dimensional analysis

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0 \qquad \rho\left(\frac{\partial \mathbf{v}}{\partial t}+v_x\frac{\partial \mathbf{v}}{\partial x}+v_y\frac{\partial \mathbf{v}}{\partial y}\right)=\rho\mathbf{g}-\nabla\rho+\mu\left(\frac{\partial^2\mathbf{v}}{\partial x^2}+\frac{\partial^2\mathrm{v}}{\partial y^2}\right)$$

$$x^*=x/L \qquad v_x^*=v_x/v_\infty \qquad \mathrm{t}^*=\frac{\mathrm{t}\,v_\infty}{\mathrm{L}} \qquad \mathrm{v}^*=\mathrm{v}/v_\infty \qquad \nabla^*=\mathrm{L}\nabla$$

$$y^*=y/L \qquad v_y^*=v_y/v_\infty$$

$$\frac{\partial \mathrm{v}^*}{\partial \mathrm{t}^*}+v_\mathrm{x}^*\frac{\partial \mathrm{v}^*}{\partial \mathrm{x}^*}+v_\mathrm{y}^*\frac{\partial \mathrm{v}^*}{\partial \mathrm{y}^*}=\mathrm{g}\frac{\mathrm{L}}{v_\infty^2}-\frac{\nabla^*\mathrm{P}}{\rho v_\infty^2}+\frac{\mu}{\mathrm{L}\,v_\infty\rho}\left(\frac{\partial^2\mathrm{v}^*}{\partial \mathrm{x}^{*2}}+\frac{\partial^2\mathrm{v}^*}{\partial \mathrm{y}^{*2}}\right)$$

$Fr\equiv v_\infty^2/gL$ (inertial to gravitational forces)

$Eu\equiv P/\rho v_\infty{}^2$ (pressure to inertial forces)

$$C_D\equiv\frac{F/A}{\rho v_\infty{}^2/2}$$

$\mathrm{Re}\equiv Lv_\infty\rho/\mu$ (inertial to viscous force)

dynamic similarity

$$\nabla \cdot \mathbf{u} = 0 \qquad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g}$$

$$x^* = x/L, \quad \mathbf{u}^* = \mathbf{u}/U, \quad t^* = Ut/L, \quad p^* = p/\rho U^2 \qquad \nabla \equiv \frac{\partial}{\partial \mathbf{x}} = \frac{1}{L}\frac{\partial}{\partial \mathbf{x}^*}$$

$$\nabla^* \cdot \mathbf{u}^* = 0 \qquad \frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^*\cdot\nabla^*\mathbf{u}^* = -\nabla^* p^* + \frac{1}{\mathrm{Re}}\nabla^{*2}\mathbf{u}^* + \frac{1}{\mathrm{Fr}}\frac{\mathbf{g}}{g}$$

$$\boxed{\mathrm{Re} = \frac{\rho U L}{\mu}} \qquad \boxed{\mathrm{Fr} = \frac{U^2}{gL}}$$

Boundary conditions

$$u_x = 0 \quad \text{on} \quad y = 0 \qquad\qquad u_x^* = 0 \quad \text{on} \quad y^* = 0$$

$$u_x = U \quad \text{on} \quad y = B \qquad\qquad u_x^* = 1 \quad \text{on} \quad y^* = \frac{B}{L}$$

$$p - \mu\Delta_{nn} - \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 0 \qquad\qquad p^* - \frac{1}{\mathrm{Re}}\Delta_{nn}^* - \frac{1}{\mathrm{We}}\left(\frac{1}{R_1^*} + \frac{1}{R_2^*}\right) = 0$$

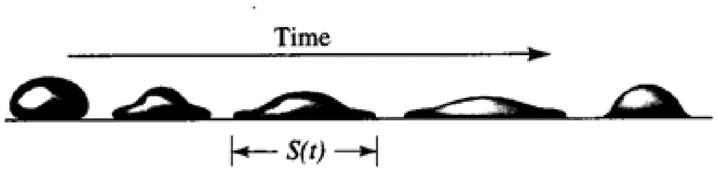
$$\boxed{\mathrm{We} = \frac{\rho U^2 L}{\sigma}}$$

Table 11.2 Common dimensionless parameters in momentum transfer

Name/Symbol	Dimensionless group	Physical meaning	Area of application
Reynolds number, Re	$Lv\rho/\mu$	$\dfrac{\text{Inertial force}}{\text{Viscous force}}$	Widely applicable in a host of fluid flow situations
Euler number, Eu	$P/\rho v^2$		
Coefficient of skin friction, C_f	$\dfrac{F/A}{\rho v^2/2}$	$\dfrac{\text{Pressure Force}}{\text{Inertial force}}$	Flows involving pressure differences due to frictional effects
Froude number, Fr	v^2/gL	$\dfrac{\text{Inertial force}}{\text{Gravitational force}}$	Flows involving free liquid surfaces
Weber number, We	$\dfrac{\rho v^2 L}{\sigma}$	$\dfrac{\text{Inertial force}}{\text{Surface tension force}}$	Flows with significant surface tension effects
Mach number, M	v/C	$\dfrac{\text{Inertial force}}{\text{Compressibility force}}$	Flows with significant compressibility effects

If two flows occur in geometrically similar systems in which viscous effects and interfacial phenomena occur, and if *Re, We, Fr* are the same in both systems, the two systems are dynamically similar

Inkjet printing



Goal: to know how much a drop spreads on impact with a surface

$$V = 8\, nanoliters = 8 \times 10^{-12} m^3, \quad \mu = 0.005 Pa \cdot s, \quad \sigma = 0.04 N/m, \quad U = 1 m/s$$

We want to design a scaled-up version of this process to facilitate observation and measurement of the drop dynamics
(designing a dynamically similar system)

Length scale $L = V^{1/3} = (8 \times 10^{-12})^{1/3}\,\mathrm{m} = 2 \times 10^{-4}\,\mathrm{m}$

Velocity scale $U = 1\,\mathrm{m/s}$

Assume gravity is not a significant factor (this may not be the case)
-> ignore *Froude* number

$$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{1000(1)0.0002}{0.005} = 40 \qquad \mathrm{We} = \frac{\rho U^2 L}{\sigma} = \frac{1000(1)^2(0.0002)}{0.04} = 5$$

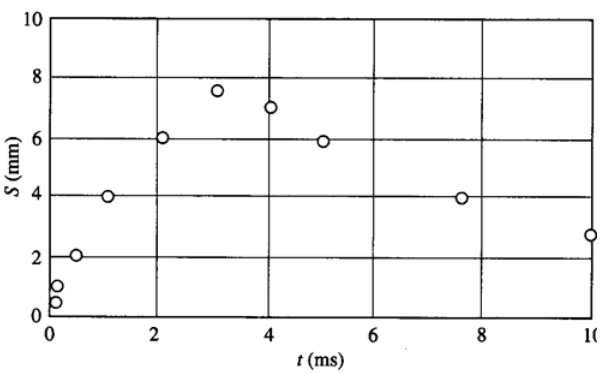
We must design an experiment such that

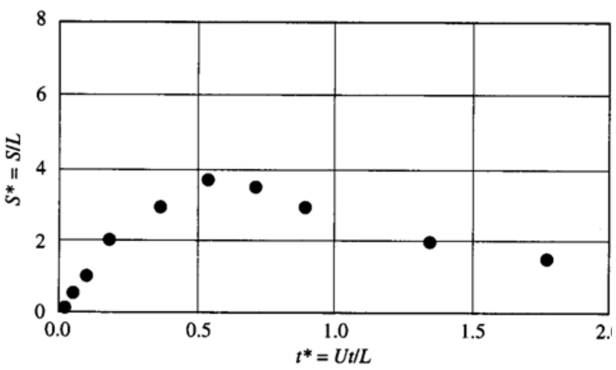
$$\mathrm{Re} = 40 = \left(\frac{\rho U L}{\mu}\right)_{\mathrm{exp}} \qquad \mathrm{We} = 5 = \left(\frac{\rho U^2 L}{\sigma}\right)_{\mathrm{exp}}$$

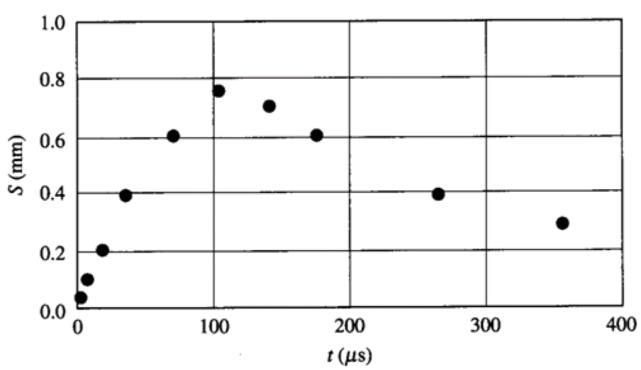
5 parameters, 2 constrains -> 3 free parameters

$$\sigma = 0.05\,\mathrm{N/m}, \quad \rho = 1000\,\mathrm{kg/m^3}, \quad \underline{L = 0.002m}$$

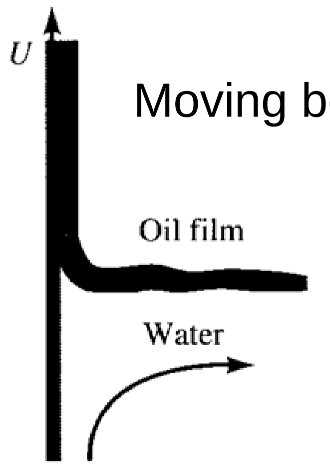
$$\Rightarrow \quad U = 0.354\,\mathrm{m/s}, \quad \mu = 0.018\ \mathrm{Pa \cdot s}$$







Removing oil from water surface



Moving belt with hydrophobic surface

Goal: design an experimental model to learn more about how the rate of oil entrainment is related to operating parameters such as belt speed and physical properties

Dynamics are dependent on the balance between inertial, viscous, surface, gravitational forces

$$\mathrm{Re} = \left(\frac{\rho U L}{\mu}\right)_{\mathrm{model}} = \left(\frac{\rho U L}{\mu}\right)_{\mathrm{system}} \quad \mathrm{We} = \left(\frac{\rho U^2 L}{\sigma}\right)_{\mathrm{model}} = \left(\frac{\rho U^2 L}{\sigma}\right)_{\mathrm{system}} \quad \mathrm{Fr} = \left(\frac{U^2}{gL}\right)_{\mathrm{model}} = \left(\frac{U^2}{gL}\right)_{\mathrm{system}}$$

5 parameters, 3 constraints -> 2 free parameters

$$L_{\mathrm{model}} = k L_{\mathrm{system}} \qquad \rho_{\mathrm{model}} = \rho_{\mathrm{system}}$$

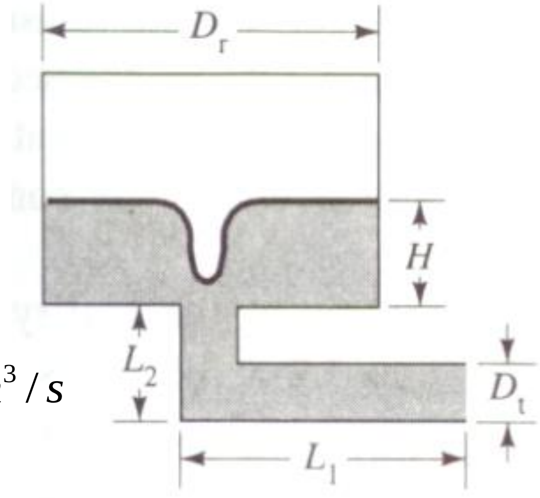
$$U_{\mathrm{model}} = k^{1/2} U_{\mathrm{system}}$$

$$\mu_{\mathrm{model}} = k^{3/2} \mu_{\mathrm{system}}$$

$$\sigma_{\mathrm{model}} = k^{2} \sigma_{\mathrm{system}}$$

1. Large k does not meet surface tension
2. Cannot take all three dimensionless group together
3. Need more physics to make knowledgeable evaluations of the relative importance of the three dynamic groups, *Re, We, Fr*

Experimental design



At some critical height, the free surface forms a vortex that is sucked into the tube, entraining air in the liquid; we wish to avoid

$$D_r = 1\text{m} \quad D_t = 0.03\text{m} \quad L_1 = 0.5\text{m} \quad L_2 = 0.3\text{m} \quad Q = 1.2\times10^{-2}\, m^3/s$$

$$\mu = 2.4\,\text{Pa}\cdot\text{s} \quad \rho = 2400\,\text{kg/m}^3 \quad \sigma = 0.25\,\text{N/m}$$

Molten ceramic at 1000K in large reservoir
-> need scale-down

Dynamic similarity $\quad \text{Re}_{\text{model}} = \text{Re}_{\text{real}} \quad \text{Fr}_{\text{model}} = \text{Fr}_{\text{real}} \quad \text{We}_{\text{model}} = \text{We}_{\text{real}}$

If the length scale is large, the radius of curvature of the vortex may be so large that surface tension effect will be negligible

$$\text{Bo} = \frac{\text{We}}{\text{Fr}} = \frac{\rho g L^2}{\sigma} = \frac{2400\times9.8\times0.03^2}{0.25} = 84.7$$

Surface tension effect is unimportant -> neglect *We*

$$\mathrm{Re} = \frac{\rho U D_{\mathrm{t}}}{\mu} = \frac{4\rho Q}{\pi D_{\mathrm{t}} \mu} = 509 \qquad \mathrm{Fr} = \frac{U^2}{g D_{\mathrm{t}}} = \frac{16 Q^2}{\pi^2 g D_{\mathrm{t}}^5} = 980$$

3 parameters (tube diameter, flow rate, kinematic viscosity), 2 constraints
-> select tube diameter 0.3*cm* (scale down by one order of magnitude)

$$\left[\frac{Q}{\nu D_{\mathrm{t}}}\right]_{\mathrm{model}} = \left[\frac{Q}{\nu D_{\mathrm{t}}}\right]_{\mathrm{real}} \qquad \left[\frac{Q^2}{D_{\mathrm{t}}^5}\right]_{\mathrm{model}} = \left[\frac{Q^2}{D_{\mathrm{t}}^5}\right]_{\mathrm{real}}$$

$$Q_{\mathrm{model}} = 3.8\times 10^{-5}\ \mathrm{m^3/s} = 38\ \mathrm{cm^3/s}$$

$$\nu_{\mathrm{model}} = 3.16\times 10^{-5}\ \mathrm{m^2/s}$$

$$\mu_{\mathrm{model}} = 3.16\times 10^{-2}\ \mathrm{Pa\cdot s}$$

Test with an aqueous solution of corn syrup or glycerol at 1/10 of full scale (geometrically and dynamically similar)

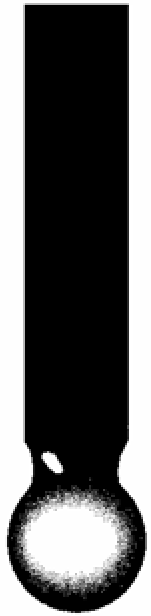
$$\mathrm{Bo} = \frac{1000(9.8)(0.003)^2}{0.06} = 1.5$$

Surface tension may be important in this small length scale

Need more experiments with liquids of several surface tensions and look for any influence

Dimensional analysis

The problem of a drop of liquid formed at the lower end of a vertical capillary



Step 1: Make a list of the relevant parameters
requires a good sense of the physics of the process

Step 2: List the fundamental dimensions of each parameter

$$V_{\text{drop}}\ [=]\ L^3 \quad D_{\text{c}}\ [=]\ L \quad \rho\ [=]\ m/L^3 \quad \sigma\ [=]\ m/t^2$$

$$g\ [=]\ L/t^2 \quad \mu\ [=]\ m/Lt \quad U\ [=]\ L/t$$

The effect of temperature arises only through its effect on the physical properties

Step 3: Determine, from the Buckingham pi theorem, the number of dimensionless groups that characterize this problem

# of independent dimensionless groups = # of parameters – $\underbrace{\text{\# of dimensions}}_{\text{rank of dimensional matrix}}$
(4 = 7 – 3)

Step 4: From the list of the independent parameters (all but V_{drop}) select a number equal to D (=3, in this case) that will be used as 'recurring parameters'.

It is wise to pick a set of parameters that include all the dimensions

$$D_{\mathrm{c}}\ , \rho\ , \sigma$$

Step 5: Form, in turn, dimensionless groups that are proportional to each of the remaining nonrecurring parameters

recurring parameters

$$\downarrow$$

$$V^*_{\mathrm{drop}} = V_{\mathrm{drop}}[D^a_{\mathrm{c}} \sigma^b \rho^c]$$

$\uparrow$ nonrecurring parameter

$$g^* = g D^a_{\mathrm{c}} \sigma^b \rho^c$$

$$U^* = U D^a_{\mathrm{c}} \sigma^b \rho^c$$

$$\mu^* = \mu D^a_{\mathrm{c}} \sigma^b \rho^c$$

a,b,c are different for each of these four equations

Step 6: For each of the four equations above, solve for the set of exponents

$$V^*_{\mathrm{drop}} = V_{\mathrm{drop}} D_{\mathrm{c}}^a \sigma^b \rho^c$$

$$m^0 L^0 t^0 = L^3 L^a (m/t^2)^b (m/L^3)^c$$

$$m: \quad 0 = b + c$$

$$L: \quad 0 = 3 + a - 3c$$

$$t: \quad 0 = -2b$$

$$b = 0, c = 0, a = -3$$

$$V^*_{\mathrm{drop}} = \frac{V_{\mathrm{drop}}}{D_{\mathrm{c}}^3}$$

$$m^0 L^0 t^0 = (L/t^2) L^a (m/t^2)^b (m/L^3)^c$$

$$0 = b + c$$

$$0 = 1 + a - 3c$$

$$0 = -2 - 2b$$

$$b = -1 \quad c = 1 \quad a = 2$$

$$g^* = \frac{g\rho D_{\mathrm{c}}^2}{\sigma}$$

$$m^0 L^0 t^0 = (L/t) L^a (m/t^2)^b (m/L^3)^c$$

$$0 = .b + c$$

$$0 = 1 + a - 3a$$

$$0 = -1 - 2b$$

$$b = -\tfrac{1}{2} \quad c = \tfrac{1}{2} \quad a = \tfrac{1}{2}$$

$$U^* = U\left(\frac{D_{\mathrm{c}}\rho}{\sigma}\right)^{1/2}$$

$$m^0 L^0 t^0 = (m/Lt) L^a (m/t^2)^b (m/L^3)^c$$

$$0 = 1 + b + c$$

$$0 = -1 + a - 3c$$

$$0 = -1 - 2b$$

$$b = -\tfrac{1}{2} \quad c = -\tfrac{1}{2} \quad a = -\tfrac{1}{2}$$

$$\mu^* = \mu\left(\mathrm{D}_{\mathrm{c}}\rho\sigma\right)^{-1/2}$$

$$V^*_{\mathrm{drop}} \equiv \frac{V_{\mathrm{drop}}}{\mathrm{D}_{\mathrm{c}}^3} = f\left[U\left(\frac{\rho D_{\mathrm{c}}}{\sigma}\right)^{1/2}, \frac{\rho g D_{\mathrm{c}}^2}{\sigma}, \frac{\mu}{(\rho\sigma D_{\mathrm{c}})^{1/2}}\right]$$

Dispersion of an oil stream in an aqueous pipe flow

Predict the mean droplet diameter as a function of the parameters that characterize this flow

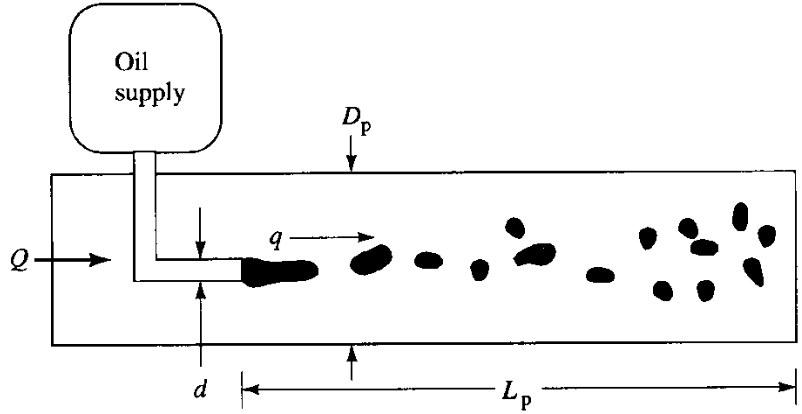


Figure 1.2.3 An oil stream is dispersed into droplets by a surrounding aqueous flow.

Step 1: Make a list of parameters.

Step 2: List the fundamental dimensions.

$$\overline{D}\,[=]\,L \quad D_p\,[=]\,L \quad L_p\,[=]\,L \quad d\,[=]\,L \quad \rho\,[=]\,m/L^3 \quad \sigma\,[=]\,m/t^2$$

$$\mu\,[=]\,m/Lt \quad Q\,[=]\,L^3/t \quad q\,[=]\,L^3/t \quad \rho'\,[=]\,m/L^3 \quad \mu'\,[=]\,m/Lt$$

Step 3: Use the Buckingham pi theorem.

The number of fundamental dimensionless groups = 11 – 3 = 8

Step 4: Select the recurring parameters. $D_{\mathrm{p}}, \rho, \sigma$

Step 5: Form, in turn, dimensionless groups.

$$\overline{D}^* = \overline{D} D_{\mathrm{p}}^a \sigma^b \rho^c \quad L_{\mathrm{p}}^* = L_{\mathrm{p}} D_{\mathrm{p}}^a \sigma^b \rho^c \quad \mu^* = \mu D_{\mathrm{p}}^a \sigma^b \rho^c \quad \mu'^* = \mu' D_{\mathrm{p}}^a \sigma^b \rho^c$$

$$\rho'^* = \rho' D_{\mathrm{p}}^a \sigma^b \rho^c \quad d^* = d D_{\mathrm{p}}^a \sigma^b \rho^c \quad q^* = q D_{\mathrm{p}}^a \sigma^b \rho^c \quad Q^* = Q D_{\mathrm{p}}^a \sigma^b \rho^c$$

Step 6: Solve for the coefficients for each of the equations.

$$\overline{D}^* = \overline{D} D_{\mathrm{p}}^a \sigma^b \rho^c$$

$$m^0 L^0 t^0 = L\, L^a (m/t^2)^b (m/L^3)^c$$

$$m: \quad 0 = b + c$$

$$L: \quad 0 = 1 + a - 3b$$

$$t: \quad 0 = -2b$$

$$b = 0 \quad c = 0 \quad a = -1$$

$$\overline{D}^* = \frac{\overline{D}}{D_{\mathrm{p}}}$$

$$m^0 L^0 t^0 = (m/Lt) L^a (m/t^2)^b (m/L^3)^c$$

$$0 = 1 + b + c$$

$$0 = -1 + a - 3c$$

$$0 = -1 - 2b$$

$$b = -\tfrac{1}{2} \quad c = -\tfrac{1}{2} \quad a = -\tfrac{1}{2}$$

$$\mu^* = \mu \left(D_{\mathrm{p}} \rho \sigma\right)^{-1/2}$$

$$\overline{D}^* = \frac{\overline{D}}{D_\mathrm{p}} = F\left[\mu\,(D_\mathrm{p}\rho\sigma)^{-1/2}, \mu'\,(D_\mathrm{p}\rho\sigma)^{-1/2}, \frac{\rho'}{\rho}, \frac{d}{D_\mathrm{p}}, \frac{L_\mathrm{p}}{D_\mathrm{p}}, Q\left(\frac{\rho}{\sigma D_\mathrm{p}^3}\right)^{1/2}, q\left(\frac{\rho}{\sigma D_\mathrm{p}^3}\right)^{1/2}\right]$$

$$\overline{D}^* = \frac{\overline{D}}{D_\mathrm{p}} = F\left[\mu'\,(D_\mathrm{p}\rho\sigma)^{-1/2}, \frac{\mu'}{\mu}, \frac{\rho'}{\rho}, \frac{d}{D_\mathrm{p}}, \frac{L_\mathrm{p}}{D_\mathrm{p}}, Q\left(\frac{\rho}{\sigma D_\mathrm{p}^3}\right)^{1/2}, \frac{q}{Q}\right]$$

Speculation about the physics of the process

- viscosity of oil is not significant if it is comparable to that of water
- inlet tube diameter is of no significance if it is large compared to the drop size
- most liquid densities lie in a narrow range -> no effect of density ratio
- as long as pipe length is large, drop size reaches equilibrium and does not change
- if *q/Q* is small, it does not affect the drop breakup

$$\overline{D}^* = \frac{\overline{D}}{D_\mathrm{p}} = F\left[Q\left(\frac{\rho}{\sigma D_\mathrm{p}^3}\right)^{1/2}\right]$$