

# 446.328 Mechanical System Analysis

## 기계항공시스템해석

- lecture 12,13,14 -

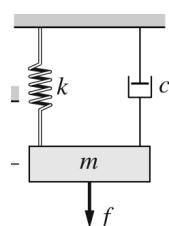
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### State-Space Example 1



system dynamics

$$m\ddot{x} + b\dot{x} + kx = f$$

input  $u = f$   
output  $y = f - b\dot{x} - kx$

transfer function

$$\frac{Y(s)}{U(s)} = \frac{ms^2}{ms^2 + bs + k}$$

state-space modeling

state variables:  $x_1 = x$ ,  $x_2 = \dot{x}$

$x_1, x_2$ : second-order system  
can completely describe  
the system's state

state-space dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

$y = [-k \quad -b] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 1 \times u$

state-space representation

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

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## State-Space Example 2

system dynamics

$$\ddot{y} + 3\dot{y} + 7y = f$$

input  $u = f$   
output  $y$

transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 3s^2 + 7s + 6}$$

state variables:  $x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y}$

state-space dynamics

$$\dot{x} = Ax + Bu, y = Cx + Du$$

first-order vector  
differential equation

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## State-Space Representation

state equation:  $\dot{x} = Ax + Bu,$

output equation:  $y = Cx + Du$

- state vector  $x \in \mathbb{R}^n$ : vector of state variables  $x = (x_1, x_2, \dots, x_n)$ , which completely describes system's state at each time ( $n =$ order of system)
- input vector  $u \in \mathbb{R}^r$ : control actuation, external force,...
- output vector  $y \in \mathbb{R}^m$ : sensor measurement, variables that we want to control,...
- state matrix  $A \in \mathbb{R}^{n \times n}$ : describe the system dynamics
- input matrix  $B \in \mathbb{R}^{n \times r}$ : characterizes how  $u$  directly affects each  $x_i$
- output matrix  $C \in \mathbb{R}^{m \times n}$ : effect of each state variable on  $y$
- direct feedthrough matrix  $D \in \mathbb{R}^{m \times r}$ : direct effect of  $u$  in  $y$  (see figure!)
- state-space: space of state-vector, that is,  $\mathbb{R}^n$  in this case;  
system evolution can then be thought of as a trajectory in its state-space

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## State-Space vs Transfer Function

$$\frac{Y(s)}{U(s)} = \frac{ms^2}{ms^2 + bs + k}$$

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \\ y &= [-k \quad -b] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 1 \times u \end{aligned}$$

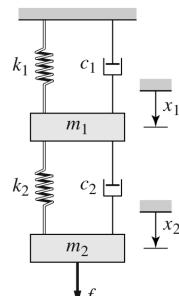
### transfer function representation

- classical frequency-domain approach (more EE...)
- easy algebra (EoM as a function; interconnected system as blocks)
- easy to see effects of parameters (e.g., stability, convergence/oscillation, FVT,...)
- **only for linear time-invariant system w/ zero initial condition (or in steady-state)**

### state-space representation

- modern time-domain approach (more ME/AE...)
- applicable to systems with nonlinearity (e.g., robots, saturation, backlash), time-varying coefficients, and non-zero initial condition
- **less transparent: effect of parameters is not so easy to see**

## Example: Two Mass Model



### equation of motion

$$\begin{aligned} m\ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) &= f \\ m\ddot{x}_1 + c\dot{x}_1 + kx_1 + c(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) &= 0 \end{aligned}$$

### 4-th order system: 4 states

$z_1 = x_1$	<u>input</u>
$z_2 = \dot{x}_1$	$u = f$
$z_3 = x_2$	<u>output</u>
$z_4 = \dot{x}_2$	$y = [x_1, x_2]^T$

completely describes  
system's state

$$\begin{aligned} m_1 &= m_2 = m \\ k_1 &= k_2 = k \\ c_1 &= c_2 = c \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m & -c/m & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m \end{bmatrix} u \\ y &= Cz + Du \end{aligned}$$

## From SS to TF

state-space representation

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$



transfer function

$$\begin{aligned}sX(s) - x(0) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s)\end{aligned}$$

$$\begin{aligned}Y(s) &= [C(sI - A)^{-1}B + D]U(s) \\ &= H(s)U(s)\end{aligned}$$

$$H(s) = C(sI - A)^{-1}B + D \in \mathbb{C}^{m \times r}$$

- for SISO (single-input-single-output) system,  $H(p) \rightarrow \infty$  if  $p$  is a pole of  $H(s)$

→ for a pole  $p$ ,  $(pI - A)^{-1}$  doesn't exist →  $\det(pI - A) = 0$

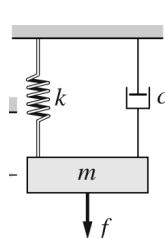
→ there is  $x \neq 0$  s.t.  $(pI - A)x = 0 \rightarrow Ax = px$  ← eigenvalue problem!

→ a pole  $p$  of  $H(s)$  is an eigenvalue of  $A$  (i.e.,  $|pI - A| = 0$ )

- some dynamics in SS can be removed (or hidden) when converted to TF

→ watch out for **unstable pole-zero cancelation!**

## From SS to TF: Examples



state-space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

$$y = [-k \quad -b] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 1 \times u$$

transfer function

$$\frac{Y(s)}{U(s)} = \frac{ms^2}{ms^2 + bs + k}$$

$$m\ddot{x} + b\dot{x} + kx = f$$

$$H(s) = C(sI - A)^{-1}B + D$$

input  $u = f$

output  $y = f - b\dot{x} - kx$

$$m = 1, b = 2, k = 5$$

$$\text{eig}(A) = -1 \pm 2j$$

## From SS to TF: Example 2

$$\begin{aligned}\dot{x} &= -3\dot{x} - 2x + u \\ y &= -2\dot{x} - 2x + u\end{aligned}$$

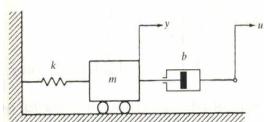
$$\dot{z} = Az + Bu, y = Cz + Du$$

$$H(s) = \frac{s(s+1)}{(s+1)(s+2)} = \frac{s}{s+2}$$

- this removed dynamics ( $s+1$ ) is unobservable from  $Y(s) = H(s)U(s)$
- if this removed dynamics were unstable (e.g.,  $s-1$  instead of  $s+1$ ),  
 $\rightarrow$  unstable pole-zero cancelation  $\rightarrow$  internal breakage!!!

## Systems with Input Derivatives

equation of motion



$$m\ddot{y} + b\dot{y} + ky = b\dot{u}$$

choose  $x_1 = y, x_2 = \dot{y}$

$$\dot{x} = Ax + Bu, y = Cx + Du$$

???

method 1: include  $u$  in state variables

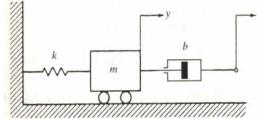
$$\ddot{y} - \frac{b}{m}\dot{u} = -\frac{b}{m}\dot{y} - \frac{k}{m}y = -\frac{b}{m}\left(\dot{y} - \frac{b}{m}u\right) - \frac{k}{m}y - \left(\frac{b}{m}\right)^2 u$$

$$\text{choose } x_2 = \dot{y} - \frac{b}{m}u, x_1 = y$$

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

## System with Input Derivatives

equation of motion



$$m\ddot{y} + b\dot{y} + ky = b\dot{u}$$

position input  $u$

method 2: use intermediate variable  $z$

$$\frac{Y(s)}{U(s)} = H(s) = \frac{bs}{ms^2 + bs + k} \times \frac{Z(s)}{Z(s)}$$

choose  $x_1 = z, x_2 = \dot{z}$

$\dot{x} = Ax + Bu, \quad y = Cx$

## Controllable Canonical Form

TF with numerator dynamics

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{ds^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + a_2 s^2 + a_1 s^1 + a_0} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s^1 + a_0} + d \\ &= C(sI - A)^{-1}B + d \rightarrow Y(s) = Y_1(s) + Y_2(s) \end{aligned}$$

$$\frac{Y_1(s)}{U(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s^1 + a_0} \times \frac{Z(s)}{Z(s)}$$

choose  $x_1 = z, x_2 = \dot{z}, x_3 = \ddot{z}$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_0 \quad b_1 \quad b_2] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + d \times u$$

controllable canonical form:  
directly derivable  
from TF!  
signal diagram?

## Observable Canonical Form

TF with numerator dynamics

$$\frac{Y(s)}{U(s)} = \frac{ds^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} + d = C(sI - A)^{-1}B + d$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 0 & -a_o \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 1] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + d \times u$$

observable  
canonical form:  
directly derivable  
from TF!

ex:  $H(s) = \frac{2s^2 + 5s + 4}{s^2 + 2s + 1} = 2 + \frac{s + 2}{s^2 + 2s + 1}$        $H(s) = \frac{2}{s^2 + 2s + 1}$

## Controllability and Observability

\* controllable: we can drive  $x(0)$  to any desired  $x(t)$  by  $u(t)$  in a finite-time  $t$   
 $\Leftrightarrow [B \ AB \ A^2B \ \dots \ A^{n-1}B]$  has rank  $n$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_o \quad b_1 \quad b_2] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + d \times u$$

controllable  
canonical form

\* observable: we can estimate  $x(t)$  if we know  $y(t)$  and  $u(t)$   
 $\Leftrightarrow [C^T \ A^T C^T \ (A^T)^2 C^T \ \dots \ (A^T)^{n-1} C^T]$  has rank  $n$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 0 & -a_o \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 1] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + d \times u$$

observable  
canonical form

## Non-Unique SS Representation

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} + d$$

controllable canonical form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_0 \quad b_1 \quad b_2] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + d \times u$$

observable canonical form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 1] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + d \times u$$

\* Kalman canonical form: controllable *and* observable with the same TF

(e.g.,  $H(s) = \frac{s+1}{(s+1)(s+2)}$  : CTRB and OBSV forms both 2-dim, yet, Kalman 1-dim)

coordinate transformation:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \quad \xrightarrow{x = P\hat{x}} \quad \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, \quad y = \hat{C}\hat{x} + Du$$

$$\hat{A} = P^{-1}AP, \hat{B} = P^{-1}B, \hat{C} = CP$$

are they really equivalent?  $A$  and  $\hat{A}$  possess the same eigenvalues, same TF, ...

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## Jordan Diagonalization

for any  $A \in \mathbb{R}^{n \times n}$ , there is a non-singular  $P \in \mathbb{R}^{n \times n}$  s.t.

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

\* each column of  $P$  is an eigenvector of  $A$  with  $\lambda_i$

\* off-diagonal "1" shows up for  $\lambda_i$  if # of eigenvectors < # of repetition of  $\lambda_i$

e.g.:  $\lambda_2$  has two eigenvectors  $v_1, v_2$  (i.e.,  $\text{rank}(\lambda_2 I - A) = 1$ )

$\lambda_3$  has only one eigenvector  $v_1$  (i.e.,  $\text{rank}(\lambda_3 I - A) = 2$ )

choose  $v_2$  s.t.  $Av_2 = v_1 + \lambda v_2$  (generalized eigenvector)

$$\text{ex)} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow \text{same } \lambda_i \text{ but different # of } v_i$$

\* if all  $\lambda_i$  are distinct,  $P^{-1}AP = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$

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## Jordan Canonical Form

coordinate transformation:

$$\begin{array}{l} \dot{x} = Ax + Bu, \quad y = Cx + Du \\ \quad \quad \quad x = P\hat{x} \end{array} \rightarrow \begin{array}{l} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, \quad y = \hat{C}\hat{x} + Du \\ \hat{A} = P^{-1}AP, \hat{B} = P^{-1}B, \hat{C} = CP \end{array}$$

\* if all  $\lambda_i$  are distinct,  $\hat{A} = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$

$$\dot{\hat{x}}_i = \lambda_i \hat{x}_i + u'_i \quad \text{each mode decoupled!}$$

$$\hat{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

$$\hat{x}_i(t) = e^{\lambda_i t} \hat{x}_i(0) + \int_0^t e^{\lambda_i(t-\tau)} u'_i(\tau) d\tau$$

free response                              forced response  
(convolution  $e^{\lambda_1 t} \Theta u'_i(t)$ )

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## Example

$$\begin{array}{l} \dot{x} = Ax + Bu, \quad y = Cx + Du \\ \quad \quad \quad x = P\hat{x} \end{array} \rightarrow \begin{array}{l} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, \quad y = \hat{C}\hat{x} + Du \\ \hat{A} = P^{-1}AP, \hat{B} = P^{-1}B, \hat{C} = CP \end{array}$$

$$\ddot{y} + 6\dot{y} + 11\dot{y} + 6y = 6u \rightarrow H(s) = \frac{6}{s^3 + 6s^2 + 11s + 6}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}, \quad \hat{A} = \Lambda = \begin{bmatrix} -1 & & \\ & -2 & \\ & & -3 \end{bmatrix}$$

$$\hat{B} = P^{-1}B = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}, \quad \hat{C} = CP = [1 \quad 1 \quad 1]$$

$$\dot{\hat{x}}_1 =$$

$$\dot{\hat{x}}_2 =$$

$$\dot{\hat{x}}_3 =$$

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## Solution of SS Equation

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

1. consider first  $\dot{x} = Ax$  with  $x(0) \in \Re^n$

$$x(t) = e^{At}x(0) \quad \text{free response}$$

$$e^{At} := I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots \quad \text{matrix exponential}$$

$$\frac{d}{dt}e^{At} = Ae^{At}, \quad e^{A(s+t)} = e^{As}e^{At}, \quad L[e^{At}] = (sI - A)^{-1}$$

2. consider then  $\dot{x} = Ax + Bu$  with  $x(0)$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

↑                      ↑  
free response        forced response

## Example

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, u = 1$$

1. using  $L[e^{At}] = (sI - A)^{-1} \rightarrow e^{At} = L^{-1}[(sI - A)^{-1}]$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \rightarrow (sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$x(t) = e^{At}x(0) + \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

2. using eigen-decomposition

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \Rightarrow \hat{A} = \Lambda, \hat{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x(t) = P\hat{x}(t), \quad \hat{x}(0) = P^{-1}x(0)$$

## Jordan Form and Multiplicity

- Consider LTI system  $\dot{x} = Ax + Bu$  with transition matrix

$$\Phi(t, 0) = e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

- Recall the eigen-problem of  $A$ , i.e.,  $A\nu_i = \lambda_i\nu_i$ , where  $\lambda_i \in \mathbb{C}$  and  $\nu_i \in \mathbb{C}^n$  are eigenvalue and eigenvector. Then, for each  $\lambda_i$ ,

algebraic multiplicity of  $\lambda_i \geq$  geometric multiplicity of  $\lambda_i$

i.e., order of  $(s - \lambda_i)$  in CE  $\geq$  number of independent eigenvectors.

- **Jordan form:** for  $A \in \mathbb{R}^{4 \times 4}$  with deficient  $\lambda_2$ ,

$$A = T \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} T^{-1} \Rightarrow e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} \end{bmatrix} T^{-1}$$

from  $AT = \Lambda T$ , where  $T \in \mathbb{R}^{4 \times 4}$  is collection of (generalized) eigenvectors.

- Stable if all  $\lambda_i \in$  LHP; marginally stable if all  $\lambda_i \in$  LHP except some non-deficient  $\lambda_i$  on  $jw$ -axis; unstable if some  $\lambda_i \in$  RHP or deficient  $\lambda_i$  on  $jw$ -axis (if deficiency = 2  $\rightarrow$  growth w/  $t^2$ :  $[0, 0; 0, 0], [0, 1; 0, 0]$ ).