induced leading-edge separation. Dynamic stall generates high oscillatory and vibratory torsion loads on the blades and the swash-plate servos. Predicting dynamic stall is necessary for initial sizing and stall flutter calculations. It is also key to achieving higher forward speed capabilities for heavily loaded rotor systems.

Numerous experiments have revealed the general sequence of events. For an airfoil pitching up, a progressive trailing edge separation due to flow reversal in the boundary layer, is accompanied by the formation of a leading edge vortex. The onset of a critical leading edge pressure triggers a leading-edge separation where the vortex detaches and starts moving downstream. This phenomenon of vortex detachment generates a strong pitching moment stall. However, as long as the vortex traverses over the airfoil, the lift does not stall and continues to increase. The lift stalls when the vortex leaves the trailing edge. At this time, the pitching moment reaches its maximum negative value. A period of progressive flow re-attachment follows as the airfoil pitches down. During this time one or more weaker vortices can be shed from the upper surface, creating additional fluctuations in lift and pitching moment. This sequence of events lead to large hysteresis loops in airloads when plotted versus the angle of incidence. Typical hysteresis loops in airfoil lift coefficient and pitching moment coefficients are shown in Fig.4.1. The figure shows the airloads on a 2D SC-1095 airfoil section undergoing pitch oscillations at a nondimensional frequency $k = \omega c/2U$, where ω is the frequency of oscillation in radians/sec, c is the airfoil chord, and U is the incident velocity. The incident velocity corresponds to a Mach number of 0.3. When the angle of attack variation is such that the airfoil goes slightly out of the static stall regime with each oscillation, it is called a light stall. When a large part of the angle of oscillation occurs outside the static stall regime, it is called a deep stall.

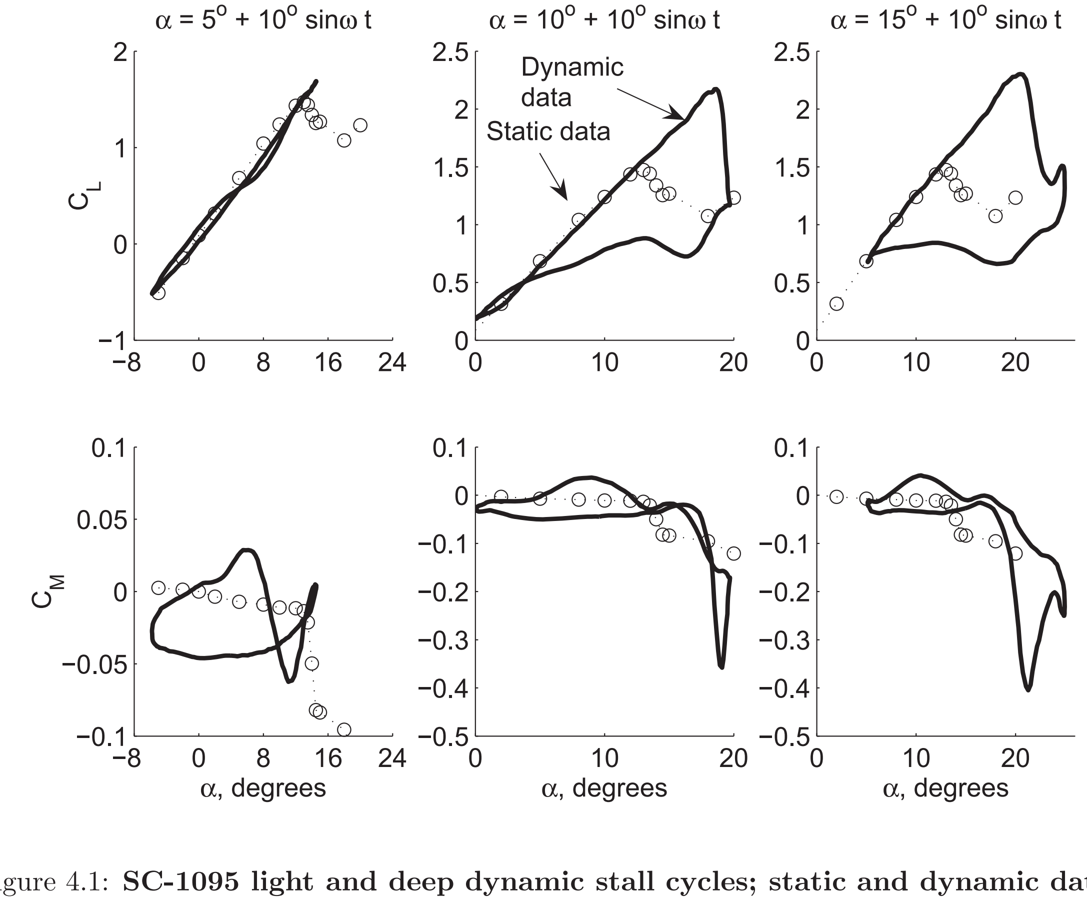


Figure 4.1: **SC-1095 light and deep dynamic stall cycles; static and dynamic data from McCroskey et al NASA TM-84245, 1982, at Mach 0.3, reduced frequency k = 0.1**

Fundamental understanding of dynamic stall began with the seminal work of Liiva [16] on helicopter rotors and Carta [17] on axial-flow turbomachines. Subsequently, many experimental investigations have provided greater insights into the phenomena.

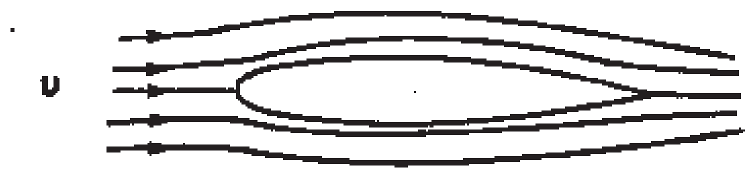
Current comprehensive analyses calculates dynamic stall using semi-empirical models. like the UTRC Method 1970, the Beddoes Time-Delay Method 1976, Gangwani's Method 1982 (all reviewed in [8]), the Boeing-Vertol gamma function method 1973 [18], Johnson's Method 1969 [19], the Leishman-Beddoes Method 1986 [12], ONERA EDLIN (Equations Differentielles Linearires) model 1990 [20] and the ONERA BH (Bifurcation de Hopf) model 1998 [21].

Dynamic stall is characterized by a delay in angle of attack before stall (or separation) and high transient loads induced by a leading edge vortex after stall. All dynamic stall models, model the delay in angle of attack and the aerodynamic coefficient increments after stall. In the Leishman-Beddoes model uses first-order differential equations for the delayed angle of attack and leading-edge vortex lift. All models are 2-D and semi-empirical in nature. The ONERA EDLIN model and BH model both use second-order differential equations to calculate delayed angle of attack and lift, drag and moment increments. The Johnson model uses an angle of attack delay proportional to the rate of change of angle of attack. The Boeing model uses an angle of attack delay proportional to the square-root of the rate of change of angle of attack. In general the agreement between different models are good considering the simplicity of the models, but correlation with test data show significant errors, as expected with empirical models. Johnson [22] compared 2D airloads, PUMA blade sectional airloads, and power predictions under stall conditions using the different models. The predictions were similar but correlation with test data showed errors, as expected of all semi-empirical models.

4.3 Unsteady Thin Airfoil Theory

Thin airfoil theory is widely used to calculate the lift force on an airfoil. The theory tries to solve the Laplace equation in two dimensions while implementing boundary conditions that produce useful aerodynamic solutions. The assumption of inviscid potential flow implies that the governing equation remains the same for both steady and unsteady flows. The treatment of unsteady flows is via boundary conditions.

Normally, the problem is divided into two parts, lift and drag. Typically, the lift problem is normally solved using the inviscid flow assumption. On the other hand, the viscosity plays an important role near the surface and it influences the drag force. The drag solution is separately obtained for the real fluid either using some empirical relations or the experimental data. For most of the problems, viscosity has little influence on the pressure solution. An airfoil is assumed sufficiently thin so that for a small angle of attack the disturbances in the flow are small perturbations.



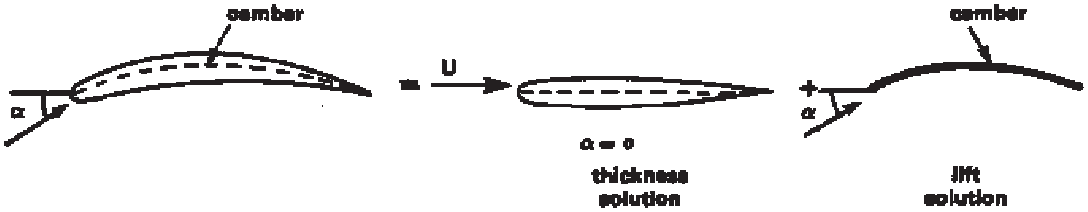
The assumptions are:

1. Flow disturbances are small perturbations.

2. Flow on the surface is tangential.

3. Flow leaves trailing edge smoothly (Kutta condition).

Break the problem into two parts.



Part I: Thickness solution

The airfoil camber as well as the angle of attack are set to zero. A symmetric airfoil at zero angle of attack gives symmetric pressure resulting in zero net lift. The airfoil is replaced by a source distribution on the chord line.

We would like to find the strength of the source distribution, and this is done using the tangential flow condition on the surface. Once the strength is known then the pressure distribution can be calculated.

Part II: Lift solution

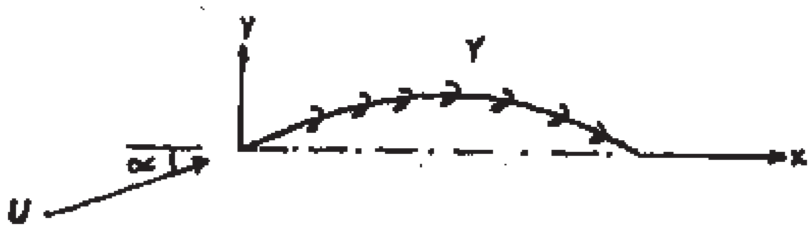
The airfoil thickness is set to zero, so the camber line is set at an angle of attack. The lift solution is obtained by replacing the camber line with the vortex sheet. The solution is anti-symmetric in character.

Using the boundary condition on the surface and the Kutta condition at the trailing edge, the strength of the vorticity distribution is evaluated. Then the pressure distribution can be calculated. Glauert used a Fourier series to solve the problem. Note that, without the Kutta condition the airfoil generates zero forces and moments. The assumption of inviscid irrotational flow gaurantees that the flow slips past the body without producing any net forces. The Kutta condition ensures that at least a lift is produced. This lift happens to be close to measured values, implying that the Kutta condition has a physical basis.

It was Helmholtz who first proposed an idea to obtain a lift solution for a thin airfoil, essentially a flat plate. It is impossible he reasoned that a real flow with viscosity would negotiate a sharp turn (zero radius of curvature) at the leading and trailing edges. One way to indirectly incorporate viscous behavior within a potential flow solution was to impose flow smoothness at the leading and trailing edges. The lift solution he obtained was far off, but the idea was correct. Kutta and Joukowski, independantly, imposed the condition only at the trailing edge. Their solution was quite accurate. We now know that the effects of viscosity is pronounced at the trailing edge, not the leading edge. The boundary layer is thick near the trailing edge.

4.3.1 Steady Airloads

Let us examine the lifting problem through the thin airfoil theory. The flow is assumed to be inviscid, irrotational (i.e. potential) and in addition incompressible.



The airfoil camber is given by $z = z(x)$, $z(x) << c$, where c is the airfoil chord. The camber line is replaced by the vorticity distribution $\gamma(x)$. For steady flow, the shed vorticity is neglected. The induced velocity $w(x)$ perpendicular to the camber line at any x is approximated to be the same as that perpendicular to the x axis. This is the thin airfoil assumption.

$$w_b(x) = \int_0^c \frac{\gamma_b(\xi)d\xi}{2\pi(\xi - x)}$$

For flow tangency, or impenetrability along the camber line, the induced velocity from the free stream should be equal and opposite to the vortex induced velocity. Thus

$$\int_0^c \frac{\gamma(\xi)d\xi}{2\pi(\xi - x)} + \left(\alpha - \frac{dz}{dx}\right) U = 0$$

or

$$\int_0^c \frac{\gamma(\xi)d\xi}{2\pi(x - \xi)} = \left(\alpha - \frac{dz}{dx}\right) U$$

The Kutta condition is given by

$$\gamma(c) = 0$$

Solve for $\gamma(x)$. Then the lift and moment about the leading edge can be calculated using

$$L = \int_0^c \rho U \gamma dx$$

$$M_{le} = \int_0^c \rho U \gamma x dx$$

The moment can be transfered to any chord-wise location based on requirements.

Glauert calculated the solution using the Fourier series. The results are summarized here. The non-dimensional lift and pitching moment coefficient at quarter chord are given by

$$Cl = 2\pi(A_0 + A_1/2)$$

$$Cm_{\frac{1}{4}c} = -\frac{\pi}{4}(A_1 - A_2)$$

where

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx} d\theta$$

$$A_n = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos n\theta d\theta$$

$$\theta = \cos^{-1}\left(1 - \frac{2x}{c}\right)$$

4.3.2 Quasi-Steady Airloads

The steady airloads results can be adapted to unsteady airfoil motions. It provides quasi-steady airloads solutions that are quite useful for simple aero-elastic stability analysis. The world quasi-steady is used because the effects of shed wake is still being neglected.

Note that the slope of the camberline dz/dx satisfies the following equation to maintain impenetrability conditions.

$$\left(\alpha - \frac{dz}{dx}\right) U = w_a(x)$$

where $w_a(x)$ is the component of free stream perpendicular to the camberline. Thus

$$\frac{dz}{dx} = \alpha - \frac{w_a(x)}{U}$$

In the case of a flat plate we have

$$\frac{dz}{dx} = 0$$

Consider a flat plate with a plunge velocity $\dot{h}$ downwards (so that the relative air velocity is $\dot{h}$ positive upwards).

$$w_a(x) = U\alpha + \dot{h}$$

$$\frac{w_a(x)}{U} = \alpha + \frac{\dot{h}}{U}$$

$$\frac{dz}{dx} = -\frac{\dot{h}}{U}$$

Now consider a flat plate pitching with a angular rate $\dot{\alpha}$. The point with zero translational velocity (center of rotation, elastic axis) is at a distance $a_h b$ from the mid-chord, where $b = c/2$.

$$w_a(x) = U\alpha + (x - b - a_h b)\,\dot{\alpha}$$

$$\frac{w_a(x)}{U} = \alpha + (x - b - a_h b)\,\frac{\dot{\alpha}}{U}$$

$$\frac{dz}{dx} = -\,(x - b - a_h b)\,\frac{\dot{\alpha}}{U}$$

For an airfoil both pitching and plunging we have

$$\frac{dz}{dx} = -\frac{\dot{h}}{U} - (x - b - a_h b)\,\frac{\dot{\alpha}}{U}$$

Using the above expression in the steady airload results and noting that $x = b(1 - \cos\theta)$, we have

$$\begin{aligned} A_0 &= \alpha + \frac{1}{\pi}\int_0^{\pi}\left[\frac{\dot{h}}{U} + (x - b - a_h b)\,\frac{\dot{\alpha}}{U}\right] d\theta \\ &= \alpha + \frac{\dot{h}}{U} - \frac{a_h b}{U}\dot{\alpha} \end{aligned} \tag{4.69}$$

For pitching about $1/4c$, i.e. if the elastic axis is at $1/4c$ then

$$a_h b = -b/2$$

$$a_h = -1/2$$
$$A_0 = \alpha + \frac{\dot{h}}{U} + \frac{b\dot{\alpha}}{2U}$$

or

$$A_0 = \alpha + \frac{\dot{h}}{U} + \frac{c\dot{\alpha}}{4U}$$

Similarly

$$\begin{aligned} A_1 &= -\frac{2}{\pi}\int_0^\pi \left[\frac{\dot{h}}{U} + (x - b - a_h b)\frac{\dot{\alpha}}{U}\right]\cos\theta d\theta \\ &= -\frac{2}{\pi}\frac{\dot{\alpha}}{U}\int_0^\pi x\cos\theta d\theta \\ &= \frac{\dot{\alpha}}{U}b \end{aligned} \tag{4.70}$$

Thus

$$\begin{aligned} Cl &= 2\pi\left[\alpha + \frac{\dot{h}}{U} + \frac{\dot{\alpha}}{U}\left(\frac{b}{2} - a_h b\right)\right] \\ L_{qs} &= \frac{1}{2}\rho U^2(2b) \\ &= 2\pi b\rho U\left[U\alpha + \dot{h} + \dot{\alpha}\left(\frac{b}{2} - a_h b\right)\right] \end{aligned} \tag{4.71}$$

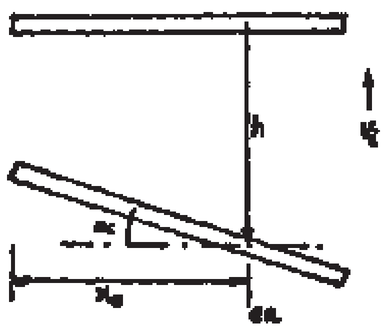
where L_{qs} is the quasi-steady lift per unit span. Note that

$$\begin{aligned} \frac{b}{2} - a_h b &= \left(b + \frac{b}{2}\right) - (b + a_h b) \\ &= \frac{3}{4}c - x_{ea} \end{aligned} \tag{4.72}$$

Thus

$$\begin{aligned} Cl &= 2\pi\left[\alpha + \frac{\dot{h}}{U} + \frac{\dot{\alpha}}{U}\left(\frac{3}{4}c - x_{ea}\right)\right] \\ &= 2\pi\left[\alpha + \frac{\text{downward velocity at 3/4 chord}}{U}\right] \\ &= 2\pi\alpha_g \end{aligned} \tag{4.73}$$

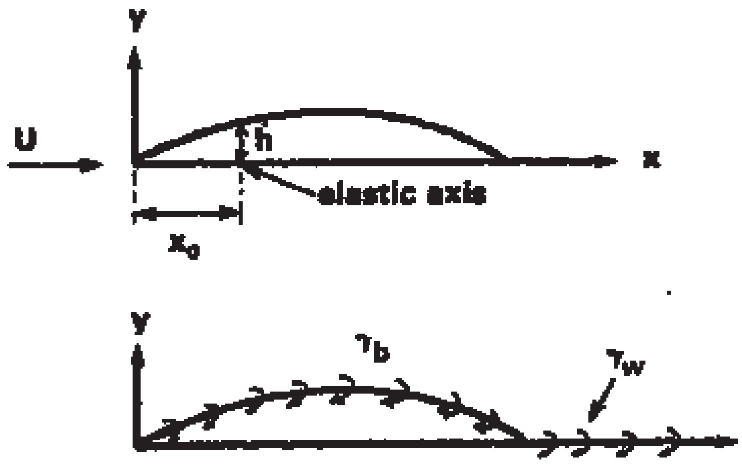
where α_g is defined as a geometric angle of attack, arising out of the unsteady blade motions. α_g is the angle of attack at $3/4c$.



Thus, the quasi-steady assumption boils down to the following. At any instant of time, freeze the motion of the body. Calculate the effective angle of attack at $3/4c$. Then use the static aerodynamic characteristics to evaluate the forces on the body.

4.3.3 Unsteady Airloads

For unsteady flow, the shed vorticity plays an important role. The Laplace solutions are retained with the addition of shed vorticity. Consider a similar pitching and plunging airfoil motion as before.



The bound vorticity strength is γ_b as before. In addition we have a shed (or wake) vorticity strength of γ_w.

$$w_a(x) = U\alpha + \dot{h} + \dot{\alpha}(x - a_h b)$$

where, as in the case of steady and quasi-steady airloads, the geometric camber has been neglected. The airfoil is assumed to behave as a flat plate.

$$w_b(x) = \int_{-b}^{b} \frac{\gamma_b(\xi) d\xi}{2\pi(\xi - x)}$$

$$\lambda_s(x) = \int_{b}^{\infty} \frac{\gamma_w(\xi) d\xi}{2\pi(\xi - x)}$$

For flow tangency or impenetrability as before we have

$$w_b + \lambda_s + w_a = 0$$

The unknown is γ_b. Note that γ_w is not an unknown. It can be related to γ_b, as follows. The total bound circulation is $\Gamma = \int_{-b}^{+b} \gamma_b dx$. The shed vorticity is the time rate of change in total bound circulation Γ. Suppose in time Δt the airfoil has traversed a distance Δs. Then

$$\gamma_w \Delta s = -\Delta\Gamma$$

It follows

$$\gamma_w \frac{\Delta s}{\Delta t} = -\frac{\Delta\Gamma}{\Delta t}$$

In differential form

$$\gamma_w = -\frac{1}{U}\frac{d\Gamma}{dt}$$

where the derivative is take at time $t - (x - b)/U$ when the vorticity was shed from the airfoil. The Kutta condition is same as before $\gamma_b(c) = 0$. In addition the condition $\gamma_w(t) = \gamma_w(x - Ut)$ is satisfied to enforce the shed vorticity to convect with free stream. This ensures that there is no pressure differential across the shed wake. The solution of the impenetrability condition, along with the above boundary conditions produce a γ_b of the following form. For details of the derivaion see Johnson [23].

$$\int_{-b}^{b} \gamma_b dx = 2\pi b \left[\left(w_0 + \frac{1}{2} w_1 \right) - \left(\lambda_0 + \frac{1}{2} \lambda_1 \right) \right]$$

where

$$w_0 = U\alpha + \dot{h} - a_h b\dot{\alpha}$$

$$w_1 = b\dot{\alpha}$$

$$\lambda_n = \frac{2}{\pi}\int_0^{\pi} \lambda_s(x)\cos n\theta d\theta = -\frac{1}{\pi}\int_b^{\infty} \gamma_w \frac{(\xi - \sqrt{\xi^2 - 1})^n}{b^n\sqrt{\xi^2 - b^2}} d\xi$$

γ_b can be broken into two parts

$$\gamma_b = \gamma_{bc} + \gamma_{bnc}$$

such that the circulatory part γ_{bc} provides the net circulation Γ but does not affect the boundary conditions, whereas the non-circulatory part γ_{bnc} does not affect the circulation but satisfies the boundary condition. Thus

$$\int_{-b}^{b} \gamma_{bc} dx = \Gamma$$

$$\int_{-b}^{b} \frac{\gamma_{bc}}{2\pi(x - \xi)} dx = 0$$

and

$$\int_{-b}^{b} \gamma_{bnc} dx = 0$$

$$\int_{-b}^{b} \frac{\gamma_{bnc}}{2\pi(x - \xi)} dx = w_a - \lambda_s$$

The solution γ_b is then related to the differential pressure on the top and bottom surfaces of the airfoil and then to lift and pitching moments. The differential pressure is obtained from the linearized form of the unsteady Bernoulli's equation. This is valid for small perturbations of the flow.

$$p = -\rho \left(U\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial t} \right)$$

$$-\Delta p = \rho \left(U\frac{\partial \Delta\phi}{\partial x} + \frac{\partial \Delta\phi}{\partial t} \right)$$

$$\frac{\partial \Delta\phi}{\partial x} = \Delta u = \gamma_b$$
$$\frac{\partial \Delta\phi}{\partial t} = \frac{\partial}{\partial t}\int_{-\infty}^{x} \Delta u dx$$

Finally

$$-\Delta p = \rho\left(U\gamma_b + \frac{\partial}{\partial t}\int_{-\infty}^{x} \gamma_{bnc} dx\right)$$

where the effect of the time derivative of γ_{bc} has already been accounted for via $\lambda_s(x)$. The lift and pitching moments about the elastic axis then become

$$L = \int_{-b}^{b} (-\Delta p) dx$$
$$M_{a_h b} = \int_{-b}^{b} (-\Delta p)(-x + a_h b) dx$$

Substitute the expression for Δp to obtain

$$L = \rho\left(U\Gamma - \frac{\partial}{\partial t}\Gamma_{nc}^{(1)}\right)$$
$$M_{a_h b} = -\rho\left(U\Gamma^{(1)} - \frac{1}{2}\frac{\partial}{\partial t}\Gamma_{nc}^{(2)}\right)$$

where

$$\Gamma^{(n)} = \int_{-b}^{b} x^n \gamma_b dx$$
$$\Gamma_{nc}^{(n)} = \int_{-b}^{b} x^n \gamma_{bnc} dx$$
$$\Gamma = \Gamma^{(0)}$$

From the solution of γ_b, and using equation 4.71 we can obtain

$$\begin{aligned}\Gamma &= 2\pi b\left[U\alpha + \dot{h} + \dot{\alpha}\left(\frac{b}{2} - a_h b\right)\right] + \int_{b}^{\infty}\left(\sqrt{\frac{\xi + b}{\xi - b}} - 1\right)\gamma_w d\xi \\ &= \frac{L_{qs}}{\rho U} + \int_{b}^{\infty}\left(\sqrt{\frac{\xi + b}{\xi - b}} - 1\right)\gamma_w d\xi\end{aligned} \tag{4.74}$$

and

$$\frac{\partial}{\partial t}\Gamma_{nc}^{(1)} = -\pi b^2(U\dot{\alpha} + \ddot{h} - a_h b\ddot{\alpha}) - U\int_{b}^{\infty}\left(1 - \frac{\xi}{\sqrt{\xi^2 - b^2}}\right)\gamma_w d\xi$$

Substituting in the lift expression we have per unit span

$$\begin{aligned}L &= 2\pi b\rho U\left[U\alpha + \dot{h} + \dot{\alpha}\left(\frac{b}{2} - a_h b\right)\right] + \rho\pi b^2(U\dot{\alpha} + \ddot{h} - a_h b\ddot{\alpha}) \\ &+ \rho U\int_{b}^{\infty}\frac{b}{\sqrt{\xi^2 - b^2}}\gamma_w d\xi \\ &= L_{qs} + L_{nc} + L_w \\ &= L_c + L_{nc}\end{aligned} \tag{4.75}$$

L_{qs} is the same expression as obtained earlier in equation 4.71. L_{nc} and L_w are the new terms. The shed wake contribution L_w can be re-arranged as follows. Note that, from conservation of vorticity we have

$$\Gamma = -\int_b^\infty \gamma_w d\xi$$

Using the above in the second line of equation 4.74 we obtain

$$L_{qs} = -\rho U \int_b^\infty \sqrt{\frac{\xi + b}{\xi - b}} \gamma_w d\xi$$

It follows

$$L_c = L_{qs} + L_w = -\rho U \int_b^\infty \frac{\xi}{\sqrt{\xi^2 - b^2}} \gamma_w d\xi$$

Finally the total lift can be expressed as

$$\begin{aligned} L &= L_c + L_{nc} \\ &= (L_{qs} + L_w) + L_{nc} \\ &= \frac{L_{qs} + L_w}{L_{qs}} L_{qs} + L_{nc} \\ &= \frac{\int_b^\infty \frac{\xi}{\sqrt{\xi^2 - b^2}} \gamma_w d\xi}{\int_b^\infty \sqrt{\frac{\xi+b}{\xi-b}} \gamma_w d\xi} L_{qs} + L_{nc} \\ &= C L_{qs} + L_{nc} \end{aligned} \tag{4.76}$$

where C is a lift deficiency function. The form of C depend on the specific time history of excitation. For example, for $\alpha = \bar{\alpha} e^{i\omega t}$ and $h = \bar{h} e^{i\omega t}$, the shed wake is of the form $\gamma_w = \bar{\gamma_w} e^{i\omega(t - \xi/U)}$ and C has the following form

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + i H_0^{(2)}(k)} \tag{4.77}$$

where $H_n^{(2)}$ are Hankel functions, expressed in terms of Bessel functions

$$H_n^{(2)} = J_n - i Y_n$$

and k is defined as the reduced frequency.

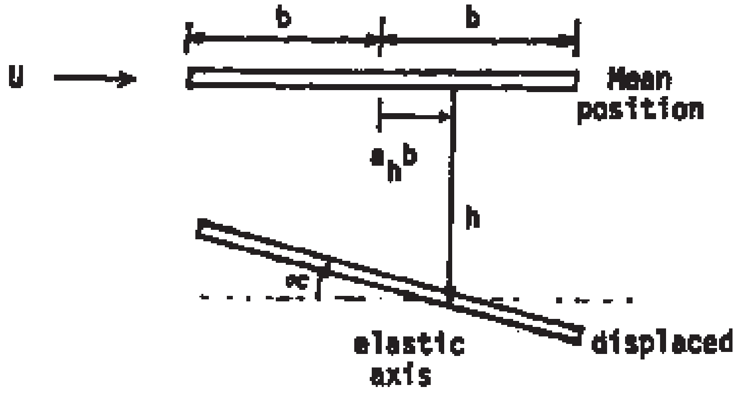
$$k = \frac{\omega b}{U}$$

$C(k)$ for this type of excitation is called the Theodorsen Lift Deficiency Function as discussed later. The circulatory lift L_c acts at quarter chord for thin airfoil theory. The moment $M_{a_h b}$ about the elastic axis is given by

$$\begin{aligned} M_{a_h b} &= L_c \cdot \left(\frac{b}{2} + a_h b \right) - \frac{1}{2} \rho \pi b^3 \left[2U\dot{\alpha} + \ddot{h} + b \left(\frac{1}{4} - a_h \right) \ddot{\alpha} \right] \\ &= L_{qs} C(k) \cdot \left(\frac{b}{2} + a_h b \right) - \frac{1}{2} \rho \pi b^3 \left[2U\dot{\alpha} + \ddot{h} + b \left(\frac{1}{4} - a_h \right) \ddot{\alpha} \right] \end{aligned} \tag{4.78}$$

4.3.4 A Simple Interpretation

The unsteady results above are often interpreted as follows. The unsteady forces generated over the wing can be classified into two categories; circulatory and non-circulatory forces. The circulatory forces are caused by circulation, which means the origin of the forces is vorticity. The non-circulatory forces are called virtual or apparent forces. Let us examine the various component of forces. The airfoil chord is 2b.



$\dot{h}$ = vertical motion, positive down
$\dot{\alpha}$ = pitch motion about elastic axis, positive nose up
1. Lift 'L_1' caused by circulation. The downwash is computed at 3/4-chord. It lies at the aerodynamic center.

$$L_1 = \frac{1}{2}\rho C_{l_\alpha} U^2 2b \left[\alpha + \frac{\dot{h}}{U} + \frac{\dot{\alpha}}{U}\left(\frac{b}{2} - a_h b\right)\right]$$

In the case of thin airfoil theory, the lift curve slope, $C_{l_\alpha} = 2\pi$. The aerodynamic center lies at $1/4c$.
2. Lift 'L_2' is noncirculatory with the center of pressure at mid-chord.

$$L_2 = \text{(apparent mass)} \times \text{(vertical acceleration at mid-chord)} = \pi\rho b^2(\ddot{h} - a_h b\ddot{\alpha})$$

For an apparent mass, a cylinder of air with diameter equal to chord and length of unity assumed to oscillate with the wing
3. Lift L_3 is noncirculatory with the center of pressure at 3/4-chord. The nature of the force is of centrifugal force type.

$$L_3 = \text{(apparent mass)} \times (U\dot{\alpha})$$

$$= \pi\rho b^2 U\dot{\alpha}$$

4. Noncirculatory nose down moment 'M_a'

$$M_a = \text{(apparent moment of inertia)} \times \text{(angular acceleration)}$$

For an apparent mass moment of inertia, a one-quarter inertia of a cylinder with diameter equal to chord and length unity is used.

$$M_a = -\frac{\pi\rho b^4}{8}\ddot{\alpha}$$

Total lift $L = L_1 + L_2 + L_3$
Total moment about elastic axis

$$= \left(\frac{b}{2} + a_h b\right) L_1 + a_h b L_2 - \left(\frac{b}{2} - a_h b\right) L_3 - \frac{\pi\rho b^4}{8}\ddot{\alpha}$$

Circulatory lift $L_Q = L_1$
Noncirculatory lift $L_{NC} = L_2 + L_3$
The effect of shed vorticity is only on circulatory lift.

4.3.5 The Theodorsen Lift Deficiency Function

The Theodorsen Lift Deficiency function is obtained for a pure harmonic excitation of a pitching and plunging airfoil. Let us consider that the wing is undergoing pure harmonic motion at frequency ω

$$h(t) = \overline{h}e^{i\omega t}$$

$$\alpha(t) = \overline{\alpha}e^{i\omega t}$$

It follows then that the wake vorticity γ_w is also periodic in time with frequency ω. The circulation lift build up depend on the reduced frequency.

$$L = C(k)L_Q + L_{NC}$$

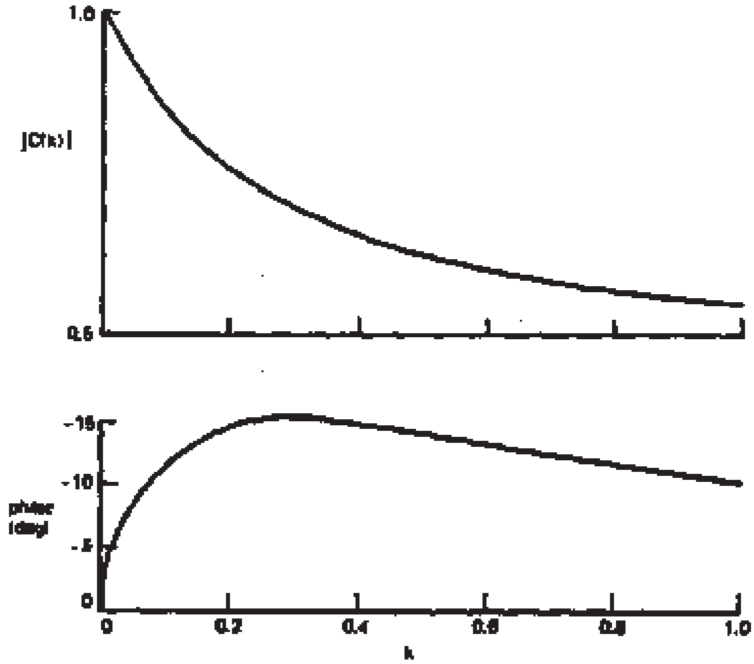
where C(k) is called Theodorsen lift deficiency function and it depends on reduced frequency

$$k = \frac{\omega b}{U}$$

where ω is the frequency of oscillation, rad/sec, b is the semi-chord, m, and U = free stream velocity, m/sec. The magnitude of C varies from 1 at low frequency to .5 at high frequency. The lift deficiency C takes care of the effect of shed vorticity on the lift due to unsteady motion and this always reduces the quasi-steady lift value. On the following figure, the lift deficiency function in terms of magnitude and phase is plotted for various k. The magnitude gives deficiency of lift and phase shows the lag in the lift build up. Thus the C(k) is a type of feed-back parameter of wake vorticity.

Lift and moment expressions are

$$\begin{aligned} L &= 2\pi b\rho U\left[U\alpha + \dot{h} + \dot{\alpha}\left(\frac{b}{2} - a_h b\right)\right] + \rho\pi b^2(U\dot{\alpha} + \ddot{h} - a_h b\ddot{\alpha}) \\ M &= 2\pi b\rho U\left[U\alpha + \dot{h} + \dot{\alpha}\left(\frac{b}{2} - a_h b\right)\right] \cdot \left(\frac{b}{2} + a_h\right) C(k) \\ &\quad + \pi\rho b^2\left[(\ddot{h} - a_h b\ddot{\alpha})a_h b - U\dot{\alpha}\left(\frac{b}{2} - a_h b\right) - \frac{b^2}{8}\ddot{\alpha}\right] \end{aligned} \tag{4.79}$$



Theodorsen lift deficiency function.

Let us examine a typical reduced frequency for a rotor blade

$$k = \frac{\omega b}{U} = \frac{\omega c/2}{\Omega r}$$

say $\omega = n\Omega$

$$k = \frac{nc}{2r}$$

Consider a representative section at 3/4-radius

$$k = \frac{nc}{2 \times \frac{3}{4}R} = \frac{2n}{3}(\frac{c}{R})$$

Assume

$$\frac{c}{R} = \frac{1}{20}$$

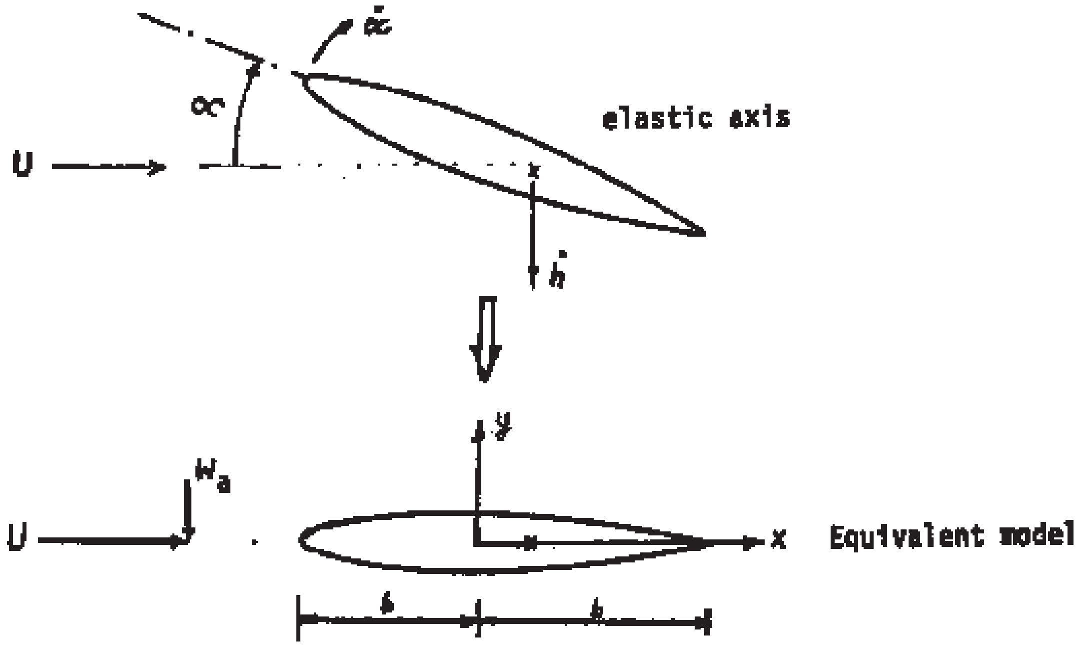
For 1/rev motion, n = 1, k = 0.033

$$|C(k)| \sim .97$$

The unsteady circulatory lift is about 97% of quasi-steady lift. This means that the unsteady effect due to shedding of vorticity are negligible. This shows that the quasi-steady assumption is quite adequate for 1/rev motion. For high frequency motion, say n = 4 (4/rev), there is about a 15% reduction in lift. Therefore unsteady effect has to be included for higher harmonic motion.

4.3.6 Application to Rotary Wings

The objective is to apply unsteady forces results derived earlier for fixed wing to rotary wing problems. For the fixed wing the blade undergoes two degrees of motion, pitching and heaving motions. The rotor blade motion as well as flow environment are complex, and for simplicity the effect of blade motion is taken care of in the velocity components. Let us examine the normal velocity due to airfoil motion.



where the first component is air velocity normal to the airfoil section at the pitch axis. The normal velocity component W_a is a function of $\dot{h}+U\alpha$ and $\dot{\alpha}$, it follows that the linear solution for aerodynamic lift and moment must also depend on these two quantities. Therefore, rewriting the lift and moment expressions.

$$\begin{aligned} L &= 2\pi\rho U b C(k)\left[\left(\dot{h}+U\alpha\right)+\left(\frac{b}{2}-a_h b\right)\dot{\alpha}\right] \\ &\quad +\pi\rho b^2\left[\left(\ddot{h}+U\dot{\alpha}\right)-a_h b\ddot{\alpha}\right] \end{aligned}$$

$$\begin{aligned} M = 2\pi\rho U b C(k)&\left[\left(\dot{h}+U\alpha\right)+\left(\frac{b}{2}-a_h b\right)\dot{\alpha}\right]\cdot\left(\frac{b}{2}+a_h b\right) \\ &+\pi\rho b^2\left[a_h b\left(\ddot{h}+U\dot{\alpha}\right)-\frac{1}{2}Ub\dot{\alpha}-b^2\left(\frac{1}{8}+a_h^2\right)\ddot{\alpha}\right] \end{aligned}$$

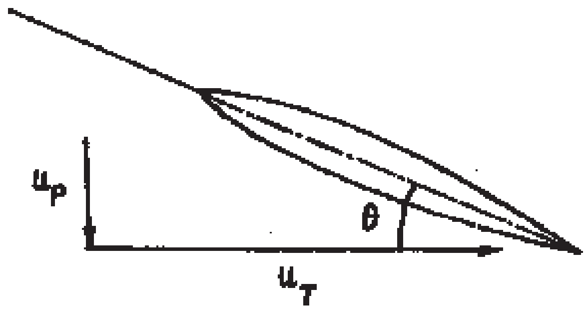
Writing the forces in this manner, one does not need to identify the section pitch and heave motions, but on the other hand one needs the mean and linear components of the normal velocity distribution over the airfoil chord. It is useful to identify, in the above expressions, the normal and inplane velocity components U_P and U_T.

$$\dot{h}+\left(\frac{b}{2}-a_h b\right)\dot{\alpha}=-U_P$$

$$U=U_T$$

$$\alpha=\theta$$

For rotor problems, h and θ are obtained based on the blade structural dynamic model. The inflow and forward velocity components are added appropriately.

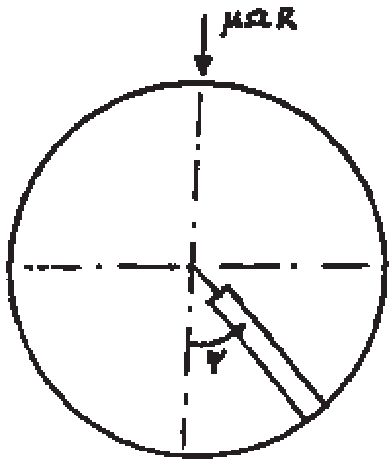


For example, consider an articulated rotor blade with rigid flap and rigid pitch motions.

$\dot{h} = -r\dot{\beta}$

$\dot{\alpha} = \dot{\theta}$

$b = c/2$



In hover

$U_T = \Omega r$

$U_P = \lambda \Omega R + r\dot{\beta} - \left(\frac{c}{4} - a_h \frac{c}{2}\right) \dot{\theta}$

In forward flight

$U_T = \Omega r + \Omega R \mu \sin\psi$

$U_P = \lambda \Omega R + \beta \Omega R \mu \cos\psi + r\dot{\beta} - \left(\frac{c}{4} - a_h \frac{c}{2}\right) \dot{\theta}$

where

λ is wake induced inflow

β is flap motion

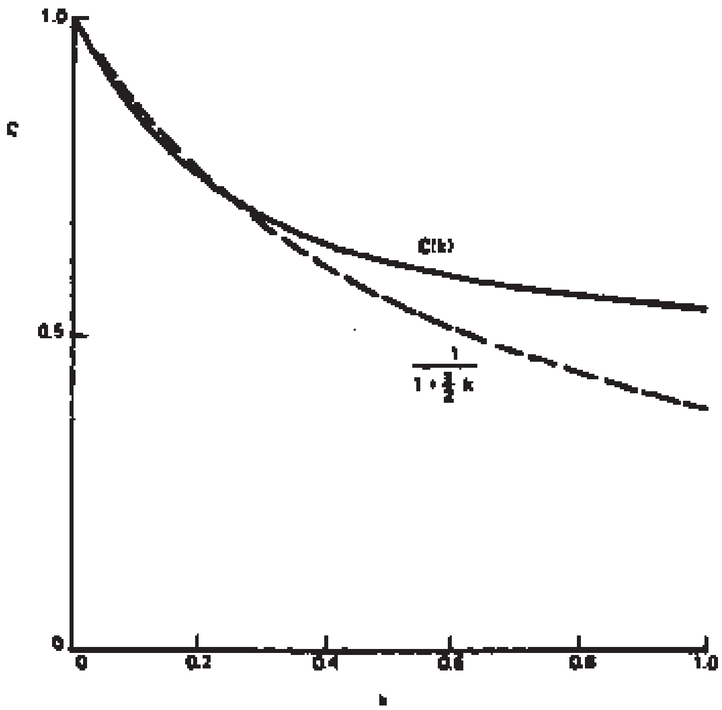
μ is advance ratio

$\lambda = \lambda_{Tpp} - \mu\beta_{1c}$

4.3.7 Near Shed Wake

Shed wake plays an important role in the determination of unsteady aerodynamic forces. The rotary shed wake is in a helical sheet behind the blade. Most of the influence on airfoil loading comes from near shed wake, extending 15° to 45° in azimuth behind the blade trailing edge. Thus considering only the near shed wake and neglecting the far wake reduces the computation to a great extent.

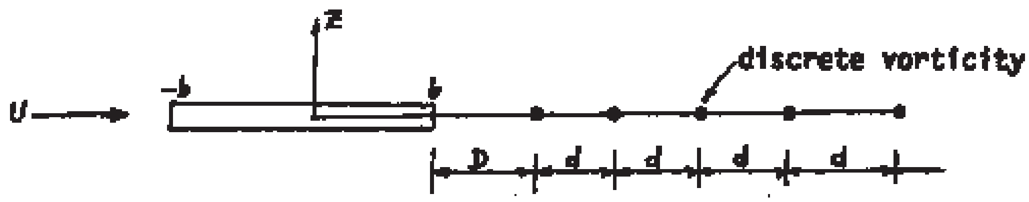
Miller (1964) considered a lifting line theory approximation for the near shed wake, implying a low reduced frequency. He derived a simple expression for the lift deficiency function.



$$C(k) = \frac{1}{1 + \frac{\pi}{2}k}$$

Pizialli Model (1966):

Pizialli made a discrete vortex approximation for the near wake.

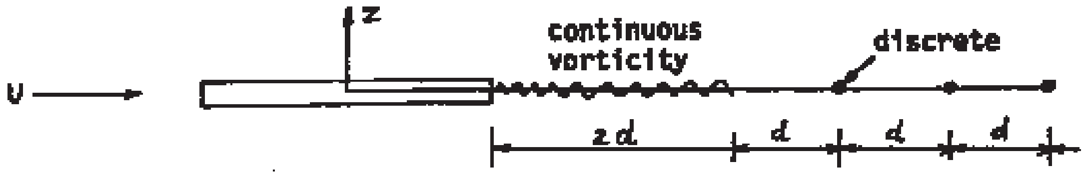


The wake is represented by a series of finite strength point vortices.

Spacing $d = \dfrac{2\pi U}{N\omega}$ for N vortices per cycle. (Typically 5-8)

Daughaday and Pizialli (1966):

They made another model for shed wake where combined continuous and discrete shed wake vorticity is used.



4.3.8 Time-Varying Free Stream

The rotating blade in forward flight has a time varying free stream velocity at a station

$$u_T = \Omega r + \Omega R \mu \sin\psi$$

This is periodic with a period of 2π rad. Since the time varying component is of the same order of magnitude as the mean component, one has to include the effect of time variation on the unsteady forces, both the direct effect as well as the shed wake effect. This results in

(a) additional noncirculating forces caused by $\frac{d}{dt}(U\alpha)$

(b) additional circulatory forces

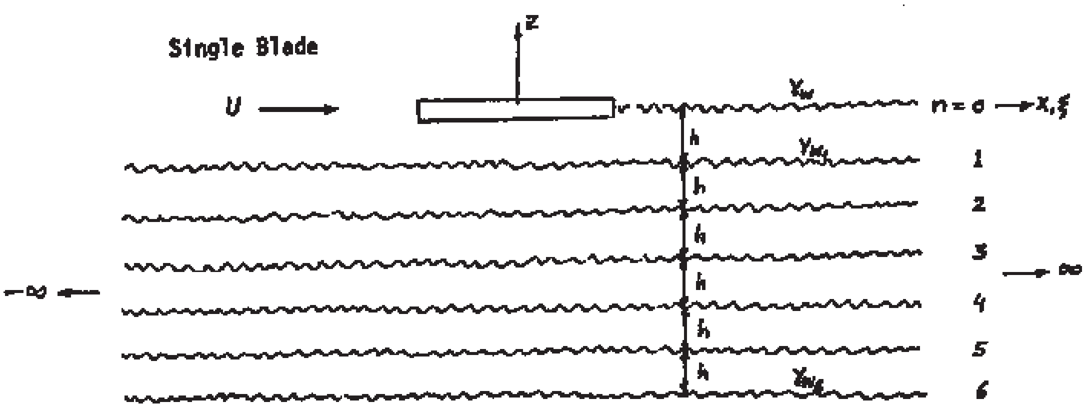
(c) additional influence of stretching and compressing of the vorticity in the shed wake.

A simple approximation sometimes can be very useful by choosing element C(k) based on mean k.

4.3.9 Returning Wake

For a hovering rotor, the wake generally moves slowly away from the rotor disk. Therefore, for the determination of unsteady loads, one needs to consider the influence of helical vortex sheets below the disk, one from each blade. For high inflow or forward speed, the rotor wake is convected away and so the influence of the returning shed wake is not important.

Loewy (1957) developed a two-dimensional model for unsteady aerodynamics of hovering rotor.



Consider a single blade rotor, so all the vorticity is originated from the same blade. The returning wake is modeled as a series of planar two-dimensional vortex sheets with a vertical separation h. For hover, the velocity $U = \Omega r$, is constant with time and the vortex sheets are parallel to free stream U. The spacing h depends on the mean flow through the rotor disk.

The wake induced velocity λ is

$$\lambda = \frac{1}{2\pi}\int_b^\infty \frac{\gamma_w d\xi}{x-\xi} + \sum_{n=1}^{\infty}\frac{1}{2\pi}\int_{-infty}^{\infty}\frac{\gamma_{w_n}(x-\xi)}{(x-\xi)^2+h^2n^2}d\xi$$

As before the strength of the shed vorticity is of the form

$$\gamma_w = \bar{\gamma_w} e^{i\omega(t-x/U)}$$

The strength of the n-th sheet is of the form

$$\gamma_{wn} = \bar{\gamma_w} e^{i\omega(t-x/U-2\pi n/\Omega)}$$

The total lift can again be written in the following form

$$L = C' L_Q + L_{NC}$$

where C' is the Loewy function and it is a function of reduced frequency k, frequency of oscillation ω/Ω and wake spacing h. The wake spacing h is such that the wake goes down by a distance $N_b h$ over a single rotor revolution. Thus

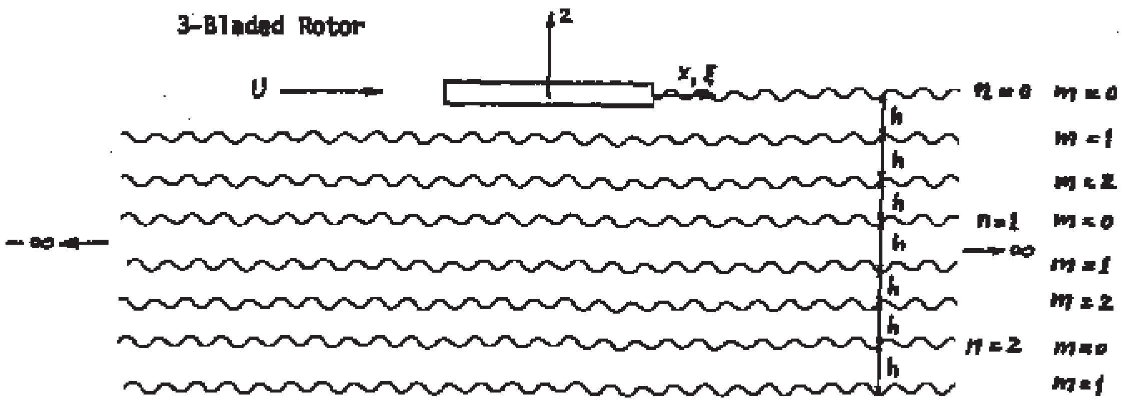
$$N_b h = v_0 \frac{2\pi}{\Omega}$$

where v_0 is the steady inflow. It follows

$$h = \frac{2\lambda_0 c}{\sigma}$$

$$\frac{h}{c} = \frac{h}{2b} = \frac{2\lambda_0}{\sigma}$$

The Loewy function is quite similar to Theodorsen function, C(k). For a N blade rotor, the returning wake model gets complicated, since the wake of other blades also has to be considered.



m = 0, 1, 2 blade index

For ω/Ω = integer and for low k (approximately < 0.4),

$$C' \simeq \frac{1}{1+\frac{\pi\sigma}{4\lambda_0}}$$

where λ is the steady inflow ratio and σ is the solidity ratio. Typically,

$$\lambda_0 = .05 \text{ to } .07$$

$$\frac{h}{b} \simeq 3 \text{ or } 4$$

$$C' \simeq .5$$

This is quite important for control loads and stability. This may reduce flap damping significantly.

4.3.10 Miller's Conclusion

When system frequencies approach integers of rotational speed Ω, unsteady flow theory must be used because of a large reduction in $\frac{dc_l}{d\alpha}$ due to spiral wake. The near wake including the first quadrant or so of vorticity behind the blade is important. For this the lift acts at 1/4-chord due to angle of attack at 3/4-chord. When frequency ω are not close to integers of Ω, the far wake contribution to C(k) are negligible.

For $\frac{\omega}{\Omega}$ = integer,

$$C' \simeq \frac{1}{1+\frac{\pi\sigma}{4\lambda_0}}$$

$$= F + iG \quad (G = 0)$$

Where

$$\lambda_0 = \text{ steady inflow.}$$

This means that there is a lift deficiency but no lag is produced.

4.4 Time Domain Methods for Unsteady Aerodynamics

Consider a unit step function at $t = h$

$$\begin{aligned} u(t-h) &= 1 \quad t \geq h \\ &= 0 \quad t < h \end{aligned}$$

Any function f at a discrete time nh can be expressed as

$$\begin{aligned} f(nh) &= f(0)u(t-0) + \sum_{i=1}^{i=n} \left[f(ih) - f(\overline{i-1}h) \right] u(t-ih) \\ &= f(0)u(t-0) + \sum_{i=1}^{i=n} \Delta f(ih) u(t-ih) \\ &= f(0)u(t-0) + \sum_{i=1}^{i=n} \frac{\Delta f(ih)}{h} u(t-ih) h \end{aligned}$$

In the limit as $h \to 0$, we have at a continuous time t

$$f(t) = f(0)u(t-0) + \int_{\sigma=0}^{\sigma=t} \frac{\partial f}{\partial \sigma} u(t-\sigma) d\sigma$$

Thus any continous and smooth function $f(t)$ can be expressed as a superimposition of a series of step functions. Note that for $t > 0$, $u(t-0) = 1$. Similarly an angle of attack variation can be expressed in the same manner, as a series of step functions

$$\alpha(t) = \alpha(0) + \int_0^t \frac{\partial \alpha}{\partial \sigma} u(t-\sigma) d\sigma \tag{4.80}$$

In order to calculate the airloads (normal force, pitching moment, and chord force) generated by the airfoil in response to this angle of attack variation, it is therefore sufficient to calculate only the response to a step input in angle of attack of unit magnitude. The response to any angle of attack variation can then be constructed by superposition of these responses. The response to a step input in angle of attack is called an indicial response. For example if $\theta(t-h)$ is the indicial lift coefficient generated in response to $u(t-h)$, an unit step input in angle of attack at $t = h$, then the lift coefficient at any time t is simply

$$C_l(t) = \alpha(0)\theta(0) + \int_0^t \frac{\partial \alpha}{\partial \sigma}\theta(t-\sigma)d\sigma \tag{4.81}$$

Note that $\theta(t-h)$, the lift increment generated in response to an unit step change in angle of attack $\alpha(t-h)$ applied at time $t = h$, finally reaches a steady state value after some time. This value is the airfoil lift curve slope $C_{l\alpha}$. In the case of a flat plate, based on calculations of thin airfoil theory with no stall, we have $C_{l\alpha} = 2\pi$. Including the Glauert correction for compressibility $C_{l\alpha} = 2\pi/\beta$, where $\beta = \sqrt{1-M^2}$. For real airfoils, $C_{l\alpha}$ depends on the initial angle of attack setting at which the step change is applied. That is, the steady state increment in C_l in response to an unit step increment in angle of attack depends on whether the unit step increment is imposed while the airfoil is at 5^o or 12^o. At 12^o, when the airfoil is already near stall, an unit step increment in angle of attack may not produce any noticeable increment in C_l at all. Thus, below stall, $C_{l\alpha}$ is same as the airfoil lift curve slope. Above stall, $C_{l\alpha}$ depends on the local angle of attack.

Instead of the lift coefficient C_l, let us consider the normal force coefficient C_n from now onwards. The direction of the normal force coefficient is defined solely by the airfoil orientation. Similarly instead of drag consider the chord force C_c. The choice is only a matter of convention, either can be used to formulate the problem without any loss in generality. For pitching moments, we consider those about the airfoil quarter-chord. In addition, we distinguish between the circulatory and non-circulatory components by the superscript C and I, where I stands for 'impulsive'. The impulsive airloads in compressible flow are similar to their the noncirculatory counterparts in incompressible flow. Let the circulatory part of the normal force indicial response to angle of attack be of the form

$$C^C_{n\alpha}(t) = C_{n\alpha}\phi^C_{n\alpha}(t)$$

where the indicial response function $\phi^C_{n\alpha}(t) \to 1$ as $t \to \infty$, so that $C^C_{n\alpha}(t) \to C_{n\alpha}$ at the steady state. Also, at $t = 0$, the indicial function must vanish, $\phi^C_{n\alpha}(0) = 0$. Consider an acceptable form as the following

$$\phi^C_{n\alpha}(t) = 1 - A_1 e^{-t/T_1} - A_2 e^{-t/T_2} \quad \text{where} \quad A_1 + A_2 = 1 \tag{4.82}$$

The normal force response of the airfoil to sinusoidal inputs can be deduced from its indicial response. The response to sinusoidal inputs is the response to inputs of the general form e^{pt}, where $p = j\omega$ for sinusoidal inputs.

The response to inputs of the form e^{pt} is, by definition, the Transfer function between input and output of the system expressed in terms of the Laplace variable p, assuming that the response is related to the input via an ODE in time. A continuous function of time $f(t)$ can be expressed as a summation of basis functions each of form e^{pt}, where p is a complex variable with frequency varying from $+\infty$ to $-\infty$, and each multiplied with a magnitude $F(p)$ independant of time t but in general a function of p. Thus

$$f(t) = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} F(p)e^{pt}dt \tag{4.83}$$

The component $F(p)$ is defined as the Laplace Transform of the function $f(t)$ and can be determined by

$$F(p) = \lim_{T\to\infty}\int_0^T f(t)e^{-pt}dt$$

Let $f(t)$ be the input to a system governed by an ODE. Let $y(t)$ be the output. The function $y(t)$ can again be expressed as a summation of basis functions as before.

$$y(t) = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} Y(p)e^{pt}dt \tag{4.84}$$

where $Y(p)$ is the Laplace Transform of $y(t)$. Now, note that for a system (an input output relationship) governed by a linear ODE with constant coefficients, the output corresponding to an input e^{pt} must necessarily be of the form $H(p)e^{pt}$. $H(p)$ is defined as the transfer function in terms of the Laplace variable. A special case is when $p = 1$. The output corresponding to an input e^t is always e^t itself, i.e., $H(p) = 1$. Thus the output $y(t)$, corresponding to $f(t)$, which is given by eqn.4.83 is necessarily of the form

$$y(t) = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} F(p)\,H(p)e^{pt}dt \tag{4.85}$$

Comparing expressions4.84 and 4.85 we have

$$Y(p) = F(p)H(p) \tag{4.86}$$

or

$$H(p) = \frac{Y(p)}{F(p)} \tag{4.87}$$

Thus the transfer function of a system $H(p)$, which is the response of the system to an input of the form e^{pt} can be determined by the ratio of the Laplace Transforms of any output-input combination.

Assuming that the airload response to an indicial input of angle of attack is governed by a linear constant coefficient system, the response to an input angle of attack e^{pt} is simply $H(p)$ where $H(p)$ is the ratio of the Laplace Transforms of any set of output-input combination. The Laplace Transforms of the unit step input and assumed indicial output are

$$L\left[u(t-0)\right] = \frac{1}{p}$$

$$L\left[C^C_{n\alpha}\right] = C_{n\alpha}\left(\frac{1}{p} - \frac{A_1T_1}{1+T_1p} - \frac{A_2T_2}{1+T_2p}\right)$$

Thus

$$\begin{aligned}
H^C_{n\alpha}(p) = \frac{L\left[C^C_{n\alpha}\right]}{L\left[u(t-0)\right]} &= C_{n\alpha}\left(1 - \frac{A_1T_1p + A_1 - A_1}{1+T_1p} - \frac{A_2T_2p + A_2 - A_2}{1+T_2p}\right) \\
&= C_{n\alpha}\left(1 - A_1 - A_2 + \frac{A_1}{1+T_1p} + \frac{A_2}{1+T_2p}\right) \\
&= C_{n\alpha}\left(\frac{A_1}{1+T_1p} + \frac{A_2}{1+T_2p}\right) \quad \text{using } A_1 + A_2 = 1
\end{aligned}$$

For frequency response, i.e. response to inputs of sine and cosine harmonics, substitute $p = j\omega$. The transfer function then takes the following form

$$\begin{aligned}
H^C_{n\alpha}(j\omega) &= C_{n\alpha}\left(\frac{A_1}{1+jT_1\omega} + \frac{A_2}{1+jT_2\omega}\right) \\
&= C_{n\alpha}\left[\frac{A_1\left(1-jT_1\omega\right)}{1+\omega^2T_1^2} + \frac{A_2\left(1-jT_2\omega\right)}{1+\omega^2T_2^2}\right] \\
&= C_{n\alpha}\left(\frac{A_1}{1+\omega^2T_1^2} + \frac{A_2}{1+\omega^2T_2^2}\right) - jC_{n\alpha}\left(\frac{A_1T_1\omega}{1+\omega^2T_1^2} + \frac{A_2T_2\omega}{1+\omega^2T_2^2}\right)
\end{aligned}$$

Consider the indicial function given in eqn.4.88. For rotor problems, the time t is often replaced with a nondimensional parameter s, where s is the distance traversed by the airfoil measured in semi-chords in time t after the step change in angle of attack.

$$s = \frac{Ut}{c/2} \quad \text{or} \quad t = \frac{c}{2U}s$$

The time constants T_1 and T_2 are replaced with constants b_1 and b_2 where

$$T_1 = \frac{c}{2U}\frac{1}{b_1\beta^2}$$

$$T_2 = \frac{c}{2U}\frac{1}{b_2\beta^2} \quad \text{where} \quad \beta^2 = 1 - M^2$$

The indicial function in terms of s and b_1 then take the following form

$$\phi^C_{n\alpha}(t) = 1 - A_1 e^{-sb_1\beta^2} - A_2 e^{-sb_2\beta^2} \quad \text{where} \quad A_1 + A_2 = 1 \tag{4.88}$$

Note that

$$\omega T_1 = \frac{\omega c}{2U}\frac{1}{b_1\beta^2} = \frac{k}{b_1\beta^2}$$

$$\omega T_2 = \frac{\omega c}{2U}\frac{1}{b_2\beta^2} = \frac{k}{b_2\beta^2} \quad \text{where } k \text{ is the reduced frequency}$$

The transfer function then takes the following form

$$H^C_{n\alpha} = C_{n\alpha}\left(\frac{A_1 b_1^2\beta^4}{b_1^2\beta^4 + k^2} + \frac{A_2 b_2^2\beta^4}{b_2^2\beta^4 + k^2}\right) - jC_{n\alpha}\left(\frac{A_1 b_1 k\beta^2}{b_1^2\beta^4 + k^2} + \frac{A_2 b_2 k\beta^2}{b_2^2\beta^4 + k^2}\right) \tag{4.89}$$

4.4.1 Leishman-Beddoes indicial model

The Leishman-Beddoes model consists of the indicial functions given by Beddoes [9, 10] and subsequently refined by Leishman and Beddoes [11, 12, 13]. They included the effects of compressibility, and later viscous flow separation. The indicial normal force due to angle of attack, the indicial pitching moment (about quarter-chord) due to angle of attack, the indicial normal force due to 'rate' of angle of attack, and the indicial pitching moment due to 'rate' of angle of attack are given by the following expressions. The first part is the impulsive part, analogous to the non-circulatory components in incompressible flow, the second part is the circulatory part due to the effects of the shed vorticity.

$$C_{n\alpha} = \frac{4}{M}\phi^I_{\alpha n} + \frac{2\pi}{\beta}\phi^C_{\alpha n} \tag{4.90}$$

$$C_{m\alpha} = -\frac{1}{M}\phi^I_{\alpha m} - \frac{2\pi}{\beta}\phi^C_{\alpha n}(x_{ac} - 0.25) \tag{4.91}$$

$$C_{nq} = \frac{1}{M}\phi^I_{qn} + \frac{\pi}{\beta}\phi^C_{qn} \tag{4.92}$$

$$C_{mq} = -\frac{7}{12M}\phi^I_{qm} - \frac{\pi}{8\beta}\phi^C_{qm} \tag{4.93}$$

Each indicial response is assumed to consist of two parts: an exponentially decaying part for the initial non-circulatory loading, and an assymptotically growing part which reaches a steady state value. The initial non-circulatory loading is taken from piston theory [14, 15]. The circulatory component of the indicial normal force due to angle of attack is

$$\phi^C_{\alpha n} = 1 - A_1 e^{-b_1\beta^2 s} - A_2 e^{-b_2\beta^2 s}$$

The impulsive component of the indicial normal force due to angle of attack is

$$\phi^I_{\alpha n} = e^{-\frac{s}{T'_\alpha}} \quad \text{where} \quad T'_\alpha = \frac{4M}{2(1-M) + 2\pi\beta M^2(A_1 b_1 + A_2 b_2)}$$

The circulatory component of the indicial normal force due to 'rate' of angle of attack is the same as that due to angle of attack

$$\phi^C_{qn} = 1 - A_1 e^{-b_1\beta^2 s} - A_2 e^{-b_2\beta^2 s}$$

The impulsive component of the indicial normal force due to 'rate' of angle of attack has the same form but with a different time constant

$$\phi^I_{qn} = e^{-\frac{s}{T'_q}} \quad \text{where} \quad T'_q = \frac{2M}{(1-M) + 2\pi\beta M^2(A_1 b_1 + A_2 b_2)}$$

The circulatory component of the indicial pitching moment due to angle of attack is assumed to be due to the aerodynamic center offset from quarter chord. The impulsive component of the indicial pitching moment due to angle of attack is

$$\phi^I_{\alpha m} = A_3 e^{-\frac{s}{b_3 T'_{\alpha m}}} + A_4 e^{-\frac{s}{b_4 T'_{\alpha m}}} \quad \text{where} \quad T'_{\alpha m} = 2M\left[\frac{A_3 b_4 + A_4 b_3}{b_3 b_4 (1-M)}\right]$$

The circulatory component of the indicial pitching moment due to 'rate' of angle of attack is

$$\phi^C_{qm} = 1 - e^{-b_5\beta^2 s}$$

The impulsive component of the indicial pitching moment due to 'rate' of angle of attack is

$$\phi^I_{qm} = e^{-\frac{s}{T'_{qm}}} \quad \text{where} \quad T'_{qm} = \frac{14M}{15(1-M) + 3\pi\beta M^2 b_5}$$

The original model parameters proposed by Beddoes are

$$A_1 = 0.3 \quad A_2 = 0.7 \quad A_3 = 1.5 \quad A_4 = -0.5$$

$$b_1 = 0.14 \quad b_2 = 0.53 \quad b_3 = 0.25 \quad b_4 = 0.1 \quad b_5 = 0.5$$

4.4.2 Frequency response of indicial model

It was shown earlier that it is possible to deduce the frequency response (i.e. response to sinusoidal inputs) from indicial response. The frequency response of the circulatory normal force was deduced in eqn.4.89. Consider the impulsive indicial normal force in response to angle of attack.

$$C^I_{n\alpha}(s) = \frac{4}{M} e^{-\frac{s}{T'_\alpha}}$$

To convert to a function in time use $s = 2Ut/c$. The nondimensional constant T'_α can be expressed as

$$T'_\alpha = \frac{2U}{c} T_\alpha$$

where T_α has the dimension of time. Then we have

$$C^I_{n\alpha}(t) = \frac{4}{M} e^{-t/T_\alpha}$$

The Laplace Transform is

$$L\left[C_{n\alpha}^{I}\right] = \frac{4}{M}\frac{T_\alpha}{1+T_\alpha p}$$

Thus

$$H_{n\alpha}^{I}(p) = \frac{L\left[C_{n\alpha}^{I}\right]}{L\left[u(t-0)\right]} = \frac{4}{M}\frac{T_\alpha p}{1+T_\alpha p}$$

Substitute $p = j\omega$ to obtain

$$H_{n\alpha}^{I}(j\omega) = \frac{4}{M}\left(\frac{\omega^2 T_\alpha^2}{1+\omega^2 T_\alpha^2}\right) + j\frac{4}{M}\left(\frac{\omega T_\alpha}{1+\omega^2 T_\alpha^2}\right)$$

The time constant T_α can be expressed as

$$T_\alpha = \frac{c}{a}K_\alpha$$

where a is the speed of sound, and K_α is a nondimensional constant. Then

$$\omega T_\alpha = \omega\frac{c}{a}K_\alpha = \frac{2Uk}{c}\frac{c}{a}K_\alpha = 2MkK_\alpha$$

The transfer function then takes the following form

$$H_{n\alpha}^{I} = \frac{4}{M}\left(\frac{4K_\alpha^2 M^2 k^2}{1+4K_\alpha^2 M^2 k^2}\right) + j\frac{4}{M}\left(\frac{2K_\alpha Mk}{1+4K_\alpha^2 M^2 k^2}\right) \tag{4.94}$$

The transfer function has been expressed as a function of incident Mach number M and reduced frequency k. Consider the impulsive indicial pitching moment in response to angle of attack.

$$C_{m\alpha}^{I}(s) = \frac{1}{M}\left(A_3 e^{-\frac{s}{b_3 T'_{\alpha m}}} + A_4 e^{-\frac{s}{b_4 T'_{\alpha m}}}\right)$$

To convert to a function in time use $s = 2Ut/c$. The nondimensional constant $T'_{\alpha m}$ can be expressed as

$$T'_{\alpha m} = \frac{2U}{c}T_{\alpha m}$$

where $T_{\alpha m}$ has the dimension of time. Then we have

$$C_{m\alpha}^{I}(t) = \frac{1}{M}\left(A_3 e^{-\frac{t}{b_3 T'_{\alpha m}}} + A_4 e^{-\frac{t}{b_4 T'_{\alpha m}}}\right)$$

The Laplace Transform is

$$L\left[C_{m\alpha}^{I}\right] = \frac{1}{M}\left(\frac{A_3 b_3 T_{\alpha m}}{1+b_3 T_{\alpha m}p} + \frac{A_4 b_4 T_{\alpha m}}{1+b_4 T_{\alpha m}p}\right)$$

Thus

$$H_{m\alpha}^{I}(p) = \frac{L\left[C_{m\alpha}^{I}\right]}{L\left[u(t-0)\right]} = \frac{1}{M}\left(\frac{A_3 b_3 T_{\alpha m}p}{1+b_3 T_{\alpha m}p} + \frac{A_4 b_4 T_{\alpha m}p}{1+b_4 T_{\alpha m}p}\right)$$

Substitute $p = j\omega$ to obtain

$$H_{m\alpha}^{I}(j\omega) = \frac{1}{M}\left(\frac{A_3 b_3^2\omega^2 T_{\alpha m}^2}{1+b_3^2\omega^2 T_{\alpha m}^2} + \frac{A_4 b_4^2\omega^2 T_{\alpha m}^2}{1+b_4^2\omega^2 T_{\alpha m}^2}\right) - j\frac{1}{M}\left(\frac{A_3 b_3\omega T_{\alpha m}}{1+b_3^2\omega^2 T_{\alpha m}^2} + \frac{A_4 b_4\omega T_{\alpha m}}{1+b_4^2\omega^2 T_{\alpha m}^2}\right)$$

The time constant T_α can be expressed as

$$T_{\alpha m} = \frac{c}{a} K_{\alpha m}$$

where a is the speed of sound, and K_α is a nondimensional constant. Then

$$\omega T_{\alpha m} = \omega \frac{c}{a} K_{\alpha m} = \frac{2Uk}{c}\frac{c}{a} K_{\alpha m} = 2MkK_{\alpha m}$$

The transfer function then takes the following form

$$H^I_{m\alpha} = \frac{1}{M}\left(\frac{4A_3 b_3^2 M^2 k^2 K_{\alpha m}^2}{1+4b_3^2 M^2 k^2 K_{\alpha m}^2} + \frac{4A_4 b_4^2 M^2 k^2 K_{\alpha m}^2}{1+4b_4^2 M^2 k^2 K_{\alpha m}^2}\right) - j\frac{1}{M}\left(\frac{2A_3 b_3 MkK_{\alpha m}}{1+4b_3^2 M^2 k^2 K_{\alpha m}^2} + \frac{2A_4 b_4 MkK_{\alpha m}^2}{1+4b_4^2 M^2 k^2 K_{\alpha m}^2}\right) \tag{4.95}$$

The transfer function has been expressed as a function of incident Mach number M and reduced frequency k.

Consider the impulsive indicial normal force in response to 'rate' of angle of attack. Note that, here, the input is still a unit step of angle of attack, and not an unit step of 'rate' of angle of attack.

$$C^I_{nq}(s) = \frac{1}{M} e^{-\frac{s}{T'_q}}$$

To convert to a function in time use $s = 2Ut/c$. The nondimensional constant T'_q can be expressed as

$$T'_q = \frac{2U}{c} T_q$$

where T_q has the dimension of time. Then we have

$$C^I_{nq}(t) = \frac{1}{M} e^{-t/T_q}$$

The Laplace Transform is

$$L\left[C^I_{nq}\right] = \frac{1}{M}\frac{T_q}{1+T_q p}$$

Note that the above expressions describes the response to a sinusoidal input in pitch rate, i.e. a pitch rate of the form e^{pt}. This is not the transform we seek. We seek the response to a sinusoidal input in angle of attack, i.e. an angle of attack variation of the form e^{pt}. The response will depend on the pitch rate this angle of attack variation generates. To this end we consider a step change in input angle of attack. Then, relate in the Laplace domain, the output response with the input pitch rate it generates. If the angle of attack variation is given by $\alpha(t)$, then the rate of angle of attack is the time derivative $\dot{\alpha}(t)$, with units of rad/sec. In nondimensional form

$$q(t) = \dot{\alpha}\frac{c}{U}$$

The Laplace transform of the pitch rate is related to the Laplace transform of the angle of attack variation as follows

$$L[q] = L[\dot{\alpha}]\frac{c}{U} = pL[\alpha]\frac{c}{U}$$

Now, the transfer function between normal force due to pitch input and pitch input is given by

$$\frac{L\left[C^I_{nq}\right]}{L[q]} = \frac{1}{M}\frac{T_q p}{1+T_q p} \tag{4.96}$$

Replace the Laplace transform of the pitch rate

$$\frac{L\left[C_{nq}^{I}\right]}{pL\left[\alpha\right]\frac{c}{U}} = \frac{1}{M}\frac{T_q p}{1+T_q p}$$

The transfer function between normal force due to pitch input and the angle of attack is then obtained by simply re-arranging the above expression

$$H_{nq}^{I}(p) = \frac{L\left[C_{nq}^{I}\right]}{L\left[\alpha\right]} = \frac{1}{M}\frac{T_q p}{1+T_q p}p\frac{c}{U} = \frac{1}{M}\frac{2T_q p}{1+T_q p}p\frac{c}{2U} \tag{4.97}$$

Substituting $p = j\omega$ in eqn.4.102 gives C_{nq}^{I} when q is sinusoidal. Substituting $p = j\omega$ in eqn.4.97 gives C_{nq}^{I} when α is sinusoidal. This is the transfer function we seek. Substitute $p = j\omega$ in eqn.4.97 to obtain

$$\begin{aligned} H_{nq}^{I}(j\omega) &= \frac{1}{M}\left(\frac{j\omega T_q}{1+j\omega T_q}j\omega\frac{c}{2V}\right) = \frac{1}{M}\left(\frac{j\omega T_q}{1+j\omega T_q}jk\right) \\ &= -\frac{1}{M}\left(\frac{2\omega T_q k}{1+\omega^2 T_q^2}\right) + j\frac{1}{M}\left(\frac{2\omega^2 T_q^2 k}{1+\omega^2 T_q^2}\right) \end{aligned}$$

The time constant T_q can be expressed as

$$T_\alpha = \frac{c}{a}K_q$$

where a is the speed of sound, and K_q is a nondimensional constant. Then

$$\omega T_\alpha = \omega\frac{c}{a}K_q = \frac{2Uk}{c}\frac{c}{a}K_q = 2MkK_q$$

The transfer function then takes the following form

$$H_{nq}^{I} = -\frac{1}{M}\left(\frac{4K_q M k^2}{1+4K_q^2 M^2 k^2}\right) + j\frac{1}{M}\left(\frac{8K_q^2 M^2 k^3}{1+4K_q^2 M^2 k^3}\right) \tag{4.98}$$

Similarly, consider the circulatory indicial pitching moment in response to 'rate' of angle of attack, i.e. pitch rate.

$$C_{mq}^{C}(s) = -\frac{\pi}{8\beta}\left(1 - e^{-b_5\beta^2 s}\right)$$

To convert to a function in time use $s = 2Ut/c$, and introduce the nondimensional constant T_5

$$T_5 = \frac{c}{2U}\frac{1}{b_5\beta^2}$$

Then we have

$$C_{mq}^{C}(t) = -\frac{\pi}{8\beta}\left(1 - e^{-t/T_5}\right)$$

Now, the transfer function with respect to the pitch input is given by

$$\frac{L\left[C_{mq}^{C}\right]}{L\left[q\right]} = -\frac{\pi}{8\beta}\left(\frac{1}{1+T_5 p}\right) \tag{4.99}$$

Following the arguments given earlier, the transfer function with respect to the angle of attack is then

$$H_{mq}^{C}(p) = \frac{L\left[C_{mq}^{C}\right]}{L\left[\alpha\right]} = -\frac{2\pi}{8\beta}\left(\frac{p}{1+T_5 p}\right)\frac{c}{2U} \tag{4.100}$$

Substitute $p = j\omega$ to obtain

$$\begin{aligned} H_{mq}^{C}(j\omega) &= -\frac{\pi}{8\beta}\left(\frac{jk}{1+j\omega T_5}\right) \\ &= -\frac{\pi}{8\beta}\left(\frac{k\omega T_5}{1+\omega^2 T_5^2}\right) - j\frac{\pi}{8\beta}\left(\frac{k}{1+\omega^2 T_5^2}\right) \end{aligned}$$

Use

$$\omega T_5 = \frac{k}{b_5 \beta^2}$$

to obtain

$$H_{mq}^{C} = -\frac{\pi}{8\beta}\left(\frac{b_5 k^2 \beta^2}{k^2 + b_5^2 \beta^4}\right) - j\frac{\pi}{8\beta}\left(\frac{k b_5^2 \beta^4}{k^2 + b_5^2 \beta^4}\right) \tag{4.101}$$

Lastly, consider the impulsive indicial pitching moment in response to pitch rate.

$$C_{mq}^{I}(s) = -\frac{7}{12M} e^{-\frac{s}{T'_{qm}}}$$

To convert to a function in time use $s = 2Ut/c$. The nondimensional constant T'_{mq} can be expressed as

$$T'_{mq} = \frac{2U}{c} T_{mq}$$

where T_q has the dimension of time. Then we have

$$C_{mq}^{I}(t) = -\frac{7}{12M} e^{-t/T_{mq}}$$

Now, the transfer function with respect to the pitch input is given by

$$\frac{L\left[C_{mq}^{I}\right]}{L\left[q\right]} = -\frac{7}{12M}\left(\frac{T_{qm} p}{1+T_{qm} p}\right) \tag{4.102}$$

Following the arguments given earlier, the transfer function with respect to the angle of attack is then

$$H_{mq}^{I}(p) = \frac{L\left[C_{mq}^{C}\right]}{L\left[\alpha\right]} = -\frac{7}{12M}\left(\frac{2T_{qm} p}{1+T_{qm} p}\right)\frac{c}{2U} \tag{4.103}$$

Substitute $p = j\omega$, and express the time constant T_{mq} as

$$T_{mq} = \frac{c}{a} K_{mq}$$

where a is the speed of sound, and K_{mq} is a nondimensional constant. Then

$$\omega T_{mq} = \omega \frac{c}{a} K_{mq} = \frac{2Uk}{c}\frac{c}{a} K_{mq} = 2MkK_{mq}$$

The transfer function then takes the following form

$$H_{nq}^{I} = \frac{7}{12M}\left(\frac{4K_{mq} M k^2}{1+4K_{mq}^2 M^2 k^2}\right) - j\frac{7}{12M}\left(\frac{8K_{mq}^2 M^2 k^3}{1+4K_{mq}^2 M^2 k^3}\right) \tag{4.104}$$

4.4.3 Recursive formulation of an indicial model

The normal force at any time t is the sum of normal forces due to angle of attack and pitch rate, each having a circulatory and an impulsive component.

$$\begin{aligned} C_N &= C_N^{\alpha} + C_N^{q} \\ &= C_N^{\alpha C} + C_N^{\alpha I} + C_N^{qC} + C_N^{qI} \end{aligned}$$

where the components are given in terms of the indicial response functions (see eqns.4.80 and 4.81) as follows

$$\begin{aligned} C_N^{\alpha C}(s,M) &= \frac{2\pi}{\beta}\phi_{\alpha n}^{C}(0)\alpha(0) + \int_0^s \frac{\partial\alpha}{\partial\sigma}\frac{2\pi}{\beta}\phi_{\alpha n}^{C}(s-\sigma)d\sigma \\ &= \frac{2\pi}{\beta}\left[\alpha(0) + \int_0^s \frac{\partial\alpha}{\partial\sigma}\phi_{\alpha n}^{C}(s-\sigma)d\sigma\right] \\ &= \frac{2\pi}{\beta}\left[\alpha(0) + \int_0^s \frac{\partial\alpha}{\partial\sigma}\left\{1 - A_1 e^{-b_1\beta^2(s-\sigma)} - A_2 e^{-b_2\beta^2(s-\sigma)}\right\}d\sigma\right] \\ &= \frac{2\pi}{\beta}\left[\alpha(0) + \int_0^s d\alpha - \int_0^s A_1\frac{\partial\alpha}{\partial\sigma}e^{-b_1\beta^2(s-\sigma)}d\sigma - \int_0^s A_2\frac{\partial\alpha}{\partial\sigma}e^{-b_2\beta^2(s-\sigma)}d\sigma\right] \\ &= \frac{2\pi}{\beta}\left[\alpha(s) - X(s) - Y(s)\right] \end{aligned} \tag{4.105}$$

where

$$\begin{aligned} X(s) &= \int_0^s A_1\frac{\partial\alpha}{\partial\sigma}e^{-b_1\beta^2(s-\sigma)}d\sigma \\ Y(s) &= \int_0^s A_2\frac{\partial\alpha}{\partial\sigma}e^{-b_2\beta^2(s-\sigma)}d\sigma \end{aligned} \tag{4.106}$$

The above formulation can be cast into a recursive form for discrete advances in time Δt, or reduced time Δs, where $\Delta s = 2U\Delta t/c$. For example, at $s + \Delta s$ we have

$$C_N^{\alpha C}(s+\Delta s, M) = \frac{2\pi}{\beta}\left[\alpha(s+\Delta s) - X(s+\Delta s) - Y(s+\Delta s)\right]$$

Using eqn4.106 it can be shown

$$X(s+\Delta s) = X(s)e^{-b_1\beta^2\Delta s} + A_1\Delta\alpha(s+\Delta s)e^{-b_1\beta^2\frac{\Delta s}{2}}$$

or in terms of current s

$$X(s) = X(s-\Delta s)e^{-b_1\beta^2\Delta s} + A_1\Delta\alpha e^{-b_1\beta^2\frac{\Delta s}{2}}$$

where $\Delta\alpha$ is at s and

$$X(0) = 0$$

Thus the recursive formulation for the circulatory normal force due to angle of attack variation can be expressed as

$$\begin{aligned} C_N^{\alpha C}(s,M) &= \frac{2\pi}{\beta}\left[\alpha(s) - X_1(s) - Y_1(s)\right] \\ X_1(s) &= X_1(s-\Delta s)e^{-b_1\beta^2\Delta s} + A_1\Delta\alpha e^{-b_1\beta^2\frac{\Delta s}{2}} \\ Y_1(s) &= Y_1(s-\Delta s)e^{-b_2\beta^2\Delta s} + A_2\Delta\alpha e^{-b_2\beta^2\frac{\Delta s}{2}} \\ X_1(0) &= Y_1(0) = 0 \end{aligned} \tag{4.107}$$

Using the same approach it can be shown that the recursive formulation for the impulsive normal force due to angle of attack variation is

$$\begin{aligned}
&C_N^{\alpha I}(s,M) = \frac{4T'_\alpha}{M}\left[\frac{\Delta\alpha(s)}{\Delta s} - D_1(s)\right] \\
&D_1(s) = D_1(s-\Delta s)e^{-\frac{\Delta s}{T'_\alpha}} + \left\{\frac{\Delta\alpha(s)}{\Delta s} - \frac{\Delta\alpha(s-\Delta s)}{\Delta s}\right\}e^{-\frac{\Delta s}{2T'_\alpha}} \\
&C_N^{\alpha I}(0,M) = 0 \ \ , \ \ D_1(0) = 0
\end{aligned} \tag{4.108}$$

The recursive formulation for the circulatory normal force due to pitch rate is

$$\begin{aligned}
&C_N^{qC}(s,M) = \frac{\pi}{\beta}\left[q(s) - X_3(s) - Y_3(s)\right] \\
&X_3(s) = X_3(s-\Delta s)e^{-b_1\beta^2\Delta s} + A_1\Delta q e^{-b_1\beta^2\frac{\Delta s}{2}} \\
&Y_3(s) = Y_3(s-\Delta s)e^{-b_2\beta^2\Delta s} + A_2\Delta q e^{-b_2\beta^2\frac{\Delta s}{2}} \\
&X_3(0) = Y_3(0) = 0
\end{aligned} \tag{4.109}$$

The recursive formulation for the impulsive normal force due to pitch rate is

$$\begin{aligned}
&C_N^{qI}(s,M) = \frac{T'_q}{M}\left[\frac{\Delta q(s)}{\Delta s} - D_3(s)\right] \\
&D_3(s) = D_3(s-\Delta s)e^{-\frac{\Delta s}{T'_q}} + \left\{\frac{\Delta q(s)}{\Delta s} - \frac{\Delta q(s-\Delta s)}{\Delta s}\right\}e^{-\frac{\Delta s}{2T'_q}} \\
&C_N^{qI}(0,M) = 0 \ \ , \ \ D_3(0) = 0
\end{aligned} \tag{4.110}$$

The circulatory pitching moment due to angle of attack is simple due to the aerodynamic center offset from quarter-chord and is given by

$$C_M^{\alpha C}(s,M) = \left(\frac{1}{4} - x_{ac}\right)C_N^{\alpha C}(s,M) \tag{4.111}$$

The recursive formulation for the circulatory pitching moment due to pitch rate is

$$\begin{aligned}
&C_M^{qC}(s,M) = -\frac{\pi}{8\beta}\left[q(s) - X_2(s)\right] \\
&X_2(s) = X_2(s-\Delta s)e^{-b_5\Delta s} + A_5\Delta q e^{-b_5\beta^2\frac{\Delta s}{2}} \\
&X(0) = 0
\end{aligned} \tag{4.112}$$

The recursive formulation for the impulsive pitching moment due to angle of attack is given by

$$\begin{aligned}
&C_M^{\alpha I}(s,M) = -\frac{A_3 b_3 T'_{m\alpha}}{M}\left[\frac{\Delta\alpha(s)}{\Delta s} - D_4(s)\right] - \frac{A_4 b_4 T'_{m\alpha}}{M}\left[\frac{\Delta\alpha(s)}{\Delta s} - D_5(s)\right] \\
&D_4(s) = D_4(s-\Delta s)e^{-\frac{\Delta s}{b_3 T'_{m\alpha}}} + \left\{\frac{\Delta\alpha(s)}{\Delta s} - \frac{\Delta\alpha(s-\Delta s)}{\Delta s}\right\}e^{-\frac{\Delta s}{2b_3 T'_{m\alpha}}} \\
&D_5(s) = D_5(s-\Delta s)e^{-\frac{\Delta s}{b_4 T'_{m\alpha}}} + \left\{\frac{\Delta\alpha(s)}{\Delta s} - \frac{\Delta\alpha(s-\Delta s)}{\Delta s}\right\}e^{-\frac{\Delta s}{2b_4 T'_{m\alpha}}} \\
&C_N^{\alpha I}(0,M) = 0 \ \ , \ \ D_4(0) = 0 \ \ , \ \ D_5(0) = 0
\end{aligned} \tag{4.113}$$

The recursive formulation for the impulsive pitching moment due to pitch rate is

$$\begin{aligned}
&C_M^{qI}(s,M) = -\frac{7T'_{mq}}{12M}\left[\frac{\Delta q(s)}{\Delta s} - D_6(s)\right] \\
&D_6(s) = D_6(s-\Delta s)e^{-\frac{\Delta s}{T'_{mq}}} + \left\{\frac{\Delta q(s)}{\Delta s} - \frac{\Delta q(s-\Delta s)}{\Delta s}\right\}e^{-\frac{\Delta s}{2T'_{mq}}} \\
&C_N^{qI}(0,M) = 0 \ \ , \ \ D_6(0) = 0
\end{aligned} \tag{4.114}$$

4.4.4 Leishman-Beddoes dynamic stall formulation

The first step is to reconstruct the static airfoil property data, normal force (or lift) and pitching moments using a theoritical model for flow separation over 2D bodies. A theory which models the separated flow regions on 2D bodies is due to Kirchhoff [7, 24]. An airfoil at an angle of attack α, normal force coefficient C_N, and force curve slope 2π for incompressible flow, has a the trailing edge separation point f given by

$$C_N = 2\pi \left(\frac{1+\sqrt{f}}{2} \right)^2 \alpha$$

For real airfoils this can be adapted to

$$C_N = C_0 + C_{n\alpha} \left(\frac{1+\sqrt{f}}{2} \right)^2 \alpha$$

Given the static airfoil properties, f can be calculated at every α. A smooth curve is then fitted through these data points

$$f = \begin{cases} f_1 + f_2 \exp\left(\frac{\alpha-\alpha_1}{S_1}\right) & \text{if } \alpha \leq \alpha_1 \\ f_3 + f_3 \exp\left(\frac{\alpha_1-\alpha}{S_2}\right) & \text{if } \alpha \geq \alpha_1 \end{cases}$$

α_1 is the static angle of attack at which the airfoil stalls. At $\alpha = \alpha_1$, the separation point $f = f_1 + f_2$. For incompressible flow this point often corresponds to $f = 0.7$. In the Leishman-Beddoes model, f is therefore described as

$$f = \begin{cases} 1 - 0.3 \exp\left(\frac{\alpha-\alpha_1}{S_1}\right) & \text{if } \alpha \leq \alpha_1 \\ 0.04 + 0.66 \exp\left(\frac{\alpha_1-\alpha}{S_2}\right) & \text{if } \alpha \geq \alpha_1 \end{cases}$$

S_1, S_2, α_1, and in general the constants f_1, f_2, f_3, f_4 can be determined from static airfoil tables at a given Mach number. The pitching moment about quarter-chord can be constructed as a function of the separation point as

$$C_M = C_{M0} + C_N \left[K_0 + K_1(1-f) + K_2 \sin\left(\pi f^m\right) \right] \tag{4.115}$$

where C_{M0} is the zero lift moment. The constant $K_0 = (0.25 - x_{ac})$ is the aerodynamic center offset from the quarter-chord. K_1 models the effect on the center of pressure due to the growth of the separated flow region. K_2 and m help describe the shape of the moment break at stall. The four constants are to be adjusted to provide the best static moment reconstruction for a particular airfoil.

Consider the circulatory normal force due to angle of attack variation as in eqn.4.107. Writting terms of a current time n we have

$$\begin{aligned} C_{N_n} &= C_{n\alpha} \left[\alpha_n - X_n - Y_n \right] \\ X_n &= X_{n-1} \exp\left(-b_1 \beta^2 \Delta s\right) + A_1 \Delta\alpha_n \exp\left(-b_1 \beta^2 \frac{\Delta s}{2}\right) \\ Y_n &= Y_{n-1} \exp\left(-b_2 \beta^2 \Delta s\right) + A_2 \Delta\alpha_n \exp\left(-b_2 \beta^2 \frac{\Delta s}{2}\right) \end{aligned} \tag{4.116}$$

where $C_N = C_N^{\alpha C}$, the superscript 'αC' is dropped for brevity. $\Delta s = s_n - s_{n-1}$ is the distance, in semi-chords, traversed by the airfoil in $\Delta t = t_n - t_{n-1}$. $\Delta\alpha_n = \alpha_n - \alpha_{n-1}$ is the step change in

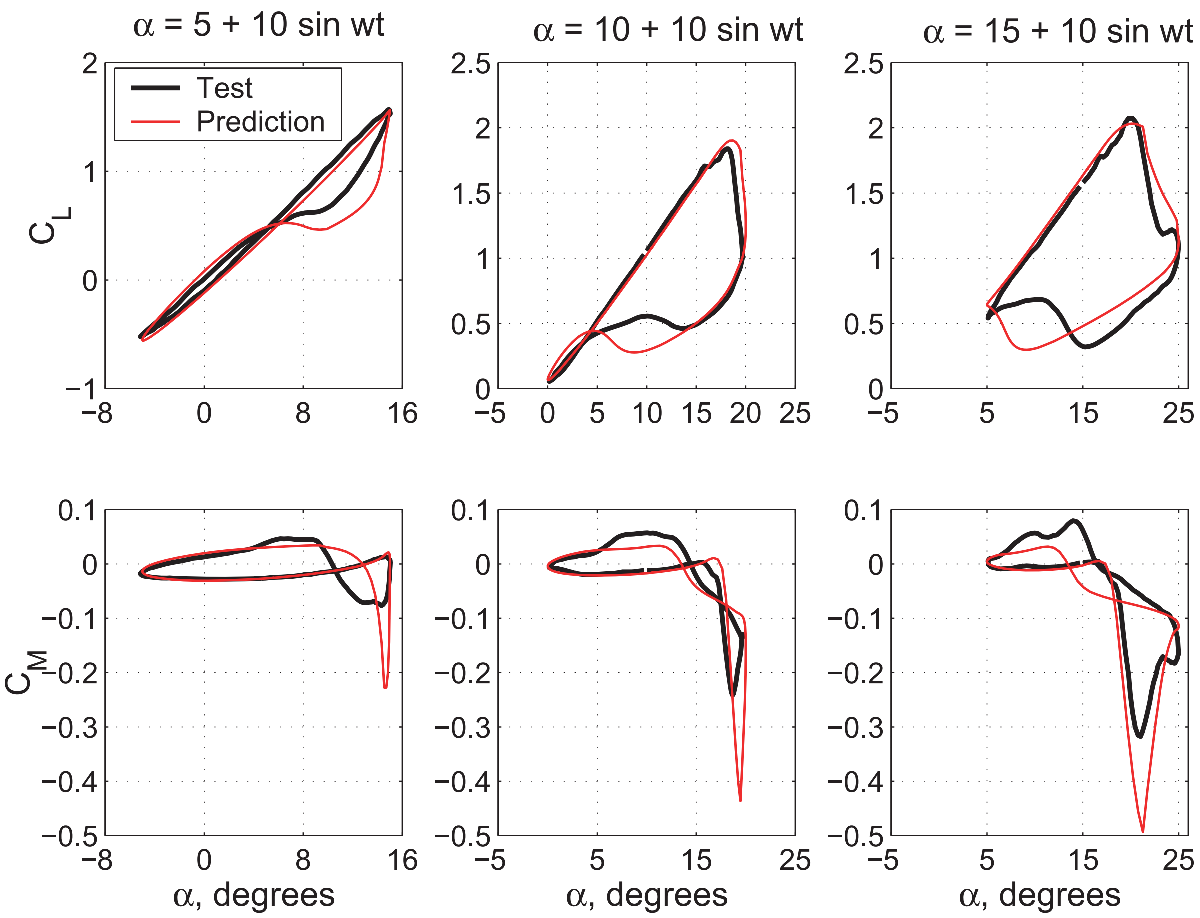


Figure 4.2: **NACA 0012 light and deep dynamic stall cycles: Test data vs. prediction using Leishman-Beddoes model (data from McCroskey et al NASA TM-84245, 1982) Mach No. 0.3, reduced freq. k = 0.1**

angle of attack at step n. Note that the flat plate lift curve slope has been replaced with a general lift curve slope $C_{n\alpha}$.

During unsteady conditions, stall is delayed due to a lag in leading edge pressure response with respect to the normal force. To implement this lag a first order reduction is applied to the circulatory normal force producing a new value

$$C'_{N_n} = C_{N_n} - D_{p_n}$$

where

$$D_{p_n} = D_{p_{n-1}} \exp\left(-\frac{\Delta s}{T_p}\right) + \left(C_{N_n} - C_{N_{n-1}}\right) \exp\left(-\frac{\Delta s}{2T_p}\right)$$

T_p is an empirical constant, a function of Mach number, and determined from unsteady experimental data. The corrected angle of attack is then

$$\alpha_f = \frac{C'_{N_n} - C_0}{C_{n\alpha}}$$

where C_0 is the normal force coefficient at zero angle of attack. The corrected angle of attack is then used to determine the effective separation point on the airfoil, f', from the static f versus α

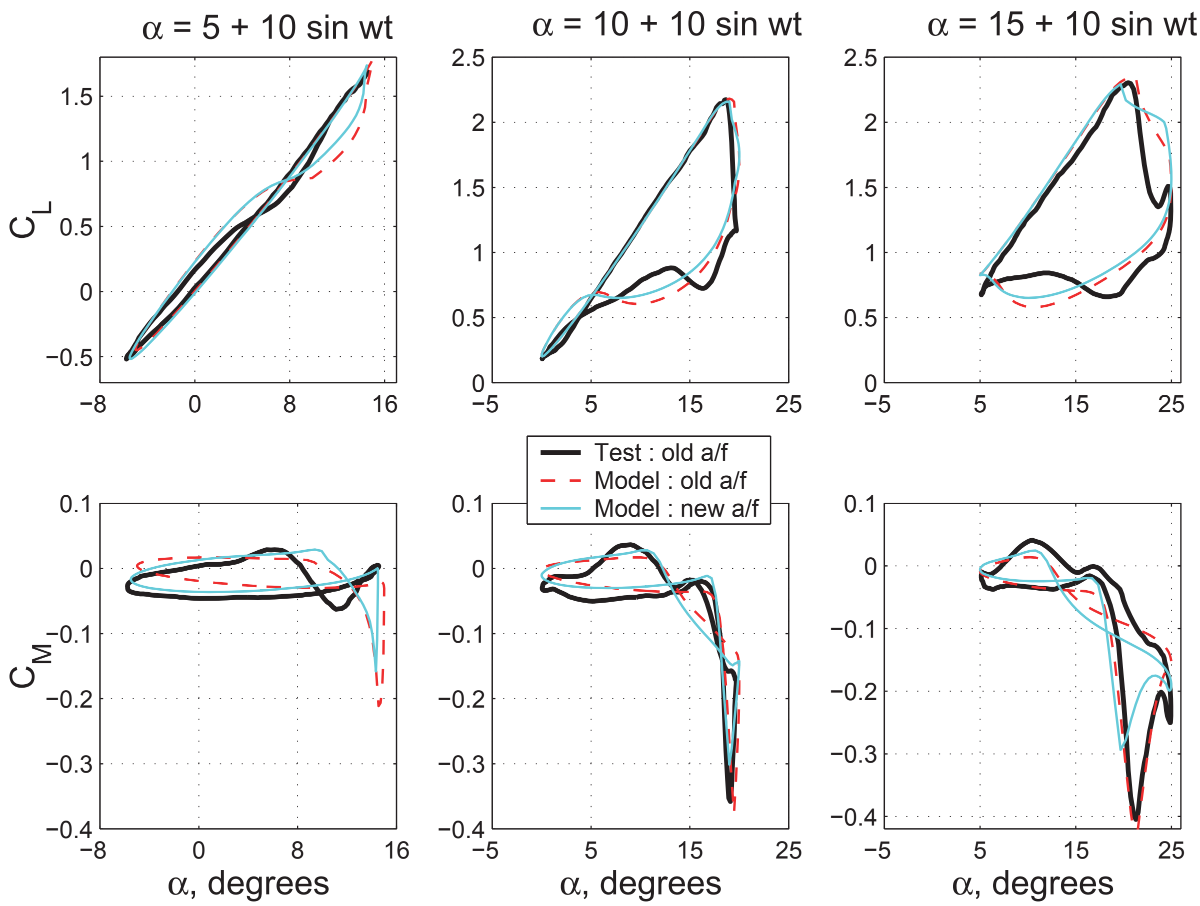


Figure 4.3: **SC-1095 light and deep dynamic stall cycles: Test data vs. prediction using Leishman-Beddoes model (data from McCroskey et al NASA TM-84245, 1982) Mach No. 0.3, reduced freq. k = 0.1; Old airfoil data is static data from McCroskey report, New airfoil data is a refined version from U.S.Army**

relationship given above. The additional effect of unsteady boundary layer response is incorporated using a first order lag

$$f''_n = f'_n - D_{f_n}$$

where

$$D_{f_n} = D_{f_{n-1}} \exp\left(-\frac{\Delta s}{T_f}\right) + \left(f'_n - f'_{n-1}\right) \exp\left(-\frac{\Delta s}{2T_f}\right)$$

T_f is an empirical constant, a function of Mach number. Can be determined from unsteady data or an unsteady boundary layer analysis, in absence of data. Once the separation parameter has been determined, the normal force can then be determined as

$$C_{N_n} = C_{n\alpha}\left(\frac{1+\sqrt{f''_n}}{2}\right)^2 \alpha_f = C^C_{N_n}$$

where $C_{n\alpha}$ is the lift curve slope, a function of Mach number. The pitching moment is given by

$$C_{M_n} = C_{M0} + C^C_{N_n}\left[K_0 + K_1(1 - f''_n) + K_2 \sin\left(\pi f''^m_n\right)\right] \tag{4.117}$$

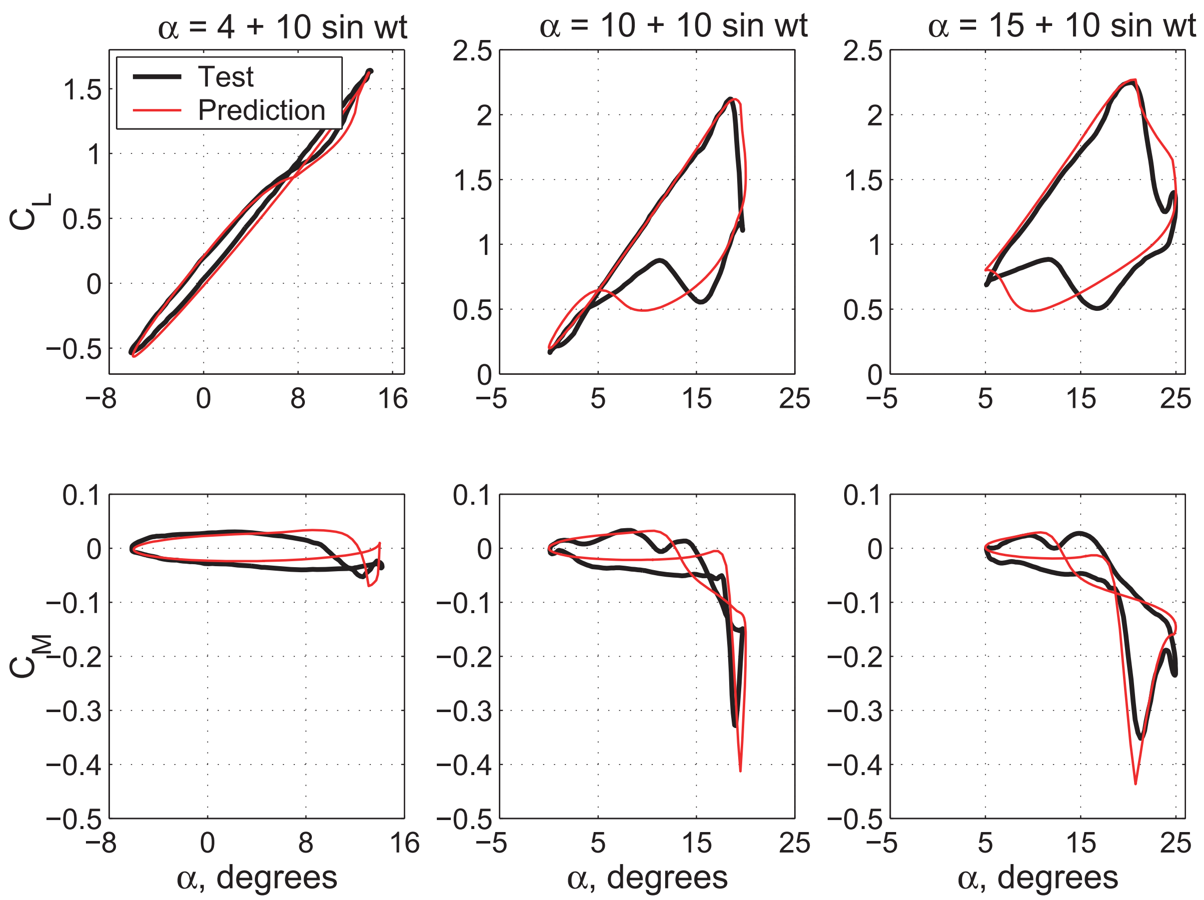


Figure 4.4: **Hughes HH-02 light and deep dynamic stall cycles: Test data vs. prediction using Leishman-Beddoes model (data from McCroskey et al NASA TM-84245, 1982) Mach No. 0.3, reduced freq. k = 0.1**

Note that, here C_M corresponds to $C_N^{\alpha C}$, the circulatory pitching moment due to angle of attack variation. The contributions of the pitch rate terms and the impulsive terms will be added later on. As the airfoil gradually pitches up, the separation point f progressively advances towards the leading edge. At the same time, a leading edge vortex is formed, gradually growing in strength. The gradual growth in its strength can be viewed as caused by an accumulation of circulation, such that, the lift induced by this gradually growing vortex accounts for the difference between the normal force given by the Kirchhoff approximation above, C_{N_n}, and a hypothetical normal force that would result if there was no separation, i.e. corresponding to $f''_n = 1$. Thus the incremental vortex lift at a time step n is given by

$$C_{V_n} = C_{N_n} - C_{n\alpha}\alpha_f = C_{n\alpha}\left(\frac{1+\sqrt{f''_n}}{2}\right)^2 \alpha_f - C_{n\alpha}\alpha_f \tag{4.118}$$

The total vortex lift, C_N^v, results from the cummulative addition of the above increments along with a simultaneous mechanism for decay.

$$C^v_{N_n} = C^v_{N_{n-1}} \exp\left(-\frac{\Delta s}{T_v}\right) + \left(C^v_n - C^v_{n-1}\right) \exp\left(-\frac{\Delta s}{2T_v}\right) \tag{4.119}$$

where T_v is another empirical constant. When a leading edge separation is triggered, the vortex lift is added to the normal force as long as the vortex traverses the chord and has not been washed aft of the trailing edge. The condition of leading edge separation, and the duration of the vortex passage over the chord are set by empirical means. The condition for leading edge separation is when C_{N_n} exceeds the value corresponding to static stall. This value is a function of Mach number and denoted by C_{N_1} in the model. At this point the accumulated vortex is assumed to start to convect over the airfoil chord. The rate of convection has been experimentally determined to be less than half of the free stream velocity. During the vortex convection, the vortex lift evolves according to eqns.4.118 and 4.119, i.e. the total vortex lift $C^v_{N_n}$ is allowed to decay exponentially with time while being constantly updated by a new increment. The duration of vortex passage, in terms of nondimensional time τ_v (distance travelled by the airfoil in semi-chords), is from $\tau_v = 0$ to $\tau_v = \tau_{vl}$. At $\tau_v = \tau_{vl}$, the vortex leaves the trailing edge. The center of pressure movement behind quarter-chord due to the vortex movement is determined empirically to be

$$C^v_P = 0.20 \left[1 - \cos\left(\frac{\pi \tau_v}{T_{vl}} \right) \right] \tag{4.120}$$

The pitching moment contribution of the moving center of pressure is then simply

$$C^v_{M_n} = -C^v_P \, C^v_{N_n} \tag{4.121}$$

The final normal force and pitching moment expressions at a given time step n is then

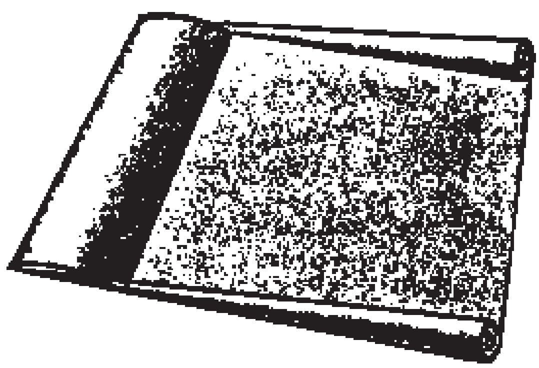
$$\begin{aligned} C_N &= C_0 + \underline{C_N^{\alpha C}} + C^v_N + C^{\alpha I}_N + C^{qC}_N + C^{qI}_N \\ C_M &= C_{M0} + \underline{C_M^{\alpha C}} + C^v_M + C^{\alpha I}_M + C^{qC}_M + C^{qI}_M \end{aligned}$$

The seperated flow model is embedded in the underlined terms. The dynamic stall effects are in C^v_N and C^v_M. Dynamic stall cycles for an oscillating 2D airfoil are shown in Figs.4.2, 4.3 and 4.4.

4.5 Wing Models

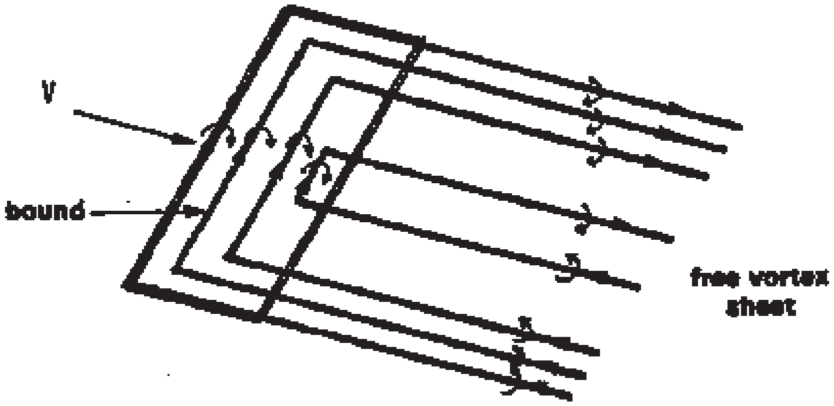
4.5.1 Prandtl Lifting Line Theory

Associated with the lift on the wing, there is a circulation around the wing. At the tip, the lift is zero and therefore the circulation must vanish at the tip. This means circulation varies along the span. Whenever there is a variation of circulation spanwise, there has to be shedding of vorticity. If there is a continuous variation of circulation along the wing span, a continuous sheet of trailing vortices must proceed from the wing.



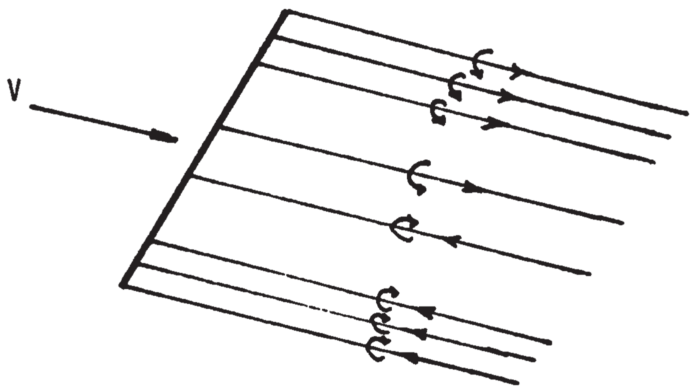
If we assume that the circulation is uniform along the span and drops to zero at the tips of the wing, then one can consider a simple model of two concentrated vorticity filaments originated at the wing tips. this concept of two tip vortices was originated by Lanchester. This model gives a good global picture but is not appropriate for analyzing flow near the wing.

A better model is to consider a continuous trailing vortex sheet distribution, as proposed by Prandtl.



The vortex sheet on the top and bottom surface is called the bound vortex sheet. Across the bound vortex sheet, a pressure difference may exist. The trailing edge sheet is called the free vortex sheet and no pressure difference exists across the sheet. The shed vortices are pulled downstream by the wind. If there is no new shedding, the old one will not have any influence on the airfoil.

For large aspect ratio wings, the bound part of the vortex sheet may be approximated by a single bound vortex line of varying strength. This is called the Prandtl lifting line theory.



For steady flow, influence on the shedding vorticity sheet on lift is negligible. For a body in motion, the lift is changing with time and so there is a continuous shedding of vorticity. The vorticity which is close to the surface plays an important role for the calculation of unsteady pressure on the surface.

4.5.2 Weissinger-L Lifting-surface Theory

The W-L model [25] is essentially a lifting-surface model with only one chord-wise element. The W-L model represents blade lift using a series of spanwise horseshoe vortex elements. The bound circulation is located at the 1/4-chord point. The flow tangency condition is imposed at the 3/4-chord point. Compared to a lifting line model, the W-L model predicts improved loading for fixed wings with arbitrary planforms.

Let the blade be divided into N aerodynamic segments. For the $i-th$ segment the flow tangency can be written as

$$\begin{aligned} V_{b_i} &= V_{\infty_i}\alpha_{e_i} \\ &= V_{\infty_i}(\alpha_i - \phi_{NW_i}) \\ &= V_{\infty_i}\alpha_i - V_{NW_i} \end{aligned} \tag{4.122}$$

where V_{b_i} is the bound vortex induced velocity at the $i-th$ control point and V_{∞_i} is the incident free stream velocity at the control point. α_{e_i} is the *effective angle of attack* at the section. The *effective angle of attack* is obtained by subtracting the near wake *induced angle of attack* from the *input angle of attack*. The later includes the effect of blade deformation and far wake inflow. V_{NW_i} is the velocity induced by the nearwake at the $i-th$ control point.

The velocities V_{b_i} and V_{NW_i} are related to the strength of the bound vortices, Γ_i through influence coefficient matrices. These matrices depend both on the blade deformations and on the blade geometry e.g., rigid twist, control angles, planform, sweep etc.

$$V_{b_i} = \sum_{j=1}^{N} I_{b_{i,j}}\Gamma_j \tag{4.123}$$

$$V_{NW_i} = \sum_{j=1}^{N} I_{NW_{i,j}}\Gamma_j \tag{4.124}$$

The linear algebraic governing equations for bound circulation (N equations, N unknowns) are thus obtained as

$$\sum_{j=1}^{N} \left\{ I_{b_{i,j}} + I_{NW_{i,j}} \right\} \Gamma_j = V_{\infty_i}\alpha_i \tag{4.125}$$

Once the bound circulation strengths, Γ_j are known they are used to calculate α_{e_i} using equations (4.124) and (4.122). Assuming thin airfoil theory, i.e., with a lift curve slope of 2π, the local lift coefficient simply becomes

$$\begin{aligned} C_l &= 2\pi\alpha_{e_i} \\ &= \frac{2\pi}{V_{\infty_i}} \sum_{j=1}^{N} I_{b_{i,j}} \end{aligned} \tag{4.126}$$

using equations (4.122) and (8.34). This is the effective angle of attack approach and is consistent with K-J theorem for 3D wings which gives

$$C_l = \frac{2\pi}{V_{\infty_i}} \sum_{j=1}^{N} I_{b_{i,j}} \tag{4.127}$$

In the present analysis, the effective angle of attack approach is used.

The radial distribution of *input angle of attack* is influenced by the far wake (rotor inflow) which in turn is governed by the bound circulation strengths. Therefore, iterations are performed between far wake and near wake until bound circulation strengths are converged. The iterations are started with a uniform inflow far wake (based on helicopter gross weight) which is subsequently replaced with non-uniform inflow.

Within the W-L near wake model, the airfoil property tables are included using the following method. The *input angle of attack* is scaled to an equivalent *flat plate angle of attack* using the lift coefficients obtained from the airfoil tables. This scaled angle of attack is used by the W-L model to calculate bound circulation strengths at 1/4-chord locations. The bound circulation strengths are then used to calculate the circulation strengths of near wake trailers. The near wake trailers are used to estimate the *induced angle of attack* at 3/4-chord locations. This induced angle of attack is subtracted from the *input angle of attack* and the resulting *effective angle of attack* is used to obtain lift (also pitching moment and drag) from the airfoil tables.

4.5.3 Unsteady Lifting-Line Analysis

An unsteady lifting-line model can be constructed using a consistent combination of the following parts: (1) A near wake model, e.g. a W-L type lifting surface model, (2) A far wake model, with free or prescribed wake geometries (3) 2D airfoil properties and (4) An unsteady aerodynamic model for attached and seperated flow flow.

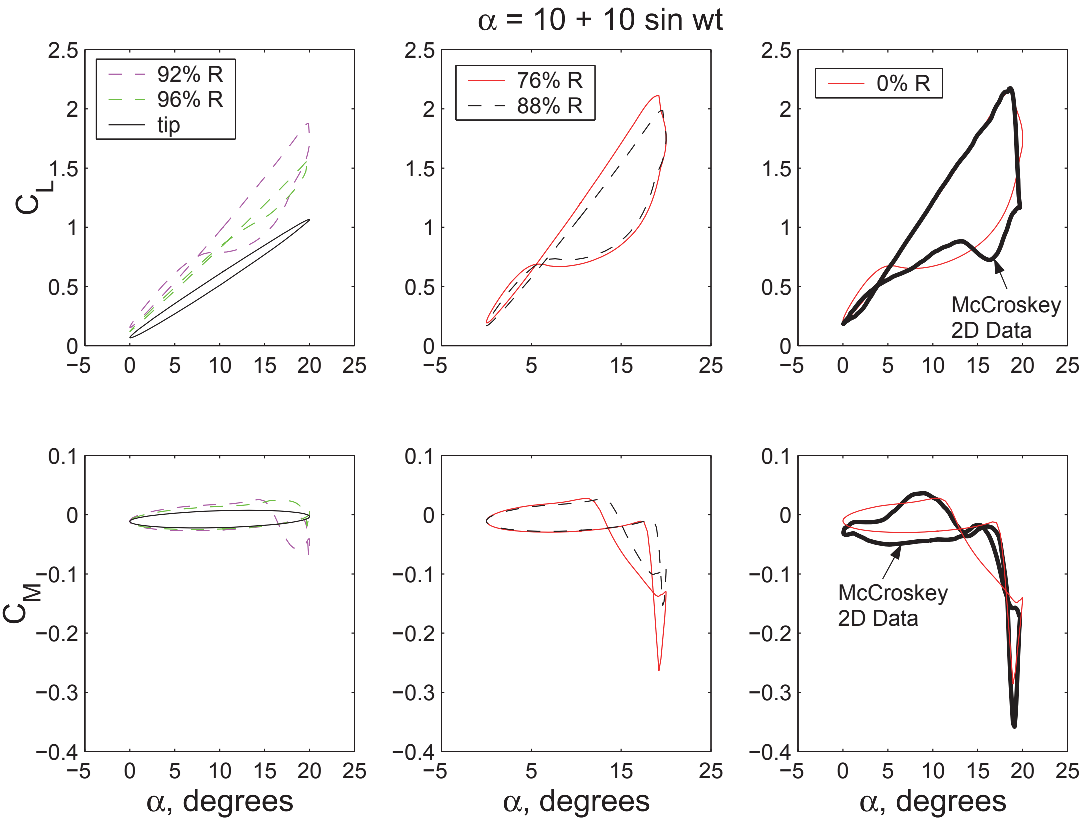


Figure 4.5: **Dynamic stall prediction on a hypothetical 3D wing of Aspect ratio 15.30 with SC-1095 airfoils; Weissinger-L and Leishman Beddoes model; Inboard predictions (almost 2D) are compared with 2D airfoil data from McCroskey et al NASA TM-84245, 1982, Mach No. 0.3, reduced freq. k = 0.1**

For a prescribed set of deformations, the airloads were calculated using the following three steps. In the first step, the blade deformations and an initial inflow distribution, for example, a uniform inflow based on the measured thrust, were used to calculate the sectional angle of attack. The sectional angle of attack was used as input to the $W - L$ near wake model, which then calculates the spanwise bound circulation distribution. The bound circulation distribution is calculated iteratively, so that it is consistent with the airfoil properties, the Kutta condition, and the

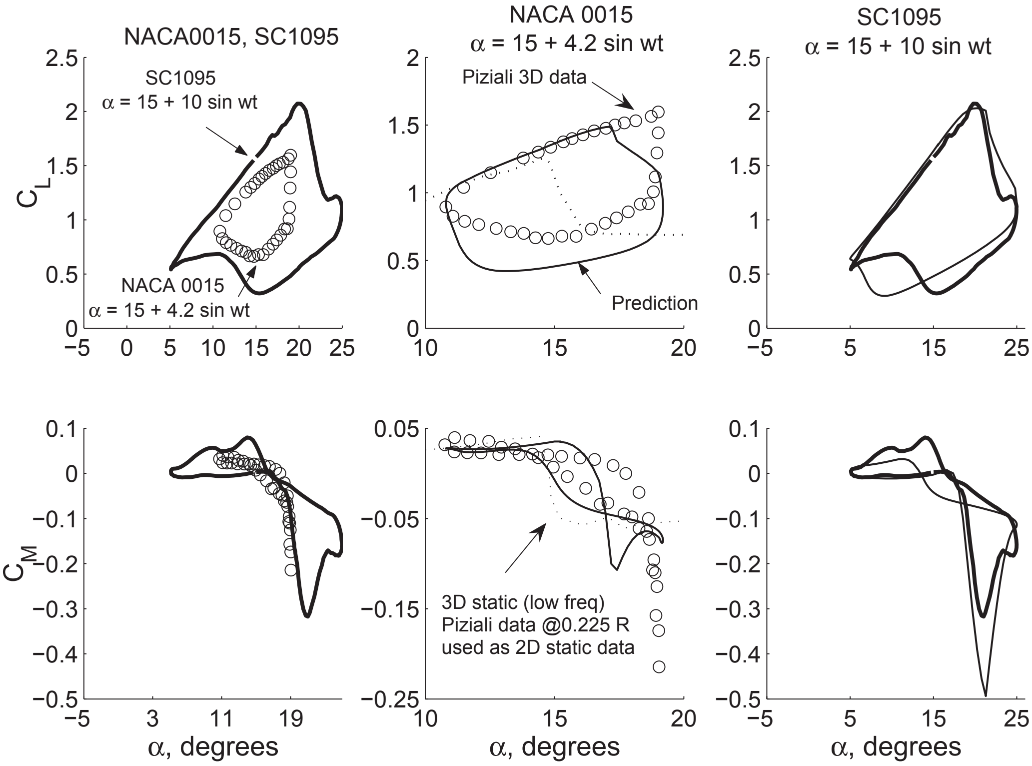


Figure 4.6: **Dynamic stall prediction on a 3D wing with NACA 0015 airfoils, Mach No. 0.3, reduced freq. k = 0.1 Piziali test data compared with McCroskey SC1095 test, (prediction for Piziali test case uses 2D static data sent to UMD on Aug. 1991)**

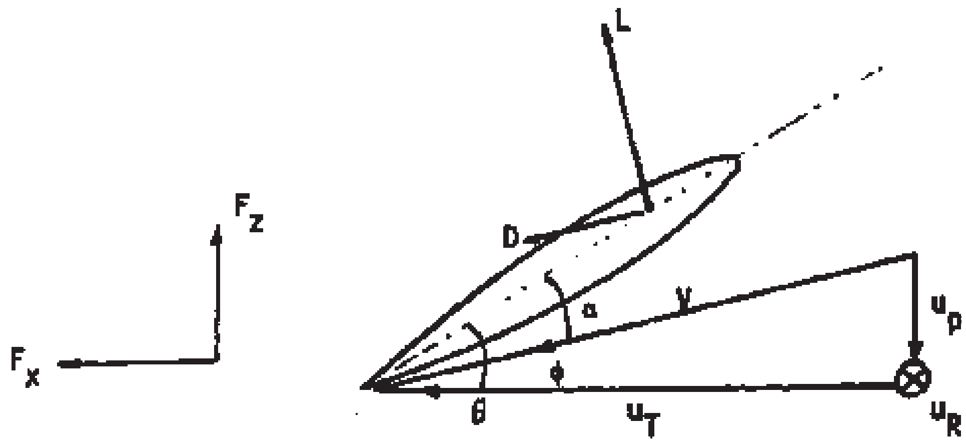
near-wake trailer sheet. This procedure is described later. In the second step, the bound circulation strengths were used to calculate the rotor far wake (free or prescribed). The far-wake generates a refined non-uniform inflow distribution. Using the non-uniform inflow, the sectional angles of attack are recalculated. In the third step, the new angles of attack are used as input to the near wake model to recalculate the bound circulation strengths. Steps one to three are repeated untill the airloads converge. Iterations are required because the bound circulation strengths calculated by the near wake model changes the far wake inflow which changes the input angle of attack distribution of the near wake model.

Within the $W - L$ model, the bound circulation strengths are obtained iteratively. First, the input angle of attack and incident Mach number are used to obtain the spanwise lift distribution from the airfoil tables. The bound circulation is obtained using the Kutta condition. Next, a near wake trailer sheet is layed out over thirty degree azimuth following the blade; 25 blade segments are used. The bound circulation line is at local $1/4-$chord and swept back at the tip. The trailer sheet follows the local incident velocity. It is allowed to trail in the reverse direction in the regions of reverse flow. The velocity induced by the trailer sheet at the local $3/4-$chord is then used to reduce the input angle of attack to an effective angle of attack. The effective angle of attack is then used to update the bound circulations using airfoil properties and the Kutta condition. The steps are repeated until the bound circulation converges. A relaxation scheme is necessary for converging the bound circulation strengths (10% used). The converged bound circulation strengths are consistent with the near wake trailers and the airfoil properties.

The W-L model can be combined with the 2D static airfoil property data and a 2D unsteady dynamic stall model. The stall model is applied at each section on the effective angle of attack distributions. Sample test data and predictions are shown in Figs.4.5 and 4.6.

4.6 Perturbation Aerodynamic Forces

The quasisteady blade element theory can used to obtain the perturbation aerodynamic forces.



u_T = tangential flow velocity, ft/sec
u_p = normal flow velocity, ft/sec
u_R = radial flow velocity acting radially outward, ft/sec
θ = pitch, rad
α = angle of attack, $\theta - \phi$, rad
ϕ = induced angle, rad
ρ = air density, slug/ft^3, lb-sec^2/ft^4
c = chord, ft
v = resultant velocity, $\sqrt{u_p^2 + u_T^2}$, ft/sec

The blade lift and drag forces per unit length

$$L = 1/2 \rho v^2 c \, c_l$$

$$D = 1/2 \rho v^2 c \, c_d$$

The moment about aerodynamic center

$$M_{ac} = 1/2 \rho v^2 c^2 c_{m_{ac}}$$

Resolving forces in hub plane

$$F_z = L \cos\phi - D \sin\phi$$

$$F_z = L \sin\phi + D \cos\phi$$

Moment about elastic axis

$$M_{ea} = 1/2 \rho v^2 c^2 c_m - L x_a$$

where x_a is the chordwise offset of aerodynamic center from elastic axis (+ ve aft). The radial force can be important for forward flight and it consists of two components; drag force due to radial velocity, and resolved component of vertical force in the radial direction.

$$F_r = D\frac{u_R}{v} - F_z\frac{dW}{dr}$$

Note

$$\sin\phi = \frac{u_p}{v}$$

$$\cos\phi = \frac{u_T}{v}$$

Thus

$$F_z = \frac{1}{2}\rho c(c_l u_T v - c_d u_p v)$$

$$F_x = \frac{1}{2}\rho c(c_l u_p v + c_d u_T v)$$

$$F_r = \frac{1}{2}\rho c(c_d v u_R) - F_z\frac{dw}{dr}$$

$$M_a = \frac{1}{2}\rho c^2(c_{m_{ac}} - \frac{x_a}{c}c_l)v^2$$

These are the forces per unit span. These forces contain blade motion and thus these are the motion dependent aerodynamic forces.

To make analysis simple, the flow components are broken into two parts, steady and perturbation components.

$$u_T = (u_T)_{\text{trim}} + \delta u_T$$

$$u_p = (u_p)_{\text{trim}} + \delta u_p$$

$$u_R = (u_R)_{\text{trim}} + \delta u_R$$

$$\theta = \theta_{\text{trim}} + \delta\theta$$

The trim or steady components are due to the operating condition of the rotor and the perturbed components are caused by the perturbed motion. Similarly, the forces are also expressed into two parts, trim and perturbation components.

$$F_z = (F_z)_{\text{trim}} + \delta F_z$$

$$F_x = (F_x)_{\text{trim}} + \delta F_x$$

$$F_r = (F_r)_{\text{trim}} + \delta F_r$$

$$M_{ea} = (M_{ea})_{\text{trim}} + \delta M_{ea}$$

for convenience, the trim word is omitted from flow components.

<u>Trim Forces</u>

$$(F_z)_{\text{trim}} = \frac{1}{2}\rho c(c_l u_T v - c_d u_p v)$$

$$(F_x)_{\text{trim}} = \frac{1}{2}\rho c(c_l u_p v + c_d u_T v)$$

$$(F_r)_{\text{trim}} = \frac{1}{2}\rho c c_d u_R v - F_z \frac{dw}{dr}$$

$$(M_a)_{\text{trim}} = \frac{1}{2}\rho c^2 (c_m - \frac{x_a}{c} c_l) v^2$$

In the above expressions, the aerodynamic coefficients are obtained for trim flight.

Perturbations

Let us first examine the perturbation of resultant velocity v and pitch θ.

$$\delta v = \delta(u_p^2 + u_T^2)^{1/2}$$

$$= \frac{u_p \delta u_p + u_T \delta u_T}{v}$$

$$\delta\alpha = \delta(\theta - \tan^{-1}\frac{u_p}{u_T}) \simeq \delta(\theta - \frac{u_p}{u_T})$$

$$= \delta\theta + \frac{u_P \delta u_T - u_T \delta u_p}{u_T}$$

The aerodynamic coefficients are the functions of the angle of attack and Mach number.

$$c_l = c_l(\alpha, M)$$

$$c_d = c_d(\alpha, M)$$

$$c_m = c_m(\alpha, M)$$

The perturbation in aerodynamic coefficients are

$$\delta c_l = \frac{\partial c_l}{\partial \alpha}\delta\alpha + \frac{\partial c_l}{\partial M}\delta M$$

$$= c_{l_\alpha}\delta\alpha + c_{l_M}\delta M$$

$$\delta c_d = c_{d_\alpha}\delta\alpha + c_{d_M}\delta M$$

$$\delta c_m = c_{m_\alpha}\delta\alpha + c_{m_M}\delta M$$

The Mach number at any radial station is

$$M = \frac{M_{tip}}{v_{tip}} v$$

$$= M_{tip}\frac{v}{\Omega R}$$

and the perturbation in the Mach number is

$$\delta M = \frac{M_{tip}}{\Omega R}\delta v$$

Let us now look at the perturbation in forces

$$F_z = \frac{1}{2}\rho c\{c_l u_T v - c_d u_p v\}$$

$$\delta F_z = \frac{1}{2}\rho c\{\delta c_l u_T v + c_l \delta u_T v + c_l u_T \delta v$$

$$-\delta c_d u_p v - c_d \delta u_p v - c_d u_p \delta v\}$$

$$= \frac{1}{2}\rho c\{(c_{l_\alpha}\delta\alpha + c_{l_M}\delta M)u_T v + c_l \delta u_T v + c_l u_T \delta v$$
$$-(c_{d_\alpha}\delta\alpha + c_{d_M}\delta M)u_p v - c_d \delta u_p v - c_d u_p \delta v\}$$

$$\frac{\delta F_z}{\frac{1}{2}\rho c} =$$
$$\delta u_T\{\frac{u_p}{v}(u_T c_{l_\alpha} - u_p c_{d_\alpha}) + \frac{u_T^2}{2}(c_l + M c_{l_M}) + c_l v$$
$$-(c_d + M c_{d_M})\frac{u_p u_T}{v} + \delta u_p\{-\frac{u_T}{v}(u_T c_{l_\alpha} - u_p c_{d_\alpha}) + \frac{u_p u_T}{v}(c_l + M c_{l_M})$$
$$-c_D v \frac{u_p^2}{v}(c_d + M c_{d_M})\} + \delta\theta\{c_{l_\alpha} v u_T - c_{d_\alpha} v u_p\}$$

Similarly

$$\begin{aligned}
\frac{\delta F_x}{\frac{1}{2}\rho c} &= \delta u_T[\frac{u_p}{v}(u_p c_{l_\alpha} + u_T c_{d_\alpha}) + \frac{u_T^2}{v}(c_d + M c_{d_M}) \\
&+ c_d v + (c_l + M c_{l_M})\frac{u_p u_T}{v}\} \\
&+ \delta u_p\{-\frac{u_T}{v}(u_p c_{l_\alpha} + u_T c_{d_\alpha}) + \frac{u_p u_T}{v}(c_d + m c_{d_M}) \\
&+ c_l v + \frac{u_p^2}{v}(c_l + M c_{l_M})\} \\
&\delta\theta\{c_{l_\alpha} v u_p + c_{d_\alpha} v u_T\}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta F_r}{\frac{1}{2}\rho c} &= \delta u_T\{c_{d_\alpha}\frac{u_p u_R}{v} + \frac{u_T u_R}{v}(c_d + m c_{d_M}) \\
&+ \delta u_p\{-c_{d_\alpha}\frac{u_p u_R}{v} + \frac{u_p u_R}{v}(c_d + M c_{d_M})\} \\
&+ \delta u_R\{c_d v\} \\
&\delta\theta\{c_{d_\alpha} v u_R\}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta M_{ea}}{\frac{1}{2}\rho c^2} &= \delta u_T\{2u_T(c_m - c_l\frac{x_a}{c}) + v^2\frac{u_p}{u_T^2}(c_{m_\alpha} - c_{l_\alpha}\frac{x_a}{c}) \\
&+ \mu_T(c_{m_M} - c_{l_M}\frac{x_a}{c})\} \\
&+ \delta u_p\{2u_p(c_m - c_l\frac{x_a}{c}) - \frac{v^2}{u_T}(c_{m_\alpha} - c_{l_\alpha}\frac{x_a}{c}) \\
&+ M u_p(c_{m_M} - c_{l_M}\frac{x_a}{c})\} \\
&+ \delta\theta\{v^2(c_{m_\alpha} - c_{l_\alpha}\frac{x_a}{c})\}
\end{aligned}$$

To these perturbation forces, the noncirculating forces also are added. The most important component is the virtual moment.

$$\delta M_{ea} = (\delta M_{ea})_c + M_{nc}$$

$$\begin{aligned}
M_{NC} &= \text{noncirculatory moment} \\
&= \frac{1}{4}\pi\rho\Omega^2 c^3\left[r\left(\frac{1}{4} + \frac{x_a}{c}\right)\ddot{\beta} - r\left(\frac{1}{2} + \frac{x_a}{c}\right)\dot{\theta} - c\left(\frac{3}{32} + \frac{1}{2}\frac{x_a}{c}\right)\ddot{\theta}\right]
\end{aligned}$$

Small Angle Simplification

Assume small angles

$$F_z \simeq L$$

$$F_x \simeq L\frac{u_p}{u_T} + D$$

$$\alpha \simeq \theta - \frac{u_p}{u_T}$$

Assume simplified airfoil characteristics

$$c_l = a\alpha \text{ (symmetric airfoil)}$$

$$= a(\theta - \frac{u_p}{u_T})$$

$$c_d = c_{d0}$$

$$c_m = c_{m0} \text{ (for symmetric it is zero)}$$

$$v = u_T$$

Trim Forces

$$F_z = \frac{1}{2}\rho ca(u_T^2\theta - u_p u_T)$$

$$F_x = \frac{1}{2}\rho ca(u_p u_T \theta - u_p^2 + \frac{c_d}{a}u_T^2)$$

$$M_a = \frac{1}{2}\rho c^2 a\{\frac{c_m}{a}(u_p^2 + u_T^2) - \frac{x_a}{c}(u_T^2\theta - u_p u_T)\}$$

Perturbation Forces

$$\delta F_z = \frac{1}{2}\rho ca\{\delta u_T(2u_T\theta - u_p) + \delta u_p(-u_T) + \delta\theta(u_T^2)\}$$

$$\delta F_x = \frac{1}{2}\rho ca\{\delta u_T(u_p\theta + \frac{c_d}{a}2u_T) + \delta u_p(u_T\theta - 2u_p) + \delta\theta(u_p u_T)\}$$

$$\delta M_a = \frac{1}{2}\rho c^2 a\{\delta u_T(2\frac{c_m}{a}u_T + \frac{x_a}{c}u_p - 2\frac{x_a}{c}u_T\theta)$$

$$+\delta u_p(2\frac{c_{m_a}}{a}u_p + \frac{x_a}{c}u_T) + \delta\theta(-\frac{x_a}{c}u_T^2)\}$$

Example 4.1:

In a circulation-controlled rotor, the aerodynamic lift is a function of geometric angle as well as blowing

$$c_l = c_l(\alpha, c_\mu)$$

where

$$c_\mu = \frac{\dot{m}V_j}{\frac{1}{2}\rho V^2 c} \quad (\dot{m}V_j = \text{ jet momentum})$$

calculate the perturbation in lift in terms of flow components of u_p and u_T and pitch angle θ (steady and perturbations).

Lift $L = 1/2\rho V^2 c\, c_l(\alpha, c_\mu)$

Perturbation $\delta L = \rho c\, c_l V\, \delta V + 1/2\rho c V^2 \delta c_l$

$$V = \sqrt{u_p^2 + u_T^2}$$

$$\delta V = \frac{u_p \delta u_p + u_T \delta u_T}{V}$$

$$\delta c_l = c_{l_a}\delta\alpha + c_{l_\mu}\delta c_\mu$$

$$\delta c_\mu = -2c_\mu \frac{\delta V}{V}$$

$$\alpha = \theta - \tan^{-1}\frac{u_p}{u_T}$$

$$\delta\alpha = \delta\theta - \frac{u_T\delta u_p - u_p\delta u_T}{u_T^2}$$

$$\frac{\delta L}{1/2\rho c} = \delta u_T\{2u_T c_l + \frac{V^2}{u_T^2}u_p c_{l_\alpha} - 2u_T c_\mu c_{l_\mu}\}$$

$$+\delta u_p\{2u_p c_l - \frac{V^2}{u_T^2}u_T c_{l_\alpha} - 2u_p c_\mu c_{l_\mu}\}$$

$$+\delta\theta\{V^2 c_{l_\alpha}\}$$

Example 4.2:

For an articulated rotor in hovering flight, obtain the blade flapping equation under varying pitch conditions. For unsteady aerodynamic forces, use the lift deficiency function of a typical section at 75% radius position. Assume a 6% hinge offset, and the elastic axis at mid-chord position.

$$\overset{**}{\beta} + \nu_\beta^2\beta - \frac{3}{2}\frac{x_I}{R}(\overset{**}{\theta} + \theta) = \gamma\overline{M}_\beta$$

$$\overline{M}_\beta = \frac{1}{2}\int_0^1 x[\frac{\delta u_T}{\Omega R}(2\frac{u_T}{\Omega R}\theta - \frac{u_p}{\Omega R}) + \frac{\delta u_p}{\Omega R}(-\frac{u_T}{\Omega R}) + \delta\theta(\frac{u_T}{\Omega R})^2]C(k))dx$$

$$+\int_0^1 x\, L_{NC} dx$$

$$\frac{u_T}{\Omega R} = x,\ \frac{u_p}{\Omega R} = \lambda,\ \frac{\delta u_T}{\Omega R} = 0,\ \frac{\delta u_p}{\Omega R} = x\dot{\beta} - \frac{1}{4}\frac{c}{R}\overset{*}{\theta}$$

$$\delta\theta = \theta$$

$$\overline{k} = \frac{\omega b}{U} = \frac{1}{.75}\frac{\omega}{\Omega}\frac{c}{R}$$

$$L_{NC} = \frac{\pi\rho b^2}{\rho a c\Omega^2 R^4}[(u_T\theta - u_p) - b a_h(\theta + \Omega\beta)]$$

$$\overline{M}_\beta = C(\overline{k})[-\frac{1}{8}\overset{*}{\beta} + \frac{1}{24}\frac{c}{R}\overset{*}{\theta} + \frac{1}{8}\theta] + \frac{c}{R}(\frac{\overset{*}{\theta}}{24} - \frac{\overset{**}{\beta}}{24}) - \frac{1}{64}(\frac{c}{R})^2\overset{**}{\theta}$$

4.7 Dynamic Inflow Models

Typical stability analyses normally employ steady wake induced inflow calculated from simple momentum theory. However, under unsteady flow conditions, the rotor wake will not be steady, and this will naturally result in unsteady induced inflow, called as dynamic inflow. The dynamic inflow may be a significant factor in the calculation of unsteady aerodynamic loads, and hence can have an important influence on the rotor dynamics. In fact, the dynamic inflow components should be related to the unsteady rotor loads (thrust, roll moment and pitch moment). These relationships are complex and are still subject of research. For analyses, it is important to put these relationships in simplified form. One possible way to derive these relationships is by using unsteady actuator disk theory. Let us examine the steady as well as dynamic inflow components for hover and forward flight.

4.7.1 Hover

A simple steady inflow model for hover is to assume uniform inflow over the rotor disk. Using simple momentum theory the inflow is related to the rotor thrust.

$$\lambda = \text{sign}(C_T) k_p \sqrt{\left|\frac{c_T}{2}\right|}$$

where C_T is the thrust coefficient, λ us induced inflow ($v_i/\Omega R$) and k_p is an empirical factor to cover tip losses (~ 1.15).

A simple dynamic inflow model for hover is

$$\tau\dot{\lambda} + \lambda = \text{sign}(C_T) k_p \sqrt{\left|\frac{C_T}{2}\right|}$$

or

$$\tau\Delta\dot{\lambda} + \Delta\lambda = k_p^2 \frac{\Delta C_T}{4\lambda_0}$$

where τ is time lag in seconds and can be approximately taken as $.85/4\lambda_0\Omega$. The λ_0 is the mean induced inflow and Ω is rotational speed (rad/sec). Note that the C_T here consists of total thrust, i.e., the sum of steady and perturbation thrust components.

4.7.2 Forward Flight

A simple steady inflow model for forward flight is to assume it uniform over the rotor disk.

$$\lambda_i = \frac{1}{2} \frac{C_T}{\sqrt{\mu^2 + \lambda^2}}$$

where λ_i is induced inflow ratio and μ is advance ratio and λ is inflow ratio

$$\lambda = \mu \tan\alpha + \lambda_i$$

and α is disk tilt to the free stream. An improvement over the simple uniform model is to assume a linear variation for steady induced inflow

$$\lambda_i = \lambda_m \left(1 + \kappa_x \frac{r}{R} \cos\psi + \kappa_y \frac{r}{R} \sin\psi\right)$$

where λ_m is the mean value of induced flow, and κ_x represents the longitudinal variation of inflow and κ_y represents the lateral variation of inflow.