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학습기반 공정 동적최적화

Lecture 6: Linear Quadratic Control

Stochastic Case and Practical Issues

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Optimal State Feedback for Stochastic System

System $$x(k+1) = Ax(k) + Bu(k) + \varepsilon_1(k)$$

Objective: Design a state feedback $u(k) = f(x(k))$ that minimizes

$$E\left[\sum_{i=0}^{p-1}\left\{x^T(i)Qx(i) + u^T(i)Ru(i)\right\} + x^T(p)Q_t x(p)\right]$$

Assume that the state is perfectly measured and that $\varepsilon_1(k)$ is a zero-mean Gaussian white noise with covariance R_1.

Open-loop optimal feedback law (MPC) and closed-loop optimal control law (DP) can give different results in general. Will it be in this case?

Open-loop Optimal Solution

As before,

$$
\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(p) \end{bmatrix} = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^p \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ A^{p-1}B & A^{p-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(p-1) \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ A & I & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ A^{p-1} & A^{p-2} & \cdots & I \end{bmatrix} \begin{bmatrix} \varepsilon(0) \\ \varepsilon(1) \\ \varepsilon(2) \\ \vdots \\ \varepsilon(p-1) \end{bmatrix}
$$

Short-hand notations as before:

$$\mathcal{X} = \mathcal{S}^x x(0) + \mathcal{S}^u \mathcal{U} + \mathcal{S}^\varepsilon \mathcal{E}$$

We get

$$\begin{aligned} V_0\left(x(0); \mathcal{U}\right) &= \mathbb{E}\left[\mathcal{X}^T \Gamma^x \mathcal{X} + \mathcal{U}^T \Gamma^u \mathcal{U}\right] \\ &= \mathbb{E}\left[\left(\mathcal{S}^x x(0) + \mathcal{S}^u \mathcal{U} + \mathcal{S}^\varepsilon \mathcal{E}\right)^T \Gamma^x \left(\mathcal{S}^x x(0) + \mathcal{S}^u \mathcal{U} + \mathcal{S}^\varepsilon \mathcal{E}\right) + \mathcal{U}^T \Gamma^u \mathcal{U}\right] \\ &= \left(\mathcal{S}^x x(0) + \mathcal{S}^u \mathcal{U}\right)^T \Gamma^x \left(\mathcal{S}^x x(0) + \mathcal{S}^u \mathcal{U}\right) + \mathcal{U}^T \Gamma^u \mathcal{U} + \mathbb{E}\left[\mathcal{E}^T \mathcal{S}^{\varepsilon^T} \Gamma^x \mathcal{S}^\varepsilon \mathcal{E}\right] \end{aligned}$$

This differs from the deterministic case only in the last term.

Note that

$$\mathbb{E}\left[\mathcal{E}^T \mathcal{S}^{\varepsilon^T} \Gamma^x \mathcal{S}^\varepsilon \mathcal{E}\right] = \mathbb{E}\left[\text{trace}\left\{\mathcal{S}^{\varepsilon^T} \Gamma^x \mathcal{S}^\varepsilon \mathcal{E}^T \mathcal{E}\right\}\right] = \text{trace}\left\{\mathcal{S}^{\varepsilon^T} \Gamma^x \mathcal{S}^\varepsilon R_1\right\}$$

Since the last term does not involve $\mathcal{U}$, the solution is the same as the deterministic case:

$$\mathcal{U} = -\left(S^{u^T}\Gamma^x\mathcal{S}^u\right)^{-1}\mathcal{S}^{u^T}\Gamma^x\mathcal{S}^x x_0$$

$$\begin{aligned} V_0(x_0) = & x_0^T\left[\mathcal{S}^{x^T}\mathcal{S}^x - \mathcal{S}^{x^T}\Gamma^x\mathcal{S}^u\left(\mathcal{S}^{u^T}\Gamma^x\mathcal{S}^u + \Gamma^u\right)^{-1}\mathcal{S}^{u^T}\Gamma^x\mathcal{S}^x\right]x_0 + \\ & \text{trace}\left\{\mathcal{S}^{\varepsilon^T}\Gamma^x\mathcal{S}^\varepsilon R_1\right\} \end{aligned}$$

Dynamic Programming

$$V_{p-1}(x(p-1)) = \min_{u(p-1)} \mathbb{E}\left\{x^T(p-1)Qx(p-1) + u^T(p-1)Ru(p-1)+ \right.$$
$$\left. x^T(p)S(p)x(p) \,\middle|\, x(p-1)\right\}$$

$$V_{p-2}(x(p-2)) = \min_{u(p-2)} \mathbb{E}\left\{x^T(p-2)Qx(p-2) + u^T(p-2)Ru(p-2)+ \right.$$
$$\left. V_{p-1}(x(p-1)) \,\middle|\, x(p-2)\right\}$$

CLOFC - DP

Key result: Optimal feedback law is the same as the deterministic case

$$u(k) = -\left(B^T S(k+1)B + R\right)^{-1} B^T S(k+1)Ax(k), \quad k = p-1, \ldots, 0$$

where

$$\begin{aligned} S(k) =& A^T S(k+1)A + Q \\ & - A^T S(k+1)B\left(B^T S(k+1)B + R\right)^{-1} B^T S(k+1)A \end{aligned}$$

with $S(p) = Q_t$

Optimal cost-to-go

$$V_0(x_0) = x_0^T S(0)x_0 + \sum_{j=1}^{p} \text{trace}\{S(j)R_1\}$$

Constant Setpoint Tracking

Consider the performance function of

$$\sum_{k=0}^{\infty}(r(k)-y(k))^T Q_e (r(k)-y(k)) + u^T(k) R u(k)$$

Then, one can reformulate this as a state regulation problem by writing the model as

$$\underbrace{\begin{bmatrix} x(k+1) \\ r(k+1) \end{bmatrix}}_{\tilde{x}(k+1)} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x(k) \\ r(k) \end{bmatrix}}_{\tilde{x}(k)} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}(k)} u(k)$$

$$r(k)-y(k) = \begin{bmatrix} -C & I \end{bmatrix} \begin{bmatrix} x(k) \\ r(k) \end{bmatrix}$$

$$Q = \begin{bmatrix} -C & I \end{bmatrix}^T Q_e \begin{bmatrix} -C & I \end{bmatrix}$$

At steady state, the input is not zero for offset-free tracking.

$\Delta u = 0$ at steady state for integral action. The above formulation does not guarantee integral action.

The following reformulation ensures integral action:

$$\sum_{k=0}^{\infty}(r(k)-y(k))^T Q_e (r(k)-y(k)) + \Delta u^T(k) R \Delta u(k)$$

The state needs to be augmented with previous input move

$$\tilde{x}(k) = \begin{bmatrix} x(k) & r(k) & u(k-1) \end{bmatrix}^T$$

With this definition

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} B \\ 0 \\ I \end{bmatrix}$$

The problem with this formulation is that the state $r(k)$ is not stabilizable. Hence, the RDE is not guaranteed to converge.

Remedy: Model Differencing

$$\Delta x(k+1) = A\Delta x(k) + B\Delta u(k)$$
$$\Delta y(k) = C\Delta x(k)$$

$$e(k) \overset{\Delta}{=} y(k) - r(k)$$

$$e(k+1) = y(k+1) - r(k+1)$$
$$= e(k) + \Delta y(k+1) - \underbrace{\Delta r(k+1)}_{\to 0} = e(k) + CA\Delta x(k) + CB\Delta u(k)$$

$$\underbrace{\begin{bmatrix} \Delta x(k+1) \\ e(k+1) \end{bmatrix}}_{\tilde{x}(k+1)} = \underbrace{\begin{bmatrix} A & 0 \\ CA & I \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} \Delta x(k) \\ e(k) \end{bmatrix}}_{\tilde{x}(k)} + \underbrace{\begin{bmatrix} B \\ CB \end{bmatrix}}_{\tilde{B}} \Delta u(k)$$

with this new definition:

$$Q = \begin{bmatrix} 0 & I \end{bmatrix}^T Q_e \begin{bmatrix} 0 & I \end{bmatrix}$$

Disturbance Rejection

The objective function remains the same, though the linear model is:

$$x(k+1) = Ax(k) + Bu(k) + B_d d(k)$$

As before, the steady state value of u_∞ is non-zero as long as the disturbance is non-zero.

As a result, $\Delta u(k)$ needs to be penalized as in the constant set point tracking case. The following two re-definitions will achieve this:

$$\tilde{x}(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix} \quad \text{and} \quad \tilde{x}(k) = \begin{bmatrix} \Delta x(k) \\ e(k) \end{bmatrix}$$

In this case, there are no issues with stabilizability of the new state $\tilde{x}(k)$ and both the reformulations work.