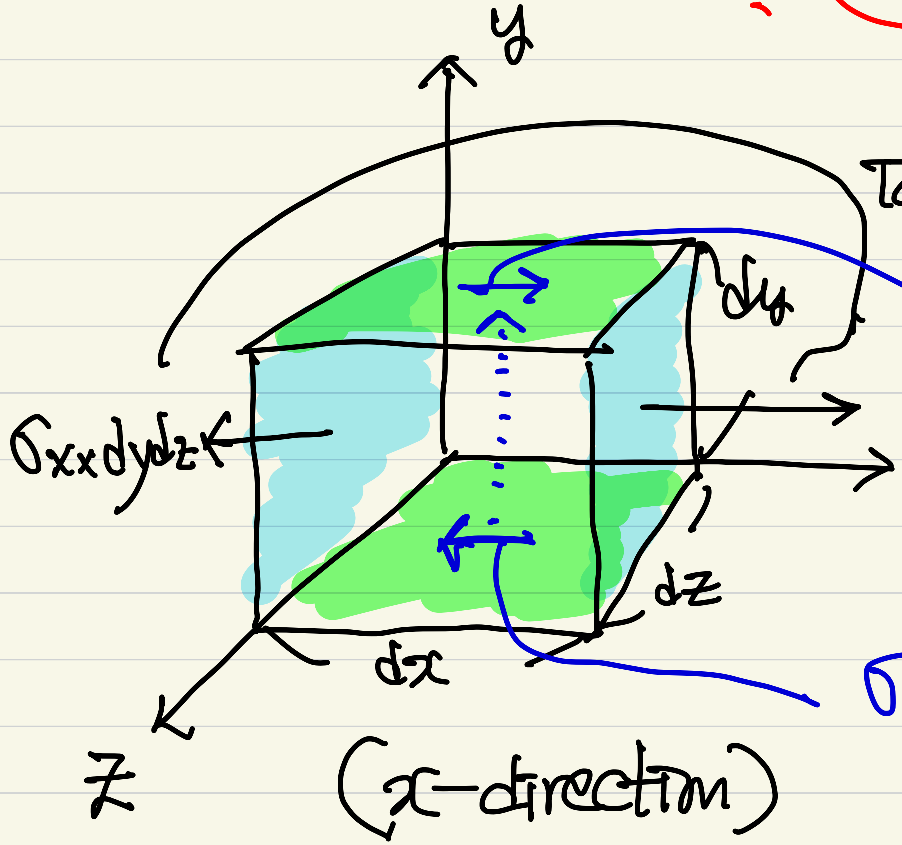
$$\boxed{\sum \underline{F} = \rho \frac{D\underline{V}}{Dt}\, dx\,dy\,dz.}$$ ①

② + ③ $\left\{\begin{array}{l} \text{body force} : \quad d\underline{F}_{grav} = \rho\, \underline{g}\, dx\,dy\,dz. \quad ② \\ \text{surface } ④ : \quad \boxed{\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}} \end{array}\right.$

viscous stress

hydrostatic pressure



Taylor series expansion

$\sigma_{xx} dydz + \frac{\partial}{\partial x}(\sigma_{xx} \cdot dydz)\cdot dx$

$\sigma_{yx} dxdz + \frac{\partial}{\partial y}(\sigma_{yx} dxdz)\, dy$

$\sigma_{yx} \cdot dx \cdot dz.$

$\Downarrow$

gradient (∇p or $\nabla \tau_{ij}$) is important!

$$\therefore\ dF_{x,surf} = \left[\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{yx} + \frac{\partial}{\partial z}\sigma_{zx} \right] dx\,dy\,dz$$

$$= \left[-\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\hat{\tau}_{xx} + \frac{\partial}{\partial y}\hat{\tau}_{yx} + \frac{\partial}{\partial z}\hat{\tau}_{zx} \right] dx\,dy\,dz.$$

$$dF_{y,surf} = \left[-\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\hat{\tau}_{xy} + \frac{\partial}{\partial y}\hat{\tau}_{yy} + \frac{\partial}{\partial z}\hat{\tau}_{zy} \right] dx\,dy\,dz$$

$$dF_{z,surf} = \left[-\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\hat{\tau}_{xz} + \frac{\partial}{\partial y}\hat{\tau}_{yz} + \frac{\partial}{\partial z}\hat{\tau}_{zz} \right] dx\,dy\,dz.$$

$+$

$$\Rightarrow d\underline{F}_{surf} = \left(-\nabla P + \nabla \cdot \underset{\sim}{\hat{\tau}} \right) dx\,dy\,dz. \quad (3) \qquad \text{(vector)}$$

$$\text{or}\quad dF_{i,surf} = \left(-\frac{\partial P}{\partial x_i} + \frac{\partial \hat{\tau}_{ij}}{\partial x_j} \right) dx\,dy\,z \qquad \text{(tensor)}$$

$$(i, j = x, y, z)$$

①+②+③ : mtm conservation eq

$$\boxed{\rho \underline{g} - \nabla P + \nabla \cdot \underset{\sim}{\tau} = \rho \frac{D\underline{v}}{Dt}}$$

$\rho \underline{g}$: gravity

∇P: pressure force

$\nabla \cdot \underset{\sim}{\tau}$: viscous force.

$$= \rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}\right)$$

$$= \rho\left(\frac{\partial \underline{v}}{\partial t} + u\frac{\partial \underline{v}}{\partial x} + v\frac{\partial \underline{v}}{\partial y} + w\frac{\partial \underline{v}}{\partial z}\right)$$

* Inviscid flow.

$$\rho\frac{D\underline{v}}{Dt} = \rho \underline{g} - \nabla P \quad : \text{Euler equation.}$$

* Newtonian Fluid.

$$\tau = \mu\frac{du}{dy}. \xrightarrow[\text{generalization}]{} \tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$(i, j = x, y, z)$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}, \quad \tau_{yy} = 2\mu\frac{\partial v}{\partial y}, \quad \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right).$$

→ mtm conservation eq.

→ Navier-Stokes eq.

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}.$$

$$= \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x} + \mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y} + \mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x} + \mu\frac{\partial u}{\partial z}\right)$$

$$= \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \mu\frac{\partial^2 u}{\partial z^2} + \mu\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)$$

$$= \boxed{\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)} + \mu\frac{\partial}{\partial x}\cancelto{0}{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}$$

incompressible flow
continuity eq.

Newtonian, incompressible

$$\nabla \cdot \tau$$
$$\Downarrow$$

$$\rho \frac{D\underline{V}}{Dt} = \rho \underline{g} - \nabla p + \mu \nabla^2 \underline{V} \quad : \text{N-S eq.}$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho \frac{\partial w}{\partial t} + \underline{\rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z}} = \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

unsteady, non-linear. 2ND order, coupled PDE.

w/ given ρ, μ, 4 unknowns (u, v, w, p)

(continuity + ③ N-S)

+ state equation
$(p = \rho R T)$.

+ energy eq (for T)

4.5 Differential eq. of Energy.

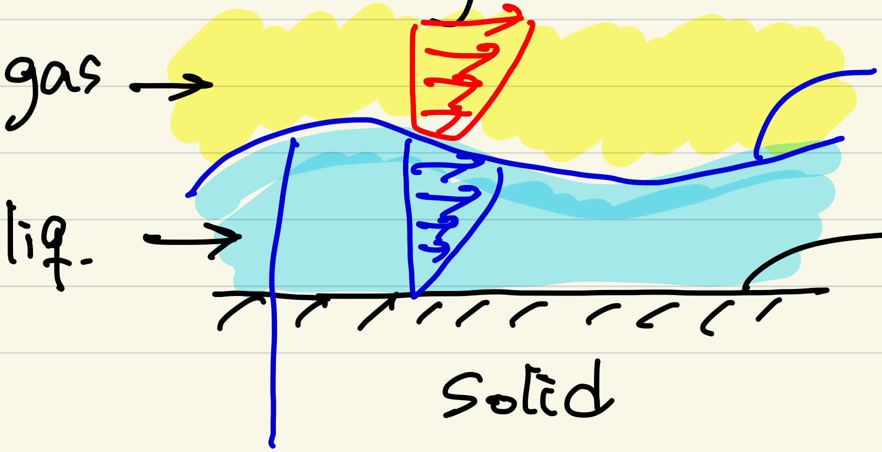
$$\rho \frac{D\hat{u}}{Dt} + p(\nabla \cdot \underline{V}) = \nabla \cdot (k \nabla T) + \Phi .$$

internal energy ($d\hat{u} \simeq C_v\, dT$)

thermal conductivity.

viscous dissipation.

4.6. Boundary conditions for basic eqns.



no-slip condition : $\underline{V}_{fluid} = \underline{V}_{solid}$

no-jump 〃 : $T_{fluid} = T_{solid}$

$$\boxed{\hat{\tau}_{liq} = \hat{\tau}_{gas}.} \Rightarrow \text{simplified free-surf. BC}$$

(valid for river, sea, ...)

$$\hat{\tau}_{liq} = 0.$$

$$(p_{liq} = p_{gas})$$

$$dF_{i,surf} = \left(-\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j}\right) dx\,dy\,dz.$$

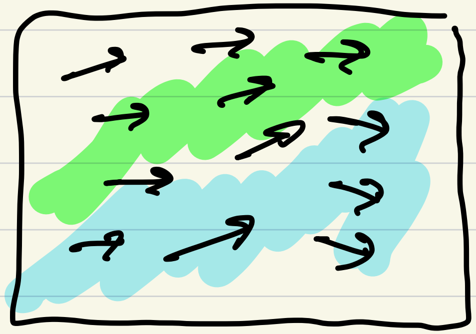
dummy index. $(i, j = x, y, z)$

$$\frac{\partial \tau_{xi}}{\partial x} + \frac{\partial \tau_{yi}}{\partial y} + \frac{\partial \tau_{zi}}{\partial z}.$$

(ρ = constant)

* incompressible flow w/ constant properties. Newtonian Fluid.

$$\begin{cases} \nabla \cdot \underline{V} = 0 & \text{(continuity eq.)} \\ \rho \dfrac{\partial \underline{V}}{\partial t} + \rho(\underline{V}\cdot\nabla)\underline{V} = \rho \underline{g} - \nabla p + \mu \nabla^2 \underline{V}. & \text{(N-S eq.)} \end{cases}$$



* Inviscid flow approximation.

(비점성 유체). $\mu \to 0$.

$$\rho \frac{D\underline{V}}{Dt} = \rho \underline{g} - \nabla P$$: Euler equation.

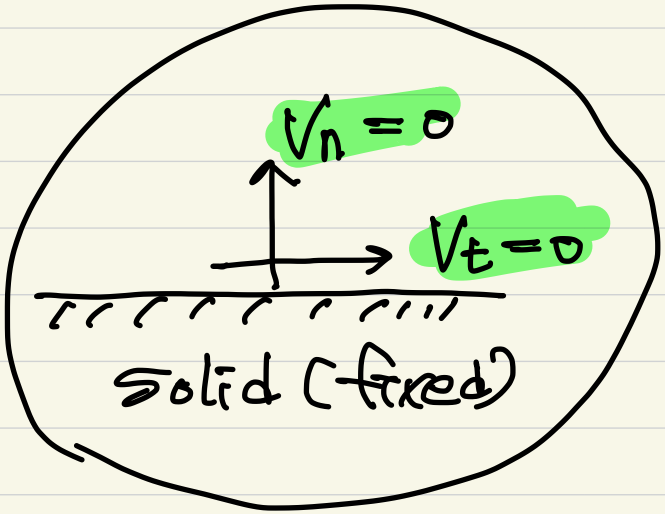
$$= \rho \frac{\partial \underline{V}}{\partial t} + \rho (\underline{V} \cdot \nabla) \underline{V}$$

⇓

1st order PDE

⇒ one BC for $\underline{V}$.

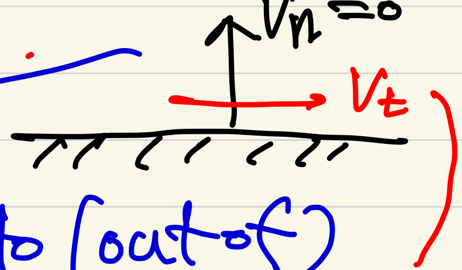
→



N-S eq.

no-slip condition. (w/ viscosity)

drop no-slip condition along the tangential direction.



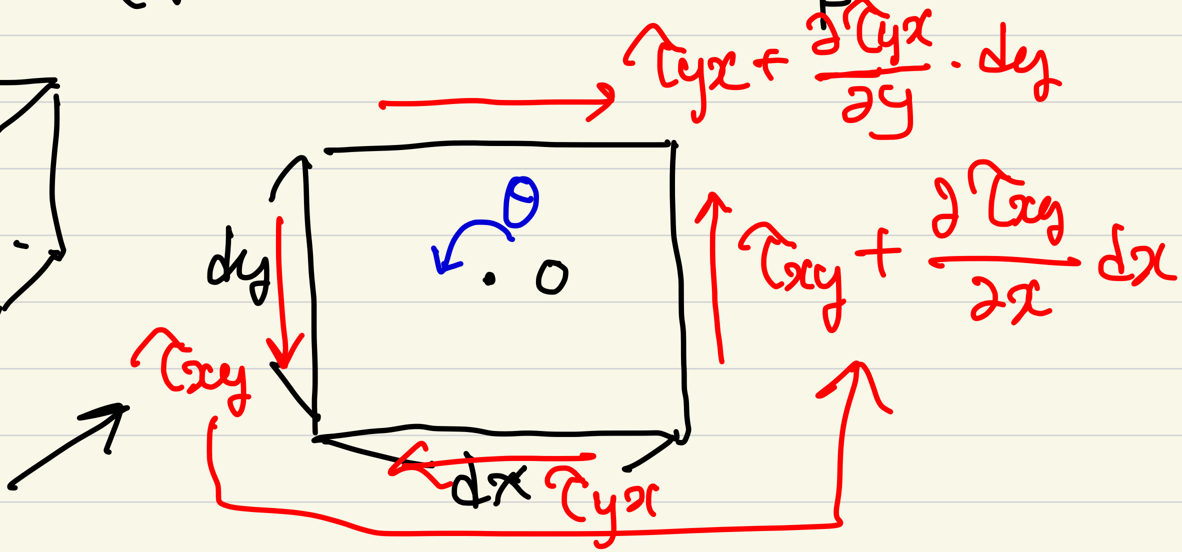
do not allow the flow into (out of)

the wall.

Not BC, part of the solution. (ch. 8)

4.4. Differential eq. of angular mntm.

$$\sum M_o = \frac{\partial}{\partial t}\int_{CV} (\underline{r}\times\underline{V})\,\rho\, d\forall + \int_{CS} (\underline{r}\times\underline{V})\,\rho\,(\underline{V}\cdot\underline{n})\,dA .$$



Taylor Series exp.

$$\sum M_o = \Big[\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx \right) dy \cdot dz \cdot \frac{1}{2} dx + \tau_{xy} \cdot dy dz \cdot \frac{1}{2} dx$$

$$- \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dx \cdot dz \cdot \frac{1}{2} dy - \tau_{yx} \cdot dx dz \cdot \frac{1}{2} dy \Big]$$

$$= I \cdot \alpha$$

$$I = \frac{1}{12} m (w^2 + d^2)$$

$$\frac{1}{12} \rho \, dx dy dz \left(dx^2 + dy^2 \right) \cdot \frac{d^2\theta}{dt^2}$$

$$(\tau_{xy} - \tau_{yx}) dx dy dz + \frac{1}{2} \frac{\partial \tau_{xy}}{\partial x} dx^2 dy dz - \frac{1}{2} \frac{\partial \tau_{yx}}{\partial y} dx dy^2 dz.$$

as $dx, dy, dz \to 0$.

if we neglect higher order terms.

$$\Rightarrow \boxed{\tau_{xy} = \tau_{yx}} \quad (\tau_{xz} = \tau_{zx}, \ \tau_{yz} = \tau_{zy} \text{ as well})$$

$\hookrightarrow \underset{\sim}{\tau}$ is symmetric.

4.7 Stream function (유선 함수).

(scalar function)

for 2D, incompressible flow (ρ = const.)

Continuity (mass conserv.)

$$: \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \quad (\nabla \cdot \underset{\sim}{V} = 0).$$

$$\text{Let } \boxed{u \equiv \frac{\partial \psi}{\partial y} \text{ and } v \equiv -\frac{\partial \psi}{\partial x}}$$

scalar function

$$\text{Then, } \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

$\rightarrow$ stream function.

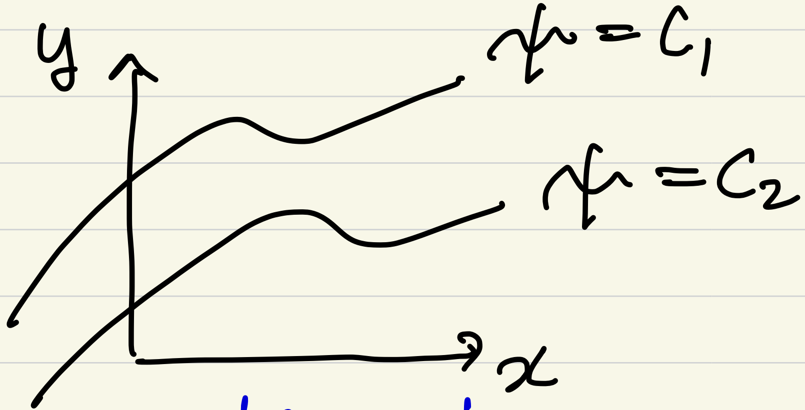
↳ automatically satisfies the continuity!

* Geometric interpretation of ψ

① Lines of constant ψ.

$$\boxed{\psi = \psi(x,y) = C}$$

streamlines



· Streamlines in 2D flow. : $\dfrac{dx}{u} = \dfrac{dy}{v}$

$u\,dy - v\,dx = 0$ along a streamline.

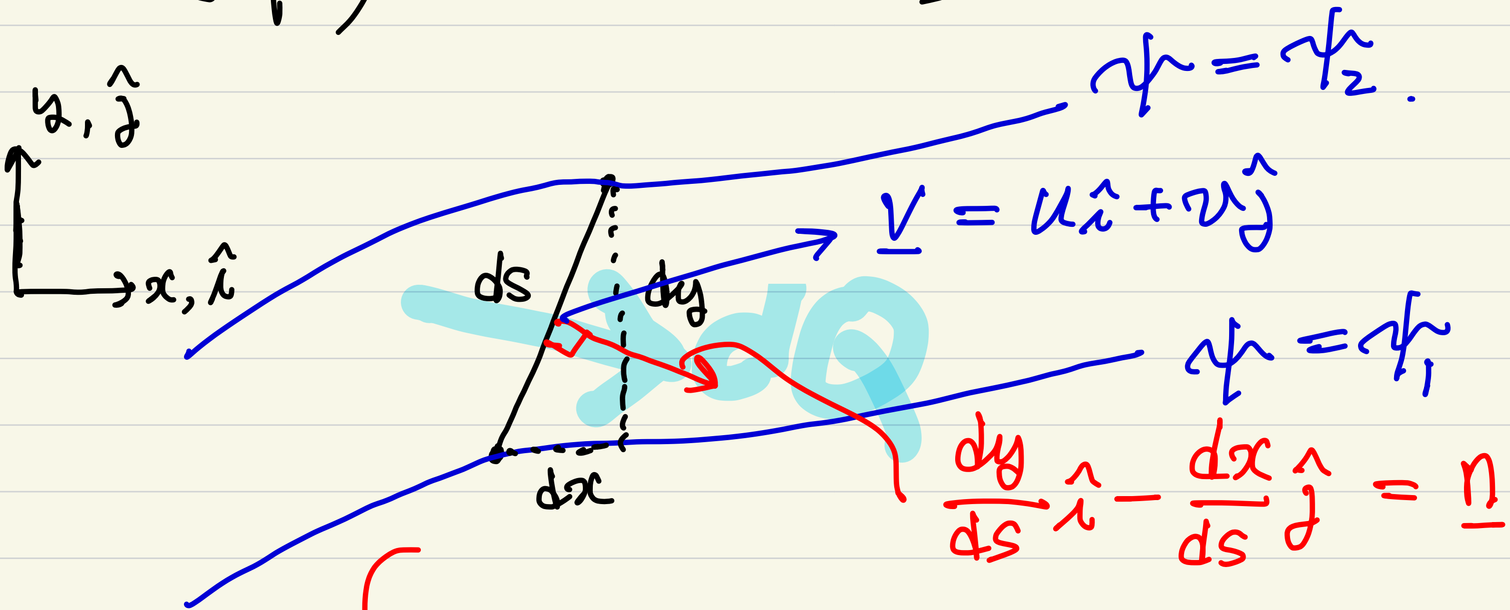
$$\boxed{d\psi = \underbrace{\frac{\partial \psi}{\partial x}}_{-v} dx + \underbrace{\frac{\partial \psi}{\partial y}}_{u} dy = 0}$$

$-v\,dx + u\,dy = 0$

$\therefore\ \psi = C$ along a streamline.

② Change in ψ across the element is equal to the volume flow through the element.

(physical meaning of ψ value) (2D)



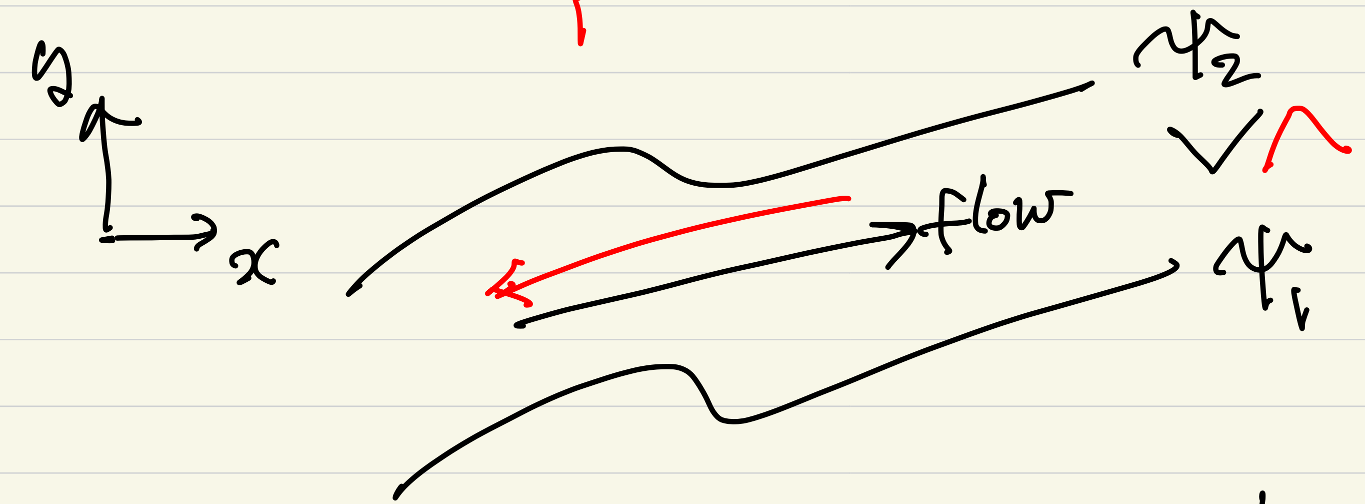
$$\int dQ = (\underline{V} \cdot \underline{n})\, dA.$$

$$= (u\hat{i} + v\hat{j}) \cdot \left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds$$

$$= \left(\frac{\partial \psi}{\partial y}\hat{i} - \frac{\partial \psi}{\partial x}\hat{j}\right) \cdot \left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds.$$

$$= \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = d\psi.$$

$$\therefore Q = \int_1^2 d\psi = \psi_2 - \psi_1.$$



$$u = \frac{\partial\psi}{\partial y}$$

$$v = -\frac{\partial\psi}{\partial x}$$

* steady plane (2D) compressible flow.

continuity : $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$

→ compressible stream function

$$\rho u \equiv \frac{\partial \psi}{\partial y}, \quad \rho v \equiv -\frac{\partial \psi}{\partial x}.$$

$$\Rightarrow d\dot{m} = \rho(\underline{V}\cdot\underline{n})\,dA = d\psi. \quad (\text{not } dQ = d\psi)$$

$$\dot{m}_{1\to 2} = \psi_2 - \psi_1.$$

* In incompressible plane flow (2D) in polar coordinate.

continuity : $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) = 0$

$\mapsto$ $r v_r = \frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \psi}{\partial r}, \quad dQ = d\psi.$

* In general, stream function is NOT defined in 3D flows, excep the axisymmetric flow. $(v_\theta = 0, \frac{\partial}{\partial \theta}(\cdot) = 0)$

continuity eq : $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z}(v_z) = 0.$

$$\Downarrow$$

$$\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z}(r v_z) = 0.$$

$$\rightarrow \quad r v_r = -\frac{\partial \psi}{\partial z}, \quad r v_z = \frac{\partial \psi}{\partial r}.$$

$$dQ = 2\pi\, d\psi. \quad \rightarrow \quad Q_{1\to 2} = 2\pi(\psi_2 - \psi_1)$$