M0000.026500

# 학습기반 공정 동적최적화

# Lecture 6: Linear Quadratic Control

## Deterministic Case

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# Outline

- Basic problem setup
- Deterministic system
- Stochastic system (Lecture 7)

# Basic Problem Setup

Linear Deterministic System:

$$x(k+1) = Ax(k) + Bu(k) \quad (1)$$
$$y(k) = Cx(k) \quad (2)$$

We consider time-invariant system for simplicity.

For a linear state feedback controller

$$u(k) = -L(k)x(k) \quad (3)$$

The closed-loop response is:

$$x(k+1) = (A - BL(k))x(k)$$

**Stability**

The state feedback controller (3) stabilizes the system if all the eigenvalues of ($A$ - $BL$) lie within the unit disk

# Objective of LQ

- A system visits a sequence of states of $x(0),\ x(1),\ \ldots,\ x(p)$ and desired sequence of states $x(0),\ \bar{x}(1),\ \ldots,\ \bar{x}(p)$

- Without loss of generality, the desired trajectory, $\bar{x}$, can be set as the origin.

- Objective function

$$\min \sum_{k=0}^{p-1} \left[ x^T(k)Qx(k) + u^T(k)Ru(k) \right] + x^T(p)Q_t x(p)$$

  - $Q$ and $R$ are symmetric positive definite; $Q_t$ is positive semi-definite

  - $Q$ provides relative importance to the errors in various states

  - $R$ accounts for the cost of implementing input moves

- If $p = \infty$, it is infinite horizon problem.

# Open-Loop Control vs. Feedback Control

- Optimal open-loop control problem
  - Find the optimal sequence of $u(0), \ldots, u(k)$ for given (as a function of) distribution of $x(0)$
- Optimal feedback control problem
  - Find the optimal feedback law $u(k) = f(x(k))$ or

$$u(k) = f(y(k),\ y(k-1),\ \ldots)$$

- For completely deterministic systems, the two should provide the same performance.
- State Feedback vs. Output Feedback

$$\begin{aligned} u(k) &= f(x(k)) \quad \Rightarrow \quad \text{State feedback} \\ u(k) &= \mathcal{F}(y(k)) \quad \Rightarrow \quad \text{Output feedback} \end{aligned}$$

$\mathcal{F}$ would be a dynamic operator in general.

# Least Squares Solution

## Open-Loop Optimal Feedback Control

Using (1) recursively gives,

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) = A\left(Ax(k-1) + Bu(k-1)\right) + Bu(k) \\ &= A^2x(k-1) + Bu(k) + ABu(k-1) \\ &= \quad \vdots \\ &= A^{k+1}x(0) + \big(Bu(k) + ABu(k-1) + \ldots + A^kBu(0)\big) \end{aligned}$$

Thus, we can write

$$\underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(p) \end{bmatrix}}_{\mathcal{X}} = \underbrace{\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^p \end{bmatrix}}_{\mathcal{S}^x} x(0) + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ A^{p-1}B & A^{p-2}B & \cdots & B \end{bmatrix}}_{\mathcal{S}^u} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(p-1) \end{bmatrix}}_{\mathcal{U}}$$

## System equation

$$\mathcal{X} = \mathcal{S}^x x(0) + \mathcal{S}^u \mathcal{U}$$

## Quadratic cost function

$$V_0(x(0); \mathcal{U}) = \sum_{k=0}^{p-1} \left[ x^T(k) Q x(k) + u^T(k) R u(k) \right] + x^T(p) Q_t x(p)$$
$$= \mathcal{X}^T \Gamma^x \mathcal{X} + \mathcal{U}^T \Gamma^u \mathcal{U}$$

$$\Gamma^x = \text{blockdiag}\left\{ Q, \ \ldots, \ Q, \ Q_t \right\}; \Gamma^u = \text{blockdiag}\left\{ R, \ \ldots, \ R \right\}$$

## Optimal cost

$$V_0(x(0)) = \min_{\mathcal{U}} \Big\{ x^T(0) \mathcal{S}^{xT} \Gamma^x \mathcal{S}^x x(0) + \mathcal{U}^T \left[ \mathcal{S}^{uT} \Gamma^x \mathcal{S}^u + \Gamma^u \right] \mathcal{U} + 2 x^T(0) \mathcal{S}^{xT} \Gamma^x \mathcal{S}^u \mathcal{U} \Big\}$$

## Optimal solution

$$\mathcal{U}^* = -\mathcal{H}^{-1} g = -\left[ \mathcal{S}^{uT} \Gamma^x \mathcal{S}^u + \Gamma^u \right]^{-1} \mathcal{S}^{uT} \Gamma^x \mathcal{S}^x x(0)$$

# OLOFC

$$\mathcal{U}^* = -\left[\mathcal{S}^{u^T}\Gamma^x\mathcal{S}^u + \Gamma^u\right]^{-1}\mathcal{S}^{u^T}\Gamma^x\mathcal{S}^x x(0) \tag{5}$$

$$V_0^*(x(0)) = x^T(0)\Big[\mathcal{S}^{x^T}\Gamma^x\mathcal{S}^x - \\ \mathcal{S}^{x^T}\Gamma^x\mathcal{S}^u\left(\mathcal{S}^{u^T}\Gamma^x\mathcal{S}^u + \Gamma^u\right)^{-1}\mathcal{S}^{u^T}\Gamma^x\mathcal{S}^x\Big]x(0)$$

- Open-loop optimal control finds a sequence

  $u^*(0),\ u^*(1),\ \ldots,\ u^*(p-1)$ for a given $x(0)$

- Recursively use (5) as in Receding Horizon Control
- Not efficient computationally
- Not generalizable to the stochastic case

# Closed-Loop Optimal Feedback Control

## CLOFC

- One obtains the optimal control move as a function of state at each time
- Solved using Dynamic Programming
- More elegant and closed-loop optimal solution

# Dynamic Programming

At the stage p-1, Bellman's equation is

$$V_{p-1}(x(p-1)) = \min_{u(p-1)} \{x^T(p-1)Qx(p-1) + u^T(p-1)Ru(p-1) + x^T(p)S(p)x(p)\} \tag{6}$$

where $S(p) = Q_t$

Noting that $x(p) = Ax(p-1) + Bu(p-1)$ , we get:

$$V_{p-1}(x(p-1)) = \min_{u(p-1)} \{x^T(p-1)\left(A^T S(p) A + Q\right)x(p-1) + 2x^T(p-1)A^T S(p) B u(p-1) + u^T(p-1)\left(B^T S(p) B + R\right)u(p-1)\}$$

As before, the optimal solution can be obtained as:

$$u^*(p-1) = -\underbrace{\left(B^T S(p) B + R\right)^{-1} B^T S(p) A}_{L(p-1)} x(p-1)$$

Substitution of $u^*(p-1)$ gives

$$V_{p-1}\left(x(p-1)\right) = x^T(p-1) S(p-1) x(p-1)$$

where $S(p - 1)$ is given by the following Riccati Equation

$$S(p-1) = A^T S(p) A + Q - A^T S(p) B \left(B^T S(p) B + R\right)^{-1} B^T S(p) A$$

# Stage: *p* - 2

$$
\begin{aligned}
V_{p-2}(x(p-2)) &= \min_{u(p-2)} \{x^T(p-2)Qx(p-2) + u^T(p-2)Ru(p-2) + \\
&\qquad V_{p-1}(x(p-1))\} \\
&= \min_{u(p-2)} \{x^T(p-2)Qx(p-2) + u^T(p-2)Ru(p-2) + \\
&\qquad x^T(p-1)S(p-1)x(p-1)\}
\end{aligned}
$$

This equation is in the same for as (6). The optimal solution is

$$
u^*(p-2) = -\underbrace{\left(B^T S(p-1)B + R\right)^{-1} B^T S(p-1)A}_{L(p-2)} x(p-2)
$$

# Generalization

Successively solving for cost-to-go $V_k(x(k))$, we get:

$$u^*(k) = -L(k)x(k), \qquad \text{for } k = p-1, \; \ldots, \; 0$$

where

$$\boxed{L(k) = \left(B^T S(k+1) B + R\right)^{-1} B^T S(k+1) A}$$

$$\boxed{\begin{aligned} S(k) = {} & A^T S(k+1) A + Q - \\ & A^T S(k+1) B \left(B^T S(k+1) B + R\right)^{-1} B^T S(k+1) A \end{aligned}} \tag{7}$$

Note that (7) is the familiar Riccati Difference Equation that we encounter in Kalman Filtering as well.

# Comments

- For a deterministic case, OLOFC and CLOFC yield the same solution.
- The optimal p-stage cost is: $V_0(x_0) = x^T(0)S(0)x(0)$
- Receding horizon solution to optimization is computationally demanding
- Dynamic Programming leads to the optimal control solution as an explicit linear function, $u(k) = -L(k)x(k)$
- Recursive solution of Riccati equation, required in DP, is straightforward.
- Note that the results hold only for the unconstrained system.

# Extension of DP to Infinite Horizon

Assuming the RDE solution converges to $S_\infty$,

$$u(k) = -\underbrace{\left(B^T S_\infty B + R\right)^{-1} B^T S_\infty A}_{L_\infty} x(k)$$

$$\boxed{S_\infty = A^T S_\infty A + Q - A^T S_\infty B \left(B^T S_\infty B + R\right)^{-1} B^T S_\infty A} \tag{8}$$

- Note that (8) is known as Algebraic Riccati Equation
- The RDE (7) converges to $S_\infty$ in the infinite horizon case if (A, B) is *stabilizable pair*.
- The converged solution gives stable controller if $(Q^{1/2}, A)$ is *detectable pair*.

# Extension of OLOCP to Infinite Horizon

Q. Direct extension of OLOCP to infinite horizon seems impossible because of the infinite number of inputs to optimize. What can we do?

A. For certain choices of $Q_t$, the finite horizon problem:

$$\min_{u(0),\ldots,u(p-1)} \{V_0\left(x(0);\mathcal{U}\right)\}$$

can be made equivalent to the infinite horizon problem.

# OLOCP: Equivalence with the Infinite Horizon Problem

## Option 1

We can choose $Q_t$ such that:

$$x^T(k+p)Q_t x(k+p) = \min_{u(k+p),\dots} \left\{ \sum_{i=p}^{\infty} x^T(k+i)Qx(k+i) + u^T(k+i)Ru(k+i) \right\}$$

It is clear that we can compute such $Q_t$ by solving the ARE of

$$Q_t = A^T Q_t A + Q - A^T Q_t B \left( B^T Q_t B + R \right)^{-1} B^T Q_t A$$

- With this choice of $Q_t$, the optimal solution of $p$-horizon problem is equivalent to that of $\infty$-horizon one.

# OLOCP: Equivalence with the Infinite Horizon Problem

## Option 2

We may also choose $Q_t$ such that:

$$x^T(k+p)Q_t x(k+p) = \sum_{i=p}^{\infty} x^T(k+i)Qx(k+i)$$

- The above equation is under the assumption that no control action is taken beyond the horizon $k+p$.
- Then, the autonomous system $x(k+1) = Ax(k)$ describes the evolution of the state.
- This assumption is meaningful only when the system is stable, otherwise the cost is infinite.
- We can show that the above $Q_t$ is a solution to Lyapunov Equation:

$$Q_t = Q + A^T Q_t A$$

# Option 2: Lyapunov Function

- Generalized energy function
- Zero @ equilibrium point and positive elsewhere
- The equilibrium will be stable if Lyapunov function ($V_l$) decreases along the trajectories of the system

$$\Delta V_l(x) \text{ or } \dot{V}_l(x)$$

Note that the system equation from time $k+p$ is given as

$$x(k+i+1) = A(k+i)x(k+i)$$

$$\begin{aligned} V_l(x(k+p)) &= x^T(k+p)Q_t x(k+p) \\ \Delta V_l(x(k+p)) &= V_l(x(k+p+1)) - V_l(x(k+p)) \\ &= V_l(Ax(k+p)) - V_l(x(k+p)) \\ &= x^T(k+p)\left(A^T Q_t A - Q_t\right)x(k+p) := -x(k+p)^T P x(k+p) \end{aligned}$$

- Usually, $Q_t$ is found by specifying $P$ as a positive definite matrix.

In this case, $P = Q$ from the definition of the terminal cost:

$$\begin{aligned} x^T(k+p)Q_t x(k+p) &= x^T(k+p)Qx(k+p) + \sum_{i=p+1}^{\infty} x^T(k+i)Qx(k+i) \\ &= x^T(k+p)Qx(k+p) + x^T(k+p+1)Q_t x(k+p+1) \\ &= x^T(k+p)Qx(k+p) + x^T(k+p)A^T Q_t A x(k+p) \end{aligned}$$

This gives a discrete time Lyapunov equation

$$\boxed{Q_t = A^T Q_t A + Q}$$

# OLOCP: Equivalence with the Infinite Horizon Problem

## Option 3

Solve the finite horizon problem with $x(k+p) = 0$ as a constraint.

- Note that Option 2 cannot be used if the system is unstable.
- Option 3 can then be used for unstable systems.

# Extension to Output Feedback Case

So far, we assumed that the full state feedback is available. In case of output feedback, the control actions are based on state estimates $\hat{x}(k)$.

- Observer:

$$\hat{x}(k) = A\hat{x}(k-1) + Bu(k-1) + K\left[y(k) - C\left(A\hat{x}(k-1) + Bu(k-1)\right)\right]$$

- Controller: $u(k) = -L\hat{x}(k)$

If we define $x_e(k) \triangleq x(k) - \hat{x}(k)$ , we get

$$\begin{bmatrix} x(k+1) \\ x_e(k+1) \end{bmatrix} = \begin{bmatrix} A - BL & -BL \\ 0 & A - KCA \end{bmatrix} \begin{bmatrix} x(k) \\ x_e(k) \end{bmatrix}$$

## Separation Principle

Since the above equation is one-way coupled, the system is guaranteed to be stable if the controller and the filter are guaranteed to be stable independently.