'm'} c_{lm} \int_0^{2\pi} e^{-im'\varphi} e^{im\varphi} d\varphi \int_0^{\pi} P_{l'}^{m'}(\cos\theta) P_l^m(\cos\theta)\sin\theta\, d\theta$$

$$= c_{l'm'} c_{lm}\left(2\pi\delta_{m'm}\right)\int_{-1}^{1} P_{l'}^{m'}(x) P_l^m(x)\, dx$$

Because of $\delta_{m'm}$, we just need to estimate

$$\int_{-1}^{1} P_{l'}^{m}(x) P_l^m(x)\, dx$$

Spherical Harmonics – Basis for Angular Responses

Proof for Orthogonality $\int_{-1}^{1} P_{l'}^m(x) P_l^m(x)\,dx$

$$(1-x^2)\frac{d^2P_l^m}{dx^2} - 2x\frac{dP_l^m}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m = 0 \qquad (1-x^2)\frac{d^2P_{l'}^m}{dx^2} - 2x\frac{dP_{l'}^m}{dx} + \left[l'(l'+1) - \frac{m^2}{1-x^2}\right]P_{l'}^m = 0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m = 0 \qquad \frac{d}{dx}\left[(1-x^2)\frac{dP_{l'}^m}{dx}\right] + \left[l'(l'+1) - \frac{m^2}{1-x^2}\right]P_{l'}^m = 0$$

$$P_{l'}^m\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_{l'}^m P_l^m = 0 \qquad P_l^m\frac{d}{dx}\left[(1-x^2)\frac{dP_{l'}^m}{dx}\right] + \left[l'(l'+1) - \frac{m^2}{1-x^2}\right]P_{l'}^m P_l^m = 0$$

$$P_{l'}^m\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m}{dx}\right] - P_l^m\frac{d}{dx}\left[(1-x^2)\frac{dP_{l'}^m}{dx}\right] + \left[l(l+1) - l'(l'+1)\right]P_{l'}^m P_l^m = 0$$

Spherical Harmonics – Basis for Angular Responses

$$P_{l'}^m \frac{d}{dx}\left[(1-x^2)\frac{dP_l^m}{dx}\right] - P_l^m \frac{d}{dx}\left[(1-x^2)\frac{dP_{l'}^m}{dx}\right] + \left[l(l+1) - l'(l'+1)\right]P_{l'}^m P_l^m = 0$$

$$\left[P_{l'}^m (1-x^2)\frac{dP_l^m}{dx}\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)\frac{dP_{l'}^m}{dx}\frac{dP_l^m}{dx}dx \right] - \left[P_l^m (1-x^2)\frac{dP_{l'}^m}{dx}\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)\frac{dP_{l'}^m}{dx}\frac{dP_l^m}{dx}dx \right]$$

$$= -\left[l(l+1) - l'(l'+1)\right]\int_{-1}^{1} P_{l'}^m P_l^m dx$$

$$\left[l(l+1) - l'(l'+1)\right]\int_{-1}^{1} P_{l'}^m P_l^m dx = 0$$

$$(l-l')(l+l'+1)\int_{-1}^{1} P_{l'}^m P_l^m dx = 0 \Rightarrow \int_{-1}^{1} P_{l'}^m P_l^m dx = 0 \quad \text{when} \quad l \neq l'$$

Spherical Harmonics – Basis for Angular Responses

We'll skip the derivation of normalization constant...

$$\int_{-1}^{1} \left| P_l^m(x) \right|^2 dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}$$

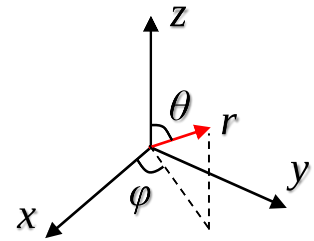
$$\int_0^{2\pi} \int_0^{\pi} \left[e^{im'\varphi} P_{l'}^{m'}(\cos\theta) \right]^* \left[e^{im\varphi} P_l^m(\cos\theta) \right] \sin\theta d\theta d\varphi = \frac{4\pi}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{m'm} \delta_{l'l}$$

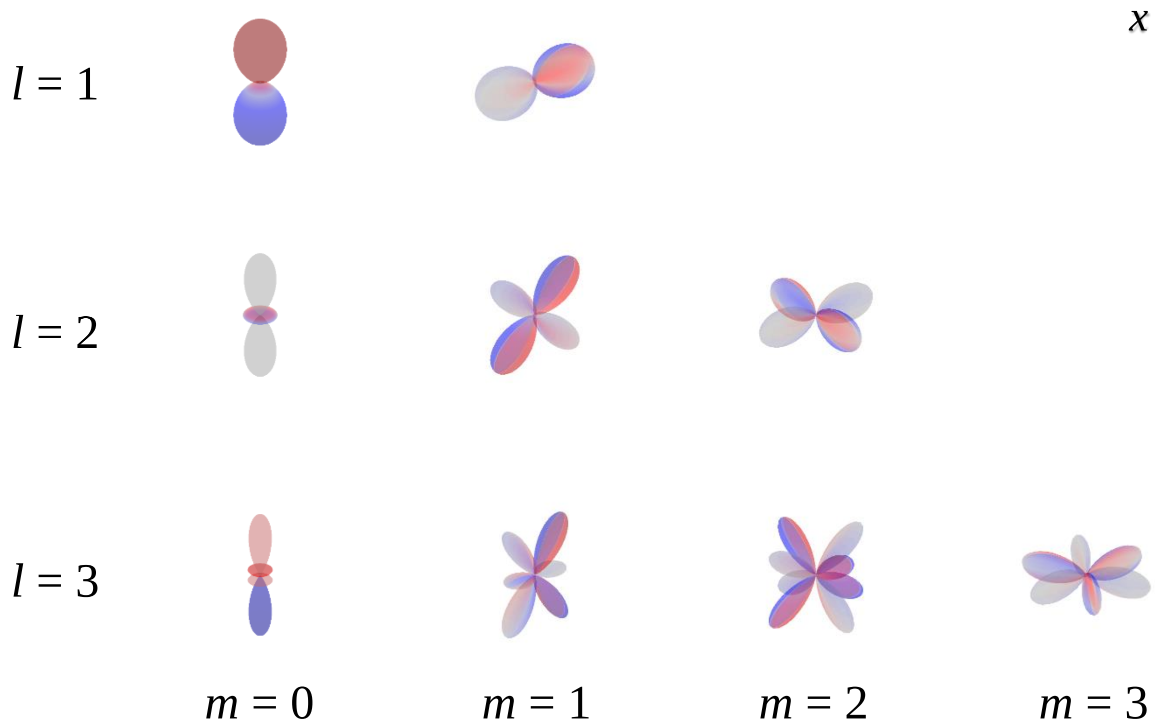
Spherical Harmonics

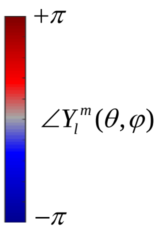
$$Y_l^m(\theta, \varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}$$

Spherical Harmonics – $\mathrm{Re}[Y_l^m]^2$

$$f(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm} Y_l^m(\theta,\varphi)$$

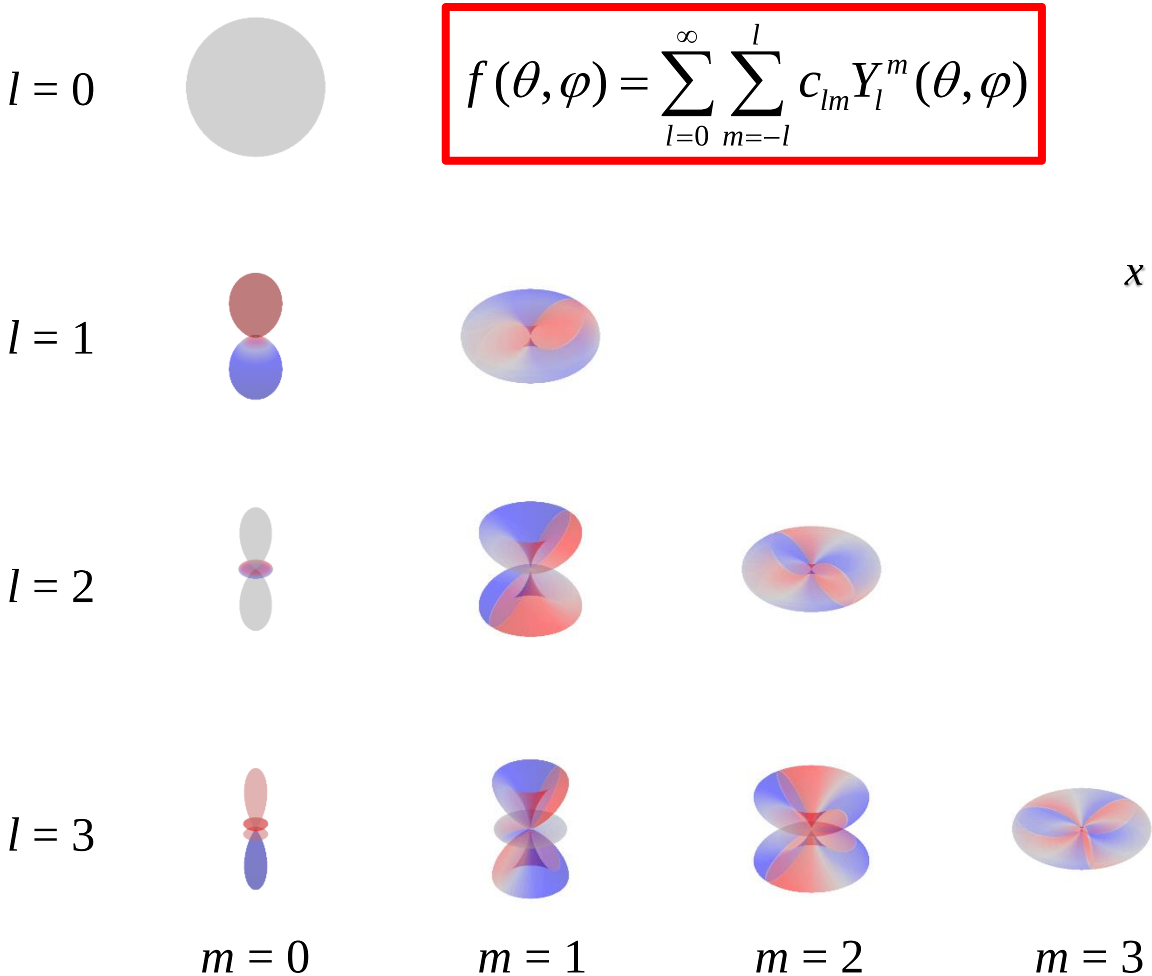


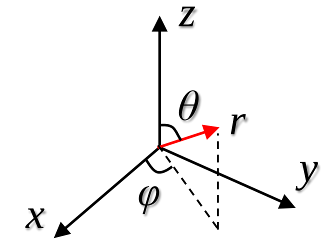


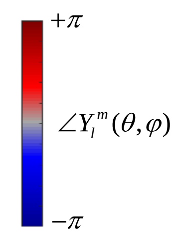


Spherical Harmonics – $|Y_l^m|^2$

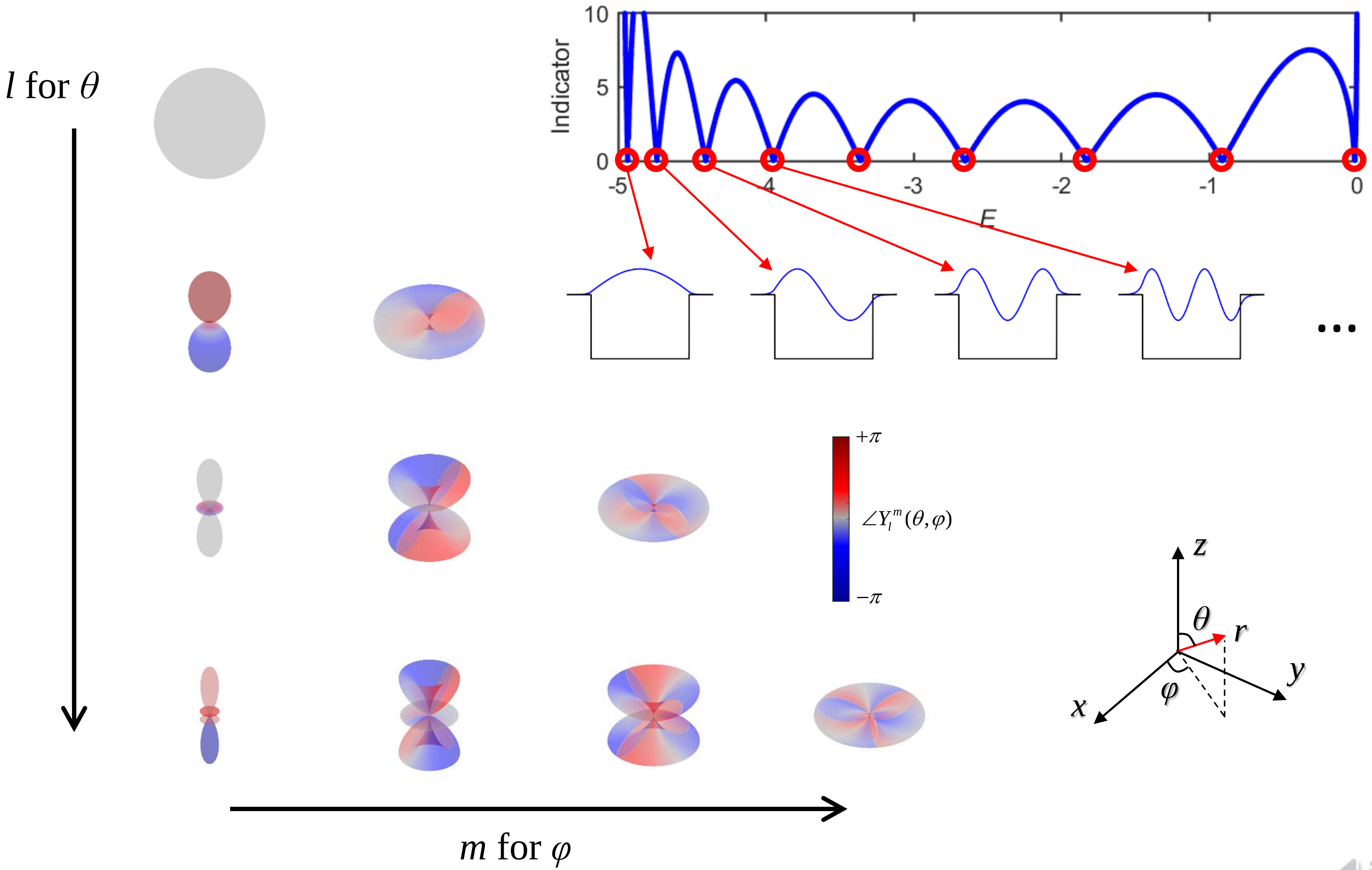
$$f(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_l^m(\theta,\varphi)$$







Understanding Higher-Order States



Summary

- SoV for Different Coordinate Systems
  - Cartesian Coordinates
  - Cylindrical Coordinates – Bessel Functions
  - Spherical Coordinates – Legendre Functions & Spherical Harmonics
- If you're familiar with special functions, you'll get more and more chances in understanding high-level physics (Q.M., EM, Solid-state physics, …) and in getting academic achievements!
  ➔ Easy problems were already solved!

Laplace's Equation & Poisson's Equation

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Sunkyu Yu

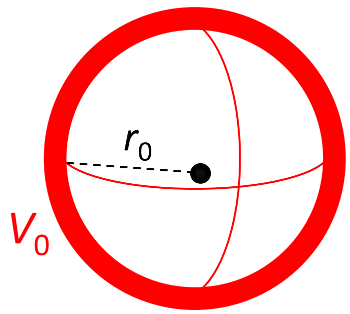
Dept. of Electrical and Computer Engineering

Seoul National University

Laplace Equations – Spherical Coordinates

Example 018

- Assume that the spherical conducting shell (radius r_0) is charged to a potential V_0. Then, calculate the difference in the surface charge density inside and outside the shell.



$$V(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} R_l^m(r) P_l^m(\cos\theta) Q_m(\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} R_l^m(r) Y_l^m(\theta,\varphi)$$

$$R_l^m(r) = c_l^m r^l + d_l^m r^{-l-1}$$

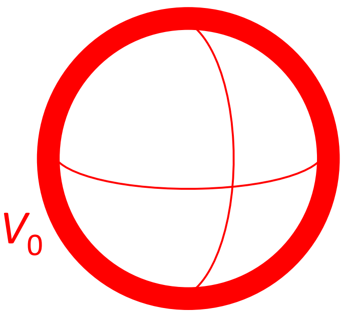
I. Prohibiting the singularity (except for a special case: *e.g. exactly* at charges)
II. An electric potential is continuous

To prohibit the singularities at $r = 0$ & $r \to \infty$:

$$\begin{aligned} R_l^m(r) &= c_l^m r^l \qquad (r \le r_0) \\ &= d_l^m r^{-l-1} \qquad (r > r_0) \end{aligned}$$

Example 018

- Assume that the spherical conducting shell (radius r_0) is charged to a potential V_0. Then, calculate the difference in the surface charge density inside and outside the shell.



$$V(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_l^{\,m} r^l Y_l^{\,m}(\theta,\varphi) \qquad (r \le r_0)$$

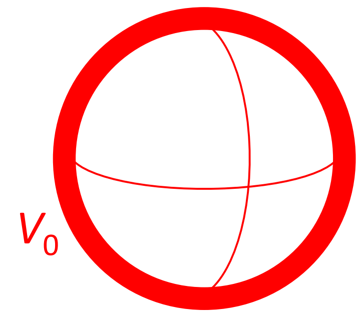
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} d_l^{\,m} \frac{1}{r^{l+1}} Y_l^{\,m}(\theta,\varphi) \qquad (r > r_0)$$

To achieve the continuity at $r = r_0$

$$c_l^{\,m} r_0^{\,l} = d_l^{\,m} \frac{1}{r_0^{\,l+1}} \;\Rightarrow\; c_l^{\,m} r_0^{\,l} = A_l^{\,m} = d_l^{\,m} \frac{1}{r_0^{\,l+1}}$$

Example 018

- Assume that the spherical conducting shell (radius r_0) is charged to a potential V_0. Then, calculate the difference in the surface charge density inside and outside the shell.



$$V(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_l^m r^l Y_l^m(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l^m \left(\frac{r}{r_0}\right)^l Y_l^m(\theta,\varphi) \qquad (r \le r_0)$$

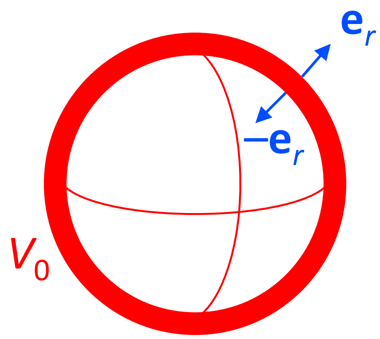
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} d_l^m \frac{1}{r^{l+1}} Y_l^m(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l^m \left(\frac{r_0}{r}\right)^{l+1} Y_l^m(\theta,\varphi) \qquad (r > r_0)$$

Applying B.C.

$$V_0 = V(r = r_0,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l^m Y_l^m(\theta,\varphi)$$

Example 018

- Assume that the spherical conducting shell (radius r_0) is charged to a potential V_0. Then, calculate the difference in the surface charge density inside and outside the shell.



$$V(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m \left(\frac{r}{r_0}\right)^l Y_l^m(\theta,\varphi) \qquad (r \le r_0)$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m \left(\frac{r_0}{r}\right)^{l+1} Y_l^m(\theta,\varphi) \qquad (r > r_0)$$

$$V_0 = V(r = r_0, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m Y_l^m(\theta,\varphi)$$

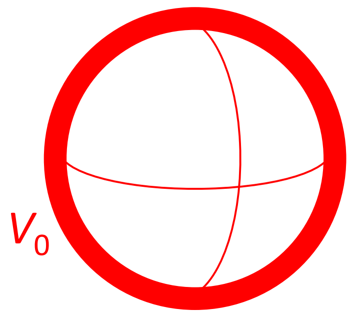
$$E_t = 0, \quad E_n = \frac{\rho_s}{\varepsilon_0}$$

$$E_{n-inside} = \frac{\rho_{s-inside}}{\varepsilon_0} = +\left.\frac{\partial V(r \le r_0)}{\partial r}\right|_{r=r_0}$$

$$E_{n-outside} = \frac{\rho_{s-outside}}{\varepsilon_0} = -\left.\frac{\partial V(r > r_0)}{\partial r}\right|_{r=r_0}$$

Example 018

- Assume that the spherical conducting shell (radius r_0) is charged to a potential V_0. Then, calculate the difference in the surface charge density inside and outside the shell.



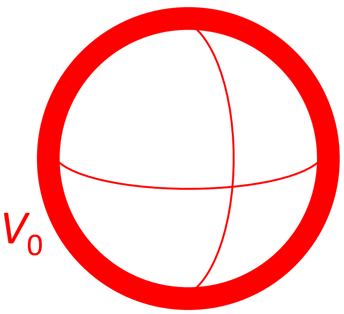
$$\rho_{s-outside} - \rho_{s-inside} = -\varepsilon_0 \left[\left. \frac{\partial V(r > r_0)}{\partial r} \right|_{r=r_0} + \left. \frac{\partial V(r \le r_0)}{\partial r} \right|_{r=r_0} \right]$$

$$\frac{\partial V}{\partial r} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m l \left(\frac{r}{r_0} \right)^{l-1} \frac{1}{r_0} Y_l^m(\theta,\varphi) \qquad (r \le r_0) \Rightarrow \left. \frac{\partial V}{\partial r} \right|_{r=r_0-} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{l}{r_0} A_l^m Y_l^m(\theta,\varphi)$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m (l+1) \left(\frac{r_0}{r} \right)^{l} \left(\frac{-r_0}{r^2} \right) Y_l^m(\theta,\varphi) \quad (r > r_0) \Rightarrow \left. \frac{\partial V}{\partial r} \right|_{r=r_0+} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{-l-1}{r_0} \right) A_l^m Y_l^m(\theta,\varphi)$$

Example 018

- Assume that the spherical conducting shell (radius r_0) is charged to a potential V_0. Then, calculate the difference in the surface charge density inside and outside the shell.



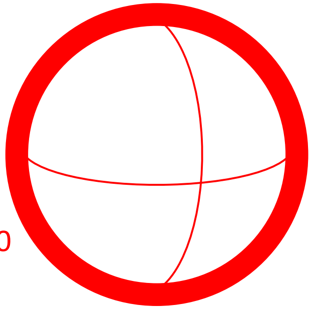
$$\left.\frac{\partial V}{\partial r}\right|_{r=r_0+} + \left.\frac{\partial V}{\partial r}\right|_{r=r_0-} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{-l-1}{r_0}\right)A_l^m Y_l^m(\theta,\varphi) + \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l}{r_0}A_l^m Y_l^m(\theta,\varphi)$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{-1}{r_0}A_l^m Y_l^m(\theta,\varphi) = -\frac{1}{r_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_l^m Y_l^m(\theta,\varphi)$$

$$\rho_{s-outside} - \rho_{s-inside} = -\varepsilon_0\left[-\frac{1}{r_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_l^m Y_l^m(\theta,\varphi)\right] = \frac{\varepsilon_0}{r_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_l^m Y_l^m(\theta,\varphi)$$

Example 018

- Assume that the spherical conducting shell (radius r_0) is charged to a potential V_0. Then, calculate the difference in the surface charge density inside and outside the shell.



$$\rho_{s-outside} - \rho_{s-inside} = \frac{\varepsilon_0}{r_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m Y_l^m(\theta, \varphi)$$

B.C. $$V_0 = V(r = r_0, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m Y_l^m(\theta, \varphi)$$

$$\boxed{\rho_{s-outside} - \rho_{s-inside} = \frac{\varepsilon_0}{r_0} V_0}$$

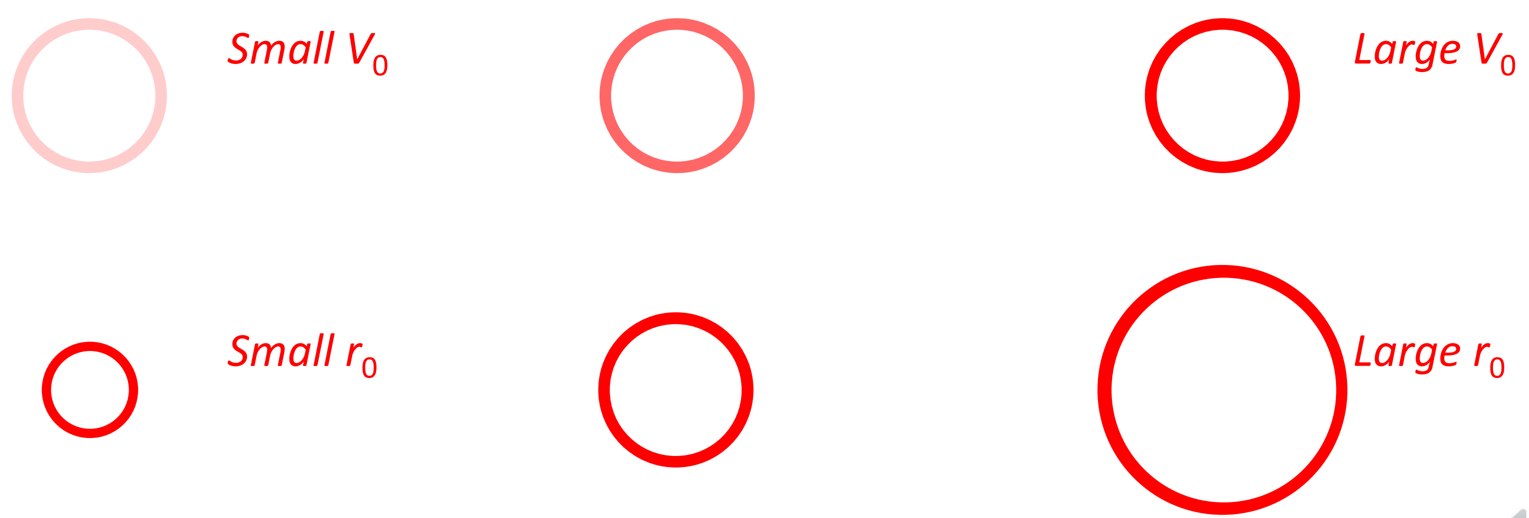
Example 018

- Assume that the spherical conducting shell (radius r_0) is charged to a potential V_0. Then, calculate the difference in the surface charge density inside and outside the shell.

Curvature

$$\kappa = \frac{1}{r_0}$$

$$\rho_{s-outside} - \rho_{s-inside} = \frac{\varepsilon_0}{r_0} V_0 \quad \Rightarrow \quad E_{n-outside} - E_{n-inside} = \frac{V_0}{r_0}$$



Poisson's Equation

Remind: Laplace & Poisson Equations

Poisson's Equation: Usually more difficult!

Green Function Method
: More general but complex &
requiring numerical analysis in many cases

$$-\nabla^2 V = \frac{\rho}{\varepsilon_0 \varepsilon_r}$$

Image Method
: Simple solutions for specialized geometries

$$\rho = 0$$

Laplace's Equation

$$\nabla^2 V = 0$$

The Method of Images

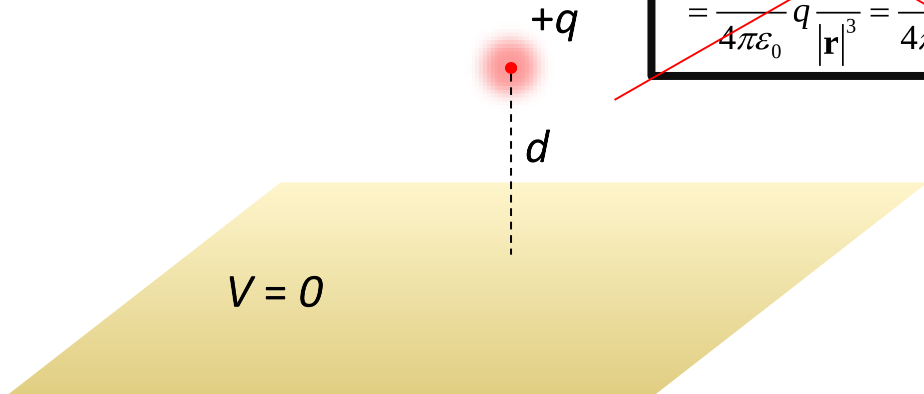
Starting from An Example

Potential $V(z > 0)$?

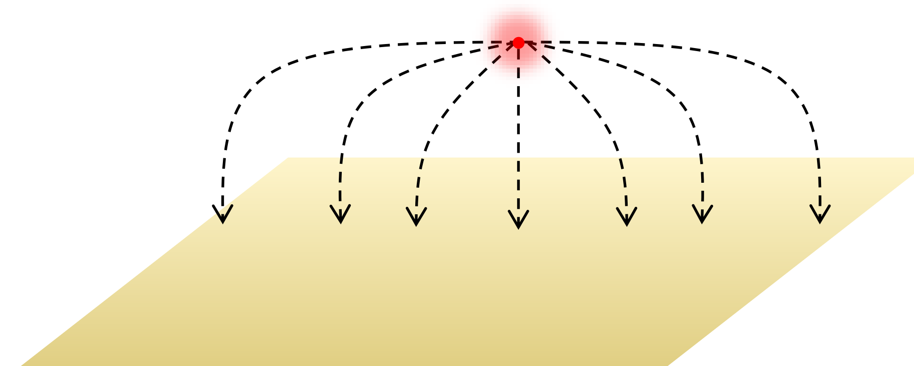
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^3} = \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{1}{4\pi\varepsilon_0} q \frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2}$$

I. $V(z = 0) = 0$

II. $V(z) \to 0$ when $x^2 + y^2 + z^2 >> d^2$



1. Positive charge ➔ Emission of a Field

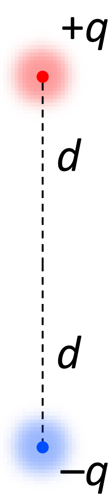


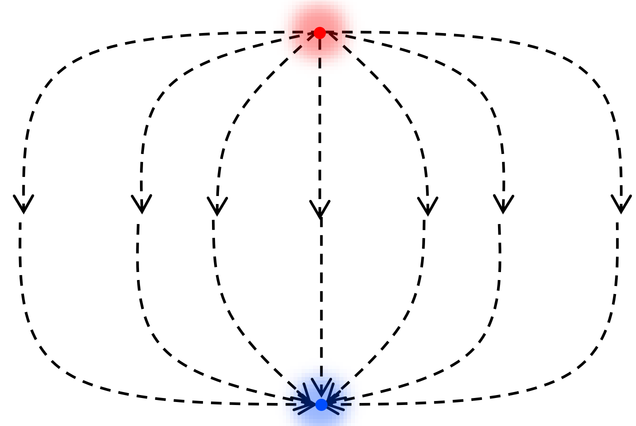
2. Normal Field at the Surface

$$E_t = 0, \quad E_n = \frac{\rho_s}{\varepsilon_0}$$

Applying a tricky (or clever) method considering these conditions?
➔ Remove the conducting plate!

Complex to Simple Problems



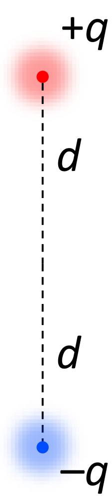


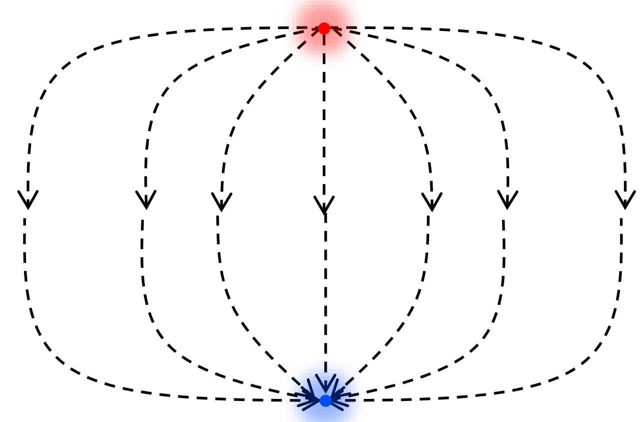
I. $V(z = 0) = 0$

II. $V(z) \to 0$ when $x^2 + y^2 + z^2 >> d^2$

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0} q \left[\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right]$$

Complex to Simple Problems





1. Positive charge ➔ Emission of a Field

2. Normal Field at the Surface

At z = 0:

$$\mathbf{E}(z=0) = E_z(z=0)\mathbf{e}_z$$
$$= -\frac{qd}{2\pi\varepsilon_0}\frac{1}{\left[x^2+y^2+d^2\right]^{3/2}}\mathbf{e}_z$$

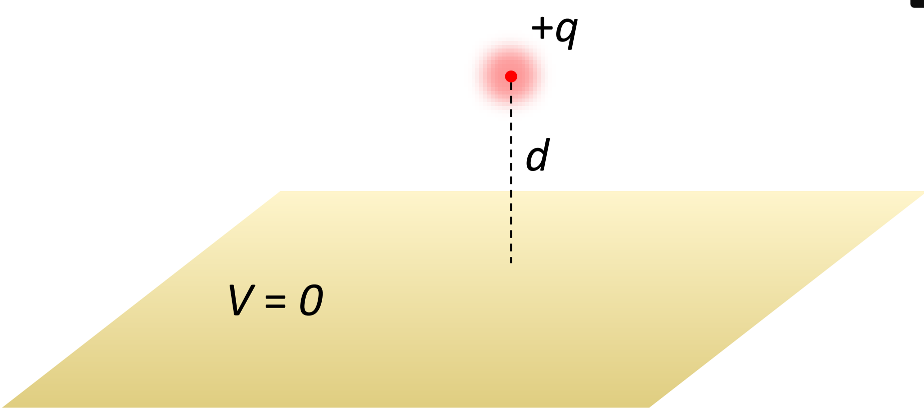
$$E_x = -\mathbf{e}_x \cdot \nabla V = \frac{q}{4\pi\varepsilon_0}\left\{\frac{x}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{x}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}$$

$$E_y = -\mathbf{e}_y \cdot \nabla V = \frac{q}{4\pi\varepsilon_0}\left\{\frac{y}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{y}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}$$

$$E_z = -\mathbf{e}_z \cdot \nabla V = \frac{q}{4\pi\varepsilon_0}\left\{\frac{z-d}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{z+d}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}$$

Back to the Original Problem

Potential $V(z>0)$?



$$V(x,y,z)=\frac{1}{4\pi\varepsilon_0}q\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

$$E_x=-\mathbf{e}_x\cdot\nabla V=\frac{q}{4\pi\varepsilon_0}\left\{\frac{x}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}-\frac{x}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}$$

$$E_y=-\mathbf{e}_y\cdot\nabla V=\frac{q}{4\pi\varepsilon_0}\left\{\frac{y}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}-\frac{y}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}$$

$$E_z=-\mathbf{e}_z\cdot\nabla V=\frac{q}{4\pi\varepsilon_0}\left\{\frac{z-d}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}-\frac{z+d}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}$$

$$\mathbf{E}(z=0)=E_z(z=0)\mathbf{e}_z=-\frac{qd}{2\pi\varepsilon_0}\frac{1}{\left[x^2+y^2+d^2\right]^{3/2}}\mathbf{e}_z$$

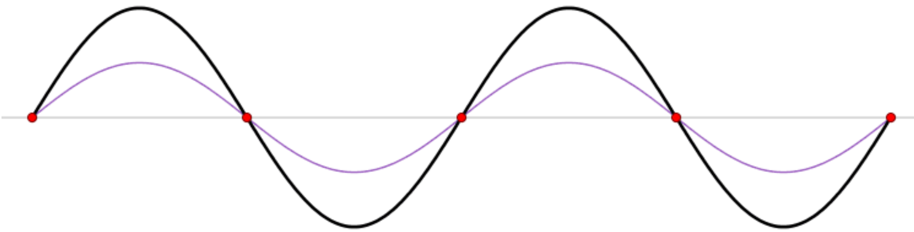
$$E_n=\frac{\rho_s}{\varepsilon_0}$$

Induced Charges on the ground plate

$$\rho_s=-\frac{q}{2\pi}\frac{d}{\left[x^2+y^2+d^2\right]^{3/2}}$$

The Green Function Method

Linear Systems: Superposition Principle



$y_1 = cx_1$

$y_2 = cx_2$

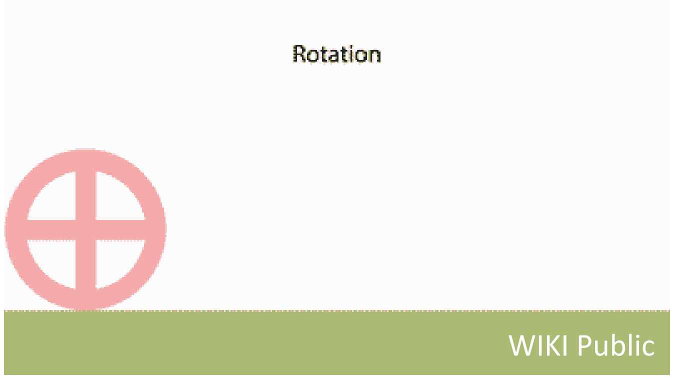
$y_1 + y_2 = c(x_1 + x_2)$

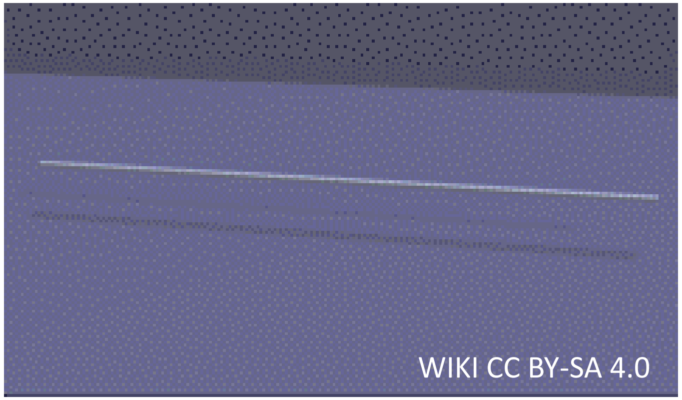
$y_1 = cx_1^2$

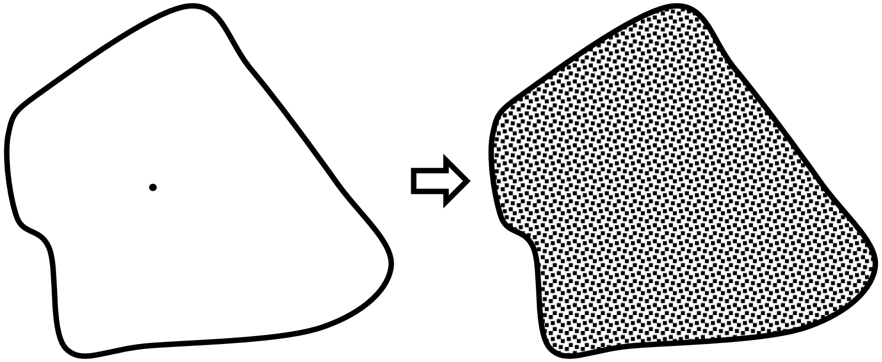
$y_2 = cx_2^2$

$y_1 + y_2 \neq c(x_1 + x_2)^2$

WIKI Public







Green's Functions for Poisson Equations

Poisson's Equation

$$-\nabla^2 V = \frac{\rho}{\varepsilon_0 \varepsilon_r} = \frac{\rho}{\varepsilon}$$

$$-\nabla^2 V(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0 \varepsilon_r(\mathbf{x})} = f(\mathbf{x})$$

How can we handle this problem?

$$\nabla^2 V(\mathbf{x}) = -f(\mathbf{x})$$

Dirichlet *Neumann*

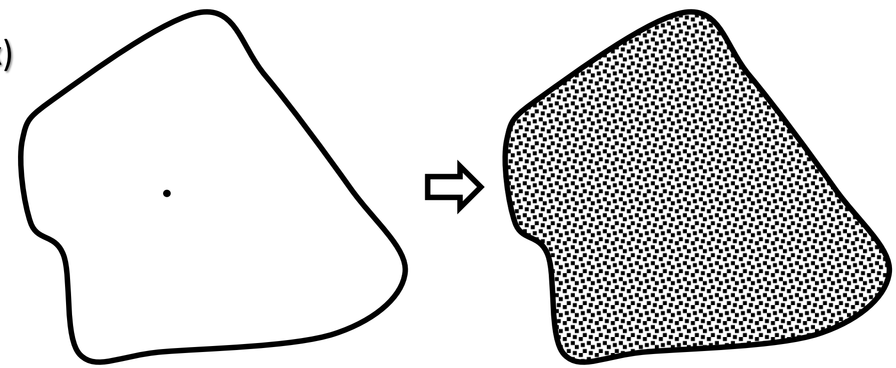
$$\mathbf{x} \in \Omega \quad (V(\mathbf{x})\big|_{\mathbf{x}\in S} \text{ or } \partial V(\mathbf{x})/\partial n\big|_{\mathbf{x}\in S})$$

$$\int \delta(\mathbf{x}-\mathbf{x}')f(\mathbf{x}')d\mathbf{x}' = f(\mathbf{x})$$

$$\int G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')d\mathbf{x}' = V(\mathbf{x})$$

More general form?

Source $f(\mathbf{x})$



$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}')$$

Green Function

I. The Simplest Case – Without *any* B.C. ~ Infinite Domains

2D

$$\nabla_T^{\,2} \ln \frac{1}{\rho} = -2\pi\delta^2(\mathbf{r})$$

$$\nabla_T^{\,2} \ln \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -2\pi\delta^2(\mathbf{x}-\mathbf{x}')$$

3D

$$\nabla^2 \frac{1}{r} = -4\pi\delta^3(\mathbf{r})$$

$$\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')$$

We already know Green functions!

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}')$$

$$\int G(\mathbf{x},\mathbf{x}') f(\mathbf{x}')d\mathbf{x}' = V(\mathbf{x})$$

$$G_{2\mathrm{D}}(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

$$G_{3\mathrm{D}}(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

Applicable only to infinite domains… How can we handle B.C.?

We know Gauss theorem…

$$\int_\Omega (\nabla \cdot \mathbf{u})\, d^3x = \oint_S \mathbf{u} \cdot d\mathbf{S}$$

$$\mathbf{u} = f \nabla g$$

Green's First Identity

$$\int_\Omega \left[f \nabla^2 g + \nabla f \cdot \nabla g \right] d^3x = \oint_S f \nabla g \cdot d\mathbf{S}$$

$$\mathbf{u} = g \nabla f$$

$$\int_\Omega \left[g \nabla^2 f + \nabla f \cdot \nabla g \right] d^3x = \oint_S g \nabla f \cdot d\mathbf{S}$$

–

$$\int_\Omega \left[f \nabla^2 g - g \nabla^2 f \right] d^3x = \oint_S \left[f \nabla g - g \nabla f \right] \cdot d\mathbf{S}$$

2nd Volume
~ 1st Surface

Green's Second Identity

II. More Complex Cases – With B.C. ~ Green's Identity

$$\int_\Omega \left[f\nabla^2 g - g\nabla^2 f \right] d^3x = \oint_S \left[f\nabla g - g\nabla f \right]\cdot d\mathbf{S}$$

We have two scalar functions *V* & *G*

$$\int_\Omega \left[V(\mathbf{x}')\nabla'^2 G(\mathbf{x}',\mathbf{x}) - G(\mathbf{x}',\mathbf{x})\nabla'^2 V(\mathbf{x}') \right] d^3x'$$

$$= \oint_S \left[V(\mathbf{x}')\nabla' G(\mathbf{x}',\mathbf{x}) - G(\mathbf{x}',\mathbf{x})\nabla' V(\mathbf{x}') \right]\cdot d\mathbf{S}'$$

$$\nabla^2 V(\mathbf{x}) = -f(\mathbf{x}) \qquad \nabla^2 G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}')$$

$$= V(\mathbf{x})$$

$$\int_\Omega \left[-V(\mathbf{x}')\delta(\mathbf{x}'-\mathbf{x}) + G(\mathbf{x}',\mathbf{x}) f(\mathbf{x}') \right] d^3x'$$

$$= \oint_S \left[V(\mathbf{x}')\nabla' G(\mathbf{x}',\mathbf{x}) - G(\mathbf{x}',\mathbf{x})\nabla' V(\mathbf{x}') \right]\cdot d\mathbf{S}'$$

II. More Complex Cases – With B.C. ~ Green's Identity

$$V(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}',\mathbf{x}) f(\mathbf{x}') d^3x' - \oint_S V(\mathbf{x}') \nabla' G(\mathbf{x}',\mathbf{x}) \cdot d\mathbf{S}' + \oint_S G(\mathbf{x}',\mathbf{x}) \nabla' V(\mathbf{x}') \cdot d\mathbf{S}'$$

$$V(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}',\mathbf{x}) f(\mathbf{x}') d^3x' - \oint_S V(\mathbf{x}') \frac{\partial G(\mathbf{x}',\mathbf{x})}{\partial n} dS' + \oint_S G(\mathbf{x}',\mathbf{x}) \frac{\partial V(\mathbf{x}')}{\partial n} dS'$$

II. More Complex Cases – With B.C. ~ Green's Identity

Superposition in the volume Ω

$$V(\mathbf{x}) = \int_\Omega G(\mathbf{x}',\mathbf{x}) \frac{\rho(\mathbf{x})}{\varepsilon_0 \varepsilon_r(\mathbf{x})} d^3x' - \oint_S V(\mathbf{x}') \frac{\partial G(\mathbf{x}',\mathbf{x})}{\partial n} dS' + \oint_S G(\mathbf{x}',\mathbf{x}) \frac{\partial V(\mathbf{x}')}{\partial n} dS'$$

Neumann for Green

Dirichlet for Green

Boundary Condition at the surface *S*

$$G_{3D}(\mathbf{x},\mathbf{x}') \neq \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

Various techniques exist…

II. Electrostatic Problems ~ Dirichlet Green Functions

$$V(\mathbf{x}) = \int_\Omega G(\mathbf{x}',\mathbf{x})\frac{\rho(\mathbf{x})}{\varepsilon_0\varepsilon_r(\mathbf{x})}d^3x' - \oint_S V(\mathbf{x}')\frac{\partial G(\mathbf{x}',\mathbf{x})}{\partial n}dS' + \oint_S G(\mathbf{x}',\mathbf{x})\frac{\partial V(\mathbf{x}')}{\partial n}dS'$$

0

because most B.C. is defined by the Dirichlet B.C. $V(x \in S) = V_S(\mathbf{x})$

$$V(\mathbf{x}) = \int_\Omega G(\mathbf{x}',\mathbf{x})\frac{\rho(\mathbf{x})}{\varepsilon_0\varepsilon_r(\mathbf{x})}d^3x' - \oint_S V_S(\mathbf{x}')\frac{\partial G(\mathbf{x}',\mathbf{x})}{\partial n}dS'$$

Known information

Various techniques exist…
: which will be discussed in high-level courses…