

M2794.25 Mechanical System Analysis

기계시스템해석

- lecture 6,7,8 -

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Lecture 6

- mechanical system modeling
- equivalent mass
- gears

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Lecture 7

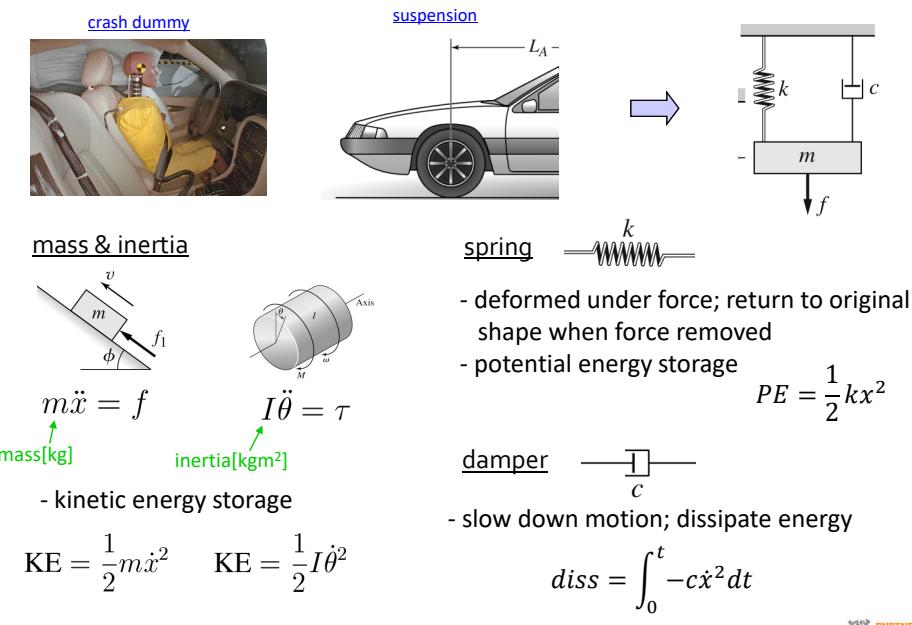
- planar dynamics
- energy method
- mechanical drives
- spring elements

Lecture 8

- natural frequency
- damper
- effect of damping

we are currently doing Chapter 4.

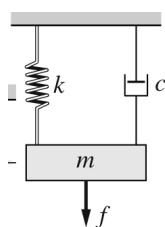
Mechanical System Modeling



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Mass-Spring-Damper Energetics



system dynamics

$$m\ddot{x} + b\dot{x} + kx = f$$

total energy

$$V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$\frac{dV}{dt} = -b\dot{x}^2 + f\dot{x}$$

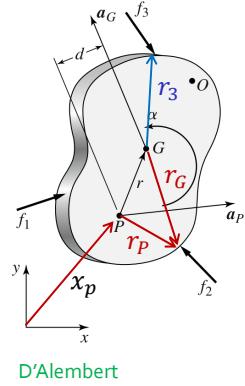
$$\Rightarrow V(t) - V(0) = - \int_0^t b\dot{x}^2 d\tau + \int_0^t f\dot{x} d\tau$$

damping dissipation energy input by f

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Mass/Inertia: 2D Case



D'Alembert

translation dynamics (w.r.t. CoM)

$$m\ddot{x}_G = f_1 + f_2 + f_3$$

rotation dynamics (w.r.t. CoM)

$$I_G \dot{\omega}_G = r_{G1} \times f_1 + r_{G2} \times f_2 + r_{G3} \times f_3 =: M_G$$

often, more convenient to use M_P or \ddot{x}_P

$$M_P = \sum r_{Pi} \times f_i = I_G \dot{\omega}_G + r \times m\ddot{x}_G = I_P \dot{\omega}_G + mr \times \ddot{x}_P$$

D'Alembert

$=0$ if P is fixed

$$\sum \vec{M}_A = \sum (\vec{M}_A)_{eff}$$

$$x_G = r + x_p \quad \dot{r} = \omega \times r$$

$$\ddot{x}_G = \ddot{x}_P + \dot{\omega} \times r + \omega \times (\omega \times r)$$

for 2D rotation about a fixed point p $\Rightarrow M_P = I_P \dot{\omega}_G, I_p = I_G + mr^2, \dot{\omega}_G = \dot{\omega}_p$

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CLASS NOTES

Example 2.4.2

rolling constraint (no sliding/bouncing)

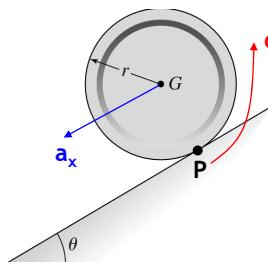
$$a_x = r\alpha \quad a_y = 0$$

translation dynamics

$$ma_x = mg \sin \theta - F$$

$$ma_y = N - mg \cos \theta$$

friction (unknown)



rotation dynamics w.r.t. G

$$I_G \alpha = Fr \quad F = \frac{I_G mg \sin \theta}{mr^2 + I_G} \quad a_x = \frac{mgr^2 \sin \theta}{mr^2 + I_G}$$

rotation dynamics w.r.t. P

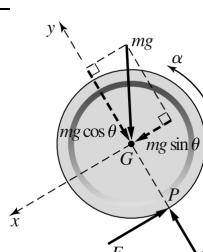
$$I_P \alpha = mgr \sin \theta \quad I_P = I_G + mr^2$$

no need to compute F!

minimum friction coefficient for no-slip

$$\mu_s N \geq F \quad \mu_s \geq \frac{I_G \tan \theta}{mr^2 + I_G}$$

F producible by $\mu_s N$



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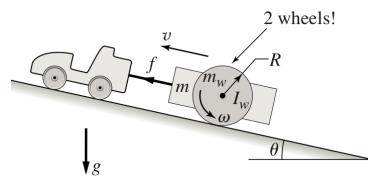
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CLASS NOTES

Energy Methods

energy conservation: a prelude to Lagrangian/Hamiltonian dynamics

$$\frac{dKE}{dt} + \frac{dPE}{dt} = fv - \text{dissipation}$$

example 2.3.1



rolling constraints: $v = R\omega$

$$\begin{aligned} KE &= \frac{1}{2}mv^2 + 2\frac{1}{2}m_wv^2 + 2\frac{1}{2}L_w\omega^2 \\ &= \frac{1}{2} \underbrace{\left(m + 2m_w + 2\frac{I_w}{R^2}\right)}_{m_e} v^2 \end{aligned}$$

can think of multiple masses
as one lumped mass

$$PE = (m + 2m_w)gh$$

system dynamics:

$$m_{eq}\dot{v} + (m + 2m_w)g \sin \theta = f$$

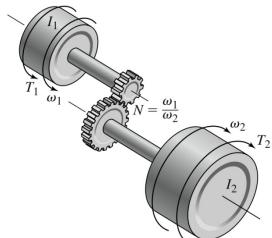
* no need to compute friction force F: why?

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Mechanical Drives

kinematic constraint: gear ratio



$$r_1w_1 = r_2w_2 \quad r_1 \approx n_1, r_2 \approx n_2$$

$$N = \frac{w_1}{w_2} = \frac{r_2}{r_1} = \frac{n_2}{n_1}$$

speed reducer
if $N > 1$

ideal gear box

$$|T_1w_1| = |T_2w_2| \quad |T_2| = |NT_1| \quad \text{torque amplification}$$

real gear box

$$|T_2| = \mu_e |NT_1| \quad \text{efficiency (friction, inertia, etc)} \approx 60 - 80\%$$

system dynamics

$$I_{eq}\dot{w}_1 = T_1 + T_2/N \rightarrow T_2 = -NT_1$$

$$\begin{aligned} I_{eq}^1 &= I_1 + \frac{I_2}{N^2} & w_2 &= \frac{1}{N}w_1 \\ I_{eq}^2 &= I_1N^2 + I_2 \end{aligned}$$

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Ex 2.3.5: Simple Robot Arm

kinematic constraint

$$w_2 = \frac{1}{N} w_1$$

energies

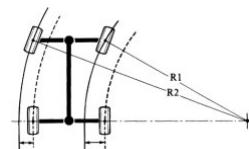
$$KE = \frac{1}{2} \left(I_m + I_{G1} + I_{S1} + \frac{I_{G2} + I_{S2} + I_L^m + mL^2}{N^2} \right) w_1^2$$

$$PE = mgL(1 - \cos \theta), \quad \dot{\theta} = w_2 \quad \text{motor feels "lighter" load}$$

system dynamics

$$I_{eq} \dot{w}_1 + \frac{mgL}{N} \sin \theta = T_m \quad \text{1-DOF system}$$

Differential Drive



- With different curvature from instantaneous center of rotation, yet, if one shaft connection:

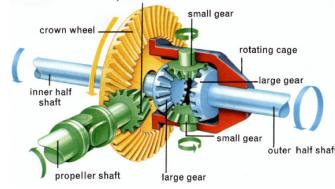
$$\dot{\theta}_1 = \dot{\theta}_2$$

thus, slip will always occur.

- Need to have different wheel speeds: Differential Drive

Differential Drive

- Crown wheel angle and torque (ϕ_{in}, τ_{in})
- Small gear angle and torque (θ_s, τ_s)
- Left shaft angle and torque (θ_1, τ_1)
- Right shaft angle and torque (θ_2, τ_2)



Differential drive modeling: with inertial effects assumed negligible,

- Kinematics relation of matched velocity:

$$v_1 = r_s \dot{\theta}_1 = r_s \dot{\phi}_i - r_s \dot{\theta}_s, \quad v_2 = r_s \dot{\theta}_2 = r_s \dot{\phi}_i + r_s \dot{\theta}_s$$

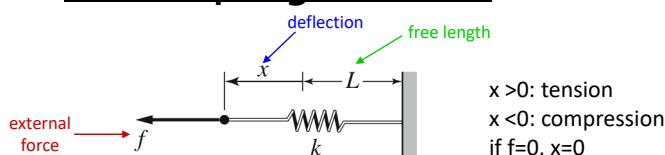
- Power relation with no energy loss assumed $\tau_i \dot{\phi}_i = \tau_1 \dot{\theta}_1 + \tau_2 \dot{\theta}_2$
- Small gear dynamics: $f_1 = \tau_1/r_s, f_2 = \tau_2/r_s, 0 = \tau_s = -f_1 r_s + f_2 r_s$
- This means τ_i is equally distributed s.t., $\tau_1 = \tau_2 = \tau_i/2$
- τ_i controlled, not $\dot{\theta}_i$, thus, possible to have $\dot{\theta}_1 \neq \dot{\theta}_2$ depending on τ_i .
- Problem when one side losses traction: locked differential or limited slip differential (LSD)

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Linear Spring Elements

- symbol



- force equation and potential energy

$$f = kx \quad \text{spring constant (or stiffness) [N/m]}$$

$$PE = \frac{1}{2} kx^2 \quad \text{quadratic potential}$$

- examples: $\sigma = E\epsilon$: small deformation, linear elastic material, etc

$$k = \frac{Gd^4}{64nR^3} \quad \text{shear modulus}$$

$$k = \frac{EA}{L} \quad \text{extensional modulus}$$

$$k = \frac{192EI_A}{L^3} \quad \text{bending modulus}$$

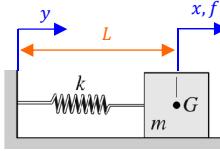
$$k = \frac{\pi G(D^4 - d^4)}{32L} \quad \text{torsional modulus}$$

* for more, see tables 4.1.1, 4.1.2

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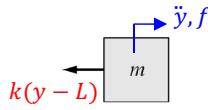
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Free Length and Equilibrium



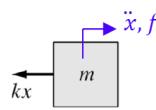
L : free-length
 x : from free-length
 y : from zero

1) w.r.t. y

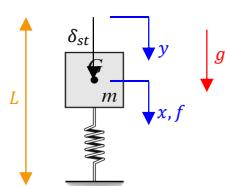


$$m\ddot{y} = -k(y - L) + f$$

2) w.r.t. x (from equilibrium L)

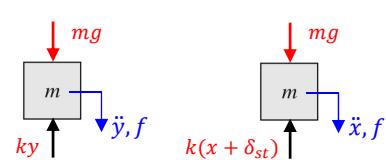


$$m\ddot{x} = -kx + f$$



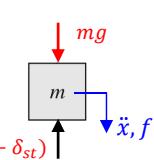
δ_{st} : static deflection = mg/k
 x : from equilibrium δ_{st}
 y : deflection from free-length

1) w.r.t. y



$$m\ddot{y} = -ky + mg + f$$

2) w.r.t. x (from equilibrium δ_{st})



$$\begin{aligned} m\ddot{x} &= -k(x + \delta_{st}) + mg + f \\ &= -kx + f \end{aligned}$$

w.r.t. equilibrium, we can write the system dynamics s.t.

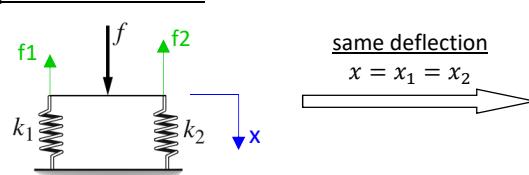
$$m\ddot{x} + kx = f$$

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Parallel and Serial Connections

- parallel connection



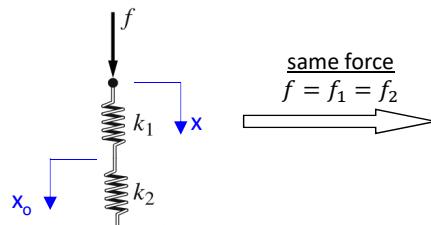
same deflection

$$x = x_1 = x_2$$

$$f = f_1 + f_2 = \underbrace{(k_1 + k_2)}_{=k_e} x$$

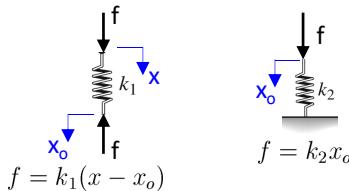
n-spring connection
 $k_e = \sum_{i=1}^n k_i$

- serial connection



same force

$$f = f_1 = f_2$$



$$f = k_1(x - x_o)$$

$$x = \frac{f}{k_1} + \frac{f}{k_2} = \underbrace{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)}_{=\frac{1}{k_e}} f$$

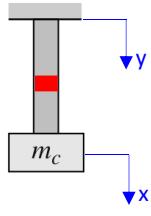
n-spring connection

$$\frac{1}{k_e} = \sum_{i=1}^n \frac{1}{k_i}$$

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ENGINEERING MECHANICS

Distributed Inertia and Equivalent Mass



$L(t)$: rod length
 $x(t)$: end position
 r : initial coordinate ($0 - L$)
 $y(r, t)$: rod coordinate ($0 - L(t)$)

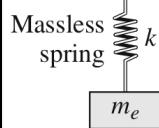
$$y(L, t) = L(t)$$

$$y(r, 0) = r, \quad y(0, t) = 0$$

mass per unit length: ρ

- infinitesimal mass ρdr moves with velocity $\dot{y}(r, t)$ at t
- linear velocity assumption

$$\dot{y}(r, t) = \dot{x}(t) \frac{r}{L}$$



$$KE(t) = \int_0^L \frac{1}{2} \dot{y}^2 \rho dr = \frac{1}{2} \left(\frac{m_d}{3}\right) \dot{x}^2(t)$$

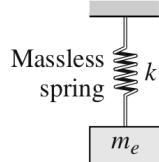
equivalent mass of rod

$$m_e = m_c + \frac{m_d}{3} \quad k = \frac{EA}{L}$$

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Natural Frequency and Rayleigh Method



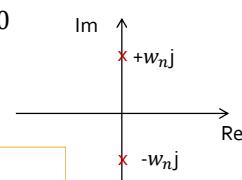
$$m\ddot{x} + kx = f, \quad x(0), \dot{x}(0), f = 0$$

$$X(s) = \frac{[initial condition]}{ms^2 + k}$$

simple harmonic oscillation

$$x(t) = A \sin(w_n t + \phi)$$

natural frequency



$$w_n = \sqrt{\frac{k}{m}}$$

- A (amplitude) and ϕ (phase) depend on the initial conditions
- if input $f(t)$ is also sinusoid with w_n , $x(t) \rightarrow \infty$ (resonance)

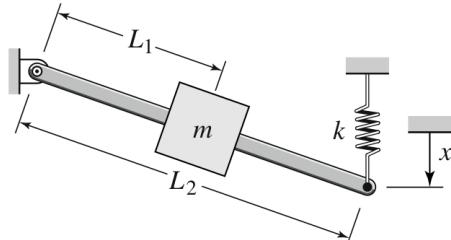
- Rayleigh Method: if $f(t) = 0$, $KE + PE$ is conserved

$$PE_{max} = \frac{1}{2} kA^2 = KE_{max} = \frac{1}{2} mw_n^2 A^2 \quad \Rightarrow \quad w_n = \sqrt{\frac{k}{m}}$$

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Example P.4.27



$x = 0$: equilibrium
 $m > 0$: mass
mass-less rod

$$KE = \frac{1}{2}m\left(\frac{L_1}{L_2}\dot{x}\right)^2$$

$$PE = \frac{1}{2}kx^2$$

$$w_n = \frac{L_2}{L_1} \sqrt{\frac{k}{m}}$$

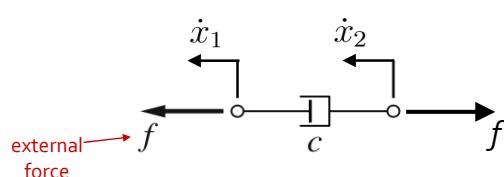
$$L_1 \rightarrow 0? L_1 \rightarrow L_2?$$

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Linear Damper

- symbol



- linear damper equation

$$f = c(\dot{x}_1 - \dot{x}_2)$$

↑ damping coefficient [N/(m/s)] ↗ relative velocity

- energy is always dissipated via damper

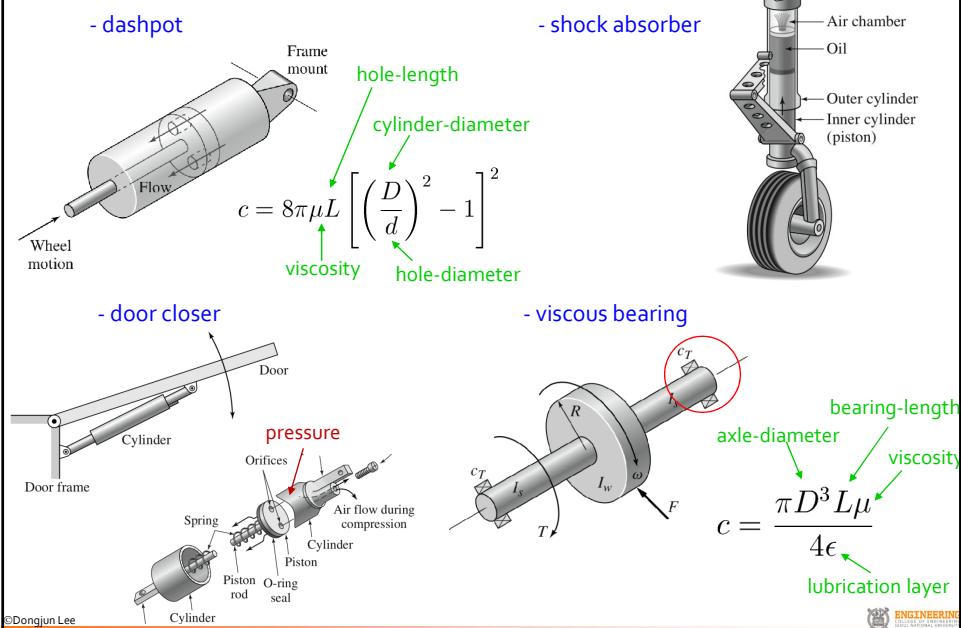
$$\text{diss} = \int_0^T cv^2 dt$$

↗ relative velocity

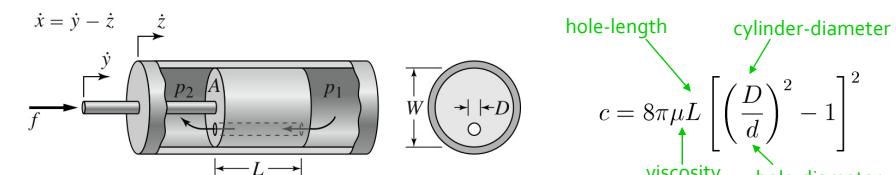
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Examples



Hydraulic Damper [Sec. 7.4]



- force equation

$$m\ddot{y} = f - A_{eq}(p_1 - p_2), \quad A_{eq} = A_{cylinder} - A_{hole}, \quad A_{rod} \approx 0$$

- continuity equation (incompressible)

$$q_v = A_{eq}\dot{y}$$

- laminar flow resistance

$$q_v = \frac{1}{\rho R} (p_1 - p_2), \quad R = \frac{128\mu L}{\pi \rho d^4}$$

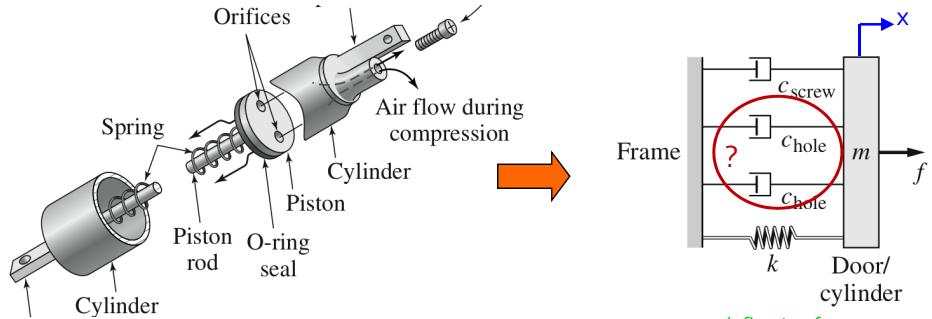
$$f_c = \rho R A_{eq}^2 \dot{y} = c\dot{y}, \quad c = \rho R A_{eq}^2$$

* if $L \rightarrow 2L$: $c_{eq} \rightarrow 2c$ (parallel)

* if two holes: $c_{eq} \rightarrow c/2$ (series)

$$c_{eq} = \frac{1}{2} 8\mu\pi L \left[\left(\frac{D}{d} \right)^2 - 2 \right]^2 \approx \frac{c}{2} \text{ if } D \gg d$$

Door Closer



- equation of motion

$$m\ddot{x} = f - \frac{1}{2}c_{hole}\dot{x} - c_{screw}\dot{x} - kx$$

- orifice equation

$$q = C_d A \sqrt{\Delta P}$$

$$m\ddot{x} + \frac{1}{2}c_{hole}\dot{x} + c_{screw}\dot{x} + kx = f$$

adjustable

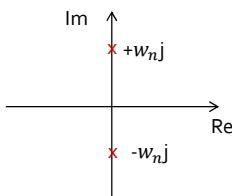
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Damped Oscillations

With non-zero initial condition $x(0), \dot{x}(0)$ w/ $f(t) = 0$

$$X(s) = \frac{[initial\ condition]}{ms^2 + bs + k}$$



- effect of damping

1) if $b = 0 \rightarrow s = \pm\sqrt{k/m}j = \pm w_n j \rightarrow$ keep oscillating (un-damped)

2) if $b < 2\sqrt{km} \rightarrow s = -\frac{b}{2m} \pm j\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ \rightarrow damped oscillation (under-damped)

$$x(t) = Ae^{-\frac{b}{2m}t} \sin(w_d t + \phi), \quad w_d = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \neq w_n$$

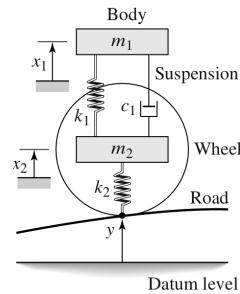
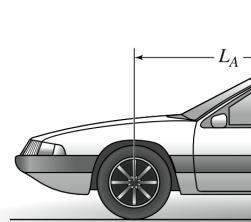
3) if $b = 2\sqrt{km} \rightarrow s = -\frac{b}{2m} \rightarrow$ no oscillation (critically-damped; optimal)

4) if $b > 2\sqrt{km} \rightarrow s = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}} \rightarrow$ no oscillation, but slow (over-damped)

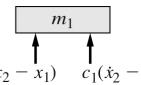
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Quad-Car Model

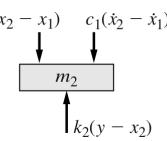


1) quarter car-body



$$m_1 \ddot{x}_1 = -c_1(\dot{x}_1 - \dot{x}_2) - k_1(x_1 - x_2)$$

2) wheel +tire +axle



$$m_2 \ddot{x}_2 = c_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) + k_2(y - x_2)$$

equation of motions are coupled:
need to be simultaneously solved

Next Lecture

- Lagrange Dynamics