- Theorem 5.22: dim(V)$< \infty$; $T : V \to V$ is a linear operator; $v \neq 0$;W =span(T-orb(v)); dim(W)$= k$. Then
  1. $\{v, T(v), T^2(v), \cdots, T^{k-1}(v)\}$ is a basis for W.
  2. $a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$
  $\Rightarrow$The characteristic polynomial of T_W is
  $g(t) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k)$.
- Theorem 5.23(Cayley-Hamilton):
  $f(t) = h(t)g(t)$, and $f(T) = h(T)g(T) = T_0$.
- Corollary 5.23(Cayley-Hamilton for Matrix):
  for $A = [T]_\beta$, $f(A) = \mathbf{0}$.

- Theorem 5.24: dim(V)$<\infty$;
  $T: V \to V$ is a linear operator;
  $f(t)$ is the characteristic polynomial of T;
  W_i is a T-invariant subspace, $i = 1, \cdots, k$;
  $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$;
  $f_i(T)$ is the characteristic polynomial of T_{W_i} Then
  $f(t) = f_1(t) f_2(t) \cdots f_k(t)$

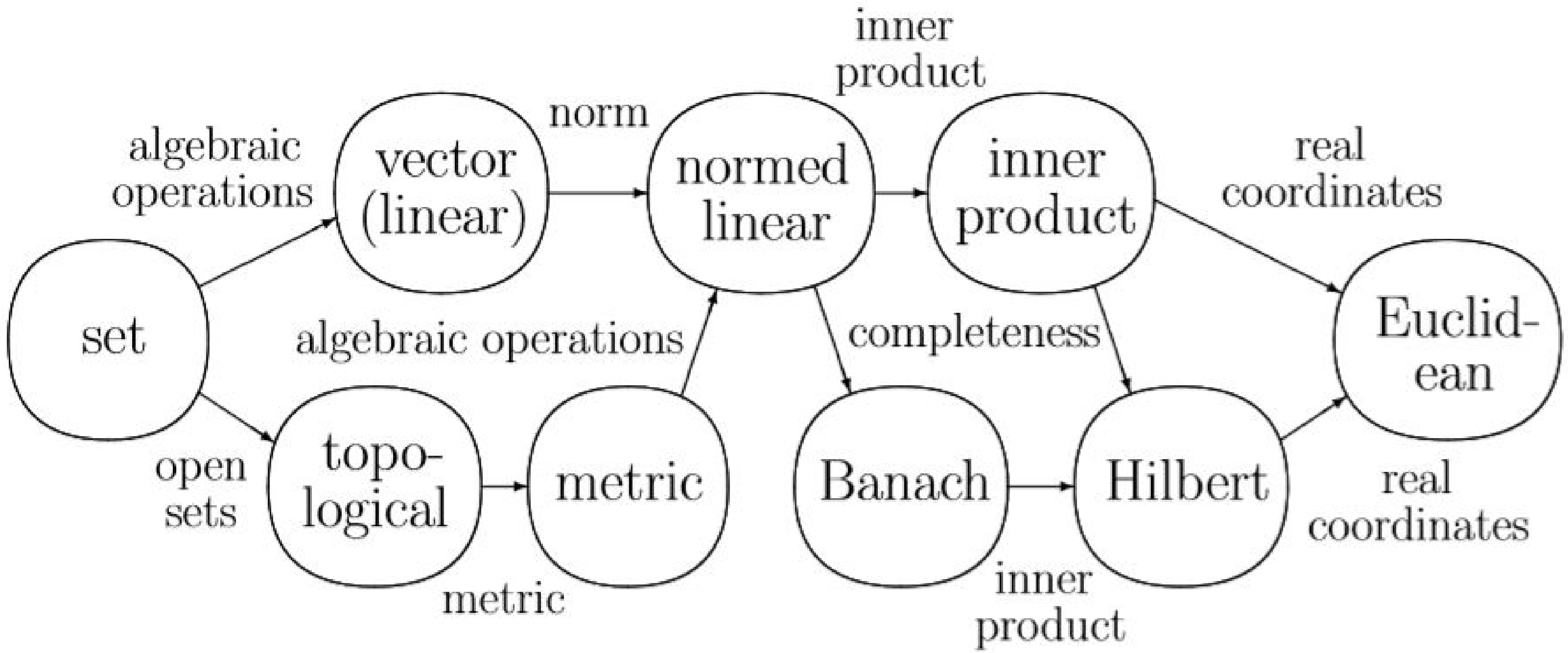
- Theorem 5.25: In addition to the above,
  β_i is a basis for W_i;
  $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis for V. [Thm 5.10]

$$\Rightarrow [T]_\beta = \begin{pmatrix} [T_{W_1}]_{\beta_1} & O_{12} & \cdots & O_{1k} \\ O_{21} & [T_{W_2}]_{\beta_2} & \cdots & O_{2k} \\ \vdots & \vdots & & \vdots \\ O_{k1} & O_{k2} & \cdots & [T_{W_k}]_{\beta_k} \end{pmatrix}$$

- [End of Review]

Inner product Space

- A norm gives length to a vector and, hence, provide the notion of distance for a vector space. $[\|x - y\|]$



- **norm** on V : $\|\cdot\| : V \to \mathbb{R}^+$ such that $\forall x, y \in V$ and $\forall a \in F$,

  1. $\|x\| \geq 0$
  2. $\|ax\| = |a| \cdot \|x\|$
  3. $\|x + y\| \leq \|x\| + \|y\|$: triangle inequality
  4. $\|x\| = 0 \Leftrightarrow x = \mathit{0}$

  - $\|x\| \geq 0$ is redundant because it is included in $\mathbb{R}^+$ and also
    $2 \Rightarrow \|\mathit{0}\| = \|0x\| = 0\|x\| = 0 \Rightarrow$ half of 4
    $3 \Rightarrow 0 = \|\mathit{0}\| = \|x - x\| \leq \|x\| + \|(-1)x\| = 2\|x\| \Rightarrow 1$
  - So norm is defined without the help of inner product.

- l_p norm, $p \geq 1$, of l_p space of $(x_1, x_2, \cdots)$ such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$: $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$
  - l_1 norm: $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ [diamond]
  - l_2 norm: $\|x\|_2 = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$ [circle]
  - l_∞ norm: $\|x\|_\infty = \mathrm{max}_i |x_i|$ [square]
  - These norms are applicable to F^n

- L_p norm, $p \geq 1$, of $L_p(a,b)$ space of functions $f(t)$ such that $\int_a^b |f(t)|^p dt < \infty : \|f\|_p = (\int_a^b |f(t)|^p dt)^{\frac{1}{p}}$
  - L_1 norm: $\|f\|_1 = \int_a^b |f(t)| dt$
  - L_2 norm: $\|f\|_2 = \sqrt{\int_a^b |f(t)|^2 dt}$
  - L_∞ norm: $\|f\|_\infty = \max_t |f(t)|$

- An inner product provides the notion of angle and orthogonality for a vector space.

- **inner product** on $V : \langle \cdot, \cdot \rangle : V \times V \to F$ such that $\forall x, y, z \in V$ and $\forall a \in F$,

  1. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$
  2. $\langle ax, y \rangle = a \langle x, y \rangle$
  3. $\overline{\langle x, y \rangle} = \langle y, x \rangle$, where $\overline{a}$ is the complex conjugate of a.
  4. $x \neq \mathit{0} \Rightarrow \langle x, x \rangle > 0$

     - 1 and 2 mean inner product is linear in the first argument.
     - 1 and 2 imply $\langle \sum_{i=1}^{k} a_i v_i, y \rangle = \sum_{i=1}^{k} a_i \langle v_i, y \rangle$.
     - 3 becomes $\langle x, y \rangle = \langle y, x \rangle$ if $F = \mathbb{R}$.
     - 3 implies that even if $F = \mathbb{C}$, $\langle x, x \rangle$ is real and nonnegative.

- example:
  - $V = F^n \Rightarrow \langle (a_1, \cdots, a_n), (b_1, \cdots, b_n) \rangle = \sum_{i=1}^n a_i \overline{b_i} = b^* a$
  - $V = C([a,b], \mathbb{R}) \Rightarrow \langle f, g \rangle = \int_a^b f(t)\overline{g(t)} dt$
  - $V = M_{n \times n}(F) \Rightarrow \langle A, B \rangle =tr(B^* A) \leftarrow$ (Frobenius inner product),
    where tr$(A) = \sum_{i=1}^n A_{ii}$ is the trace and $B^* = \overline{B^t}$ is the conjugate transpose, or adjoint, of A.

- **inner product space** : $(V(F), \langle \cdot, \cdot \rangle)$
  - In the discussion of inner product spaces, we implicitly assume that the field is complex as it encompasses the real field as a special case. These two fields are of our main interest.

- Theorem 6.1: V is an inner product space.

  Then $\forall x, y, z \in V$ and $c \in F$,

  1. $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$
  2. $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$
  3. $\langle x, \mathit{0} \rangle = \langle \mathit{0}, x \rangle = 0$
  4. $\langle x, x \rangle = 0 \Leftrightarrow x = \mathit{0}$
  5. $\forall x \in V, \langle x, y \rangle = \langle x, z \rangle \Rightarrow y = z$

     - 1,2 $\rightarrow$ conjugate linear.
     - 5 implies that $\forall x \in V, \langle x, y \rangle = 0 \Rightarrow y = \mathit{0}$
     - proof of 4: "$\Rightarrow$" contrapositive of def 4.
       "$\Leftarrow$": $\langle \mathit{0}, \mathit{0} \rangle = \langle \mathit{0}+\mathit{0}, \mathit{0} \rangle = \langle \mathit{0}, \mathit{0} \rangle + \langle \mathit{0}, \mathit{0} \rangle \Rightarrow \langle \mathit{0}, \mathit{0} \rangle = 0$, or
       $\langle \mathit{0}, \mathit{0} \rangle = \langle 0\mathit{0}, \mathit{0} \rangle = 0\langle \mathit{0}, \mathit{0} \rangle = 0$

- Theorem 6.2a: $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm.

proof: "1": "$x \neq \mathit{0} \Rightarrow \langle x, x\rangle > 0$ " [ip4] and

"$\langle x, x\rangle = 0 \Leftrightarrow x = \mathit{0}$" [Thm 6.1-4] $\Rightarrow \|x\| \geq 0$

"2": $\|ax\|^2 = \langle ax, ax\rangle = a\overline{a}\langle x, x\rangle = |a|^2\|x\|^2$

"3": We use in advance Cauchy-Schwarz inequality, $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$, which will directly follow.

$\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2 = \|x\|^2 + 2\mathfrak{R}(\langle x, y\rangle) + \|y\|^2$

$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \leq \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2$ [C-S ineq]

$= (\|x\| + \|y\|)^2$

"4": "$\langle x, x\rangle = 0 \Leftrightarrow x = \mathit{0}$" [Thm 6.1-4]

- Theorem 6.2b (Cauchy-Schwarz inequality): $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$, and equality holds if and only if $x = cy$ (or $y = cx$) for some scalar c.

  proof: if $y = 0$, equality holds with $y = 0x$ and the proof is complete.

  Assume $y \neq 0$ and c is a scalar.

$$0 \leq \|x - cy\|^2 = \langle x - cy, x - cy\rangle = \langle x, x - cy\rangle - \langle cy, x - cy\rangle$$
$$= \langle x, x\rangle - \overline{c}\langle x, y\rangle - c\langle y, x\rangle + c\overline{c}\langle y, y\rangle$$
$$= \|y\|^2(\frac{\langle x,y\rangle\langle y,x\rangle}{\|y\|^4} - \overline{c}\frac{\langle x,y\rangle}{\|y\|^2} - c\frac{\langle y,x\rangle}{\|y\|^2} + c\overline{c}) + \|x\|^2 - \frac{\langle x,y\rangle\langle y,x\rangle}{\|y\|^2}$$
$$= \|y\|^2(\frac{\langle x,y\rangle}{\|y\|^2} - c)\overline{(\frac{\langle x,y\rangle}{\|y\|^2} - c)} + \frac{\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2}{\|y\|^2}$$
$$= \|y\|^2|(\frac{\langle x,y\rangle}{\|y\|^2} - c)|^2 + \frac{\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2}{\|y\|^2}$$

Since this holds for any c, letting $c = \frac{\langle x,y \rangle}{\|y\|^2}$ proves the inequality.

"equality": If $x = cy$, then $\frac{\langle x,y \rangle}{\|y\|^2} = \frac{\langle cy,y \rangle}{\langle y,y \rangle} = c$ and equality holds in the above derivation.

Conversely, if $|\langle x, y \rangle| = \|x\| \cdot \|y\|$, we must have in the above for any

$c\|x - cy\|^2 = \|y\|^2 |(\frac{\langle x,y \rangle}{\|y\|^2} - c)|^2$.

Letting $c = \frac{\langle x,y \rangle}{\|y\|^2}$ results in $x = cy$.

- The equality condition is a very important tool for minimization or maximization.

- example:

  triangle inequality :

  l_2 space : $\sqrt{\sum_{i=1}^{n} |a_i + b_i|^2} \leq \sqrt{\sum_{i=1}^{n} |a_i|^2} + \sqrt{\sum_{i=1}^{n} |b_i|^2}$

  L_2 space : $\sqrt{\int_a^b |f(t) + g(t)|^2 dt} \leq \sqrt{\int_a^b |f(t)|^2 dt} + \sqrt{\int_a^b |g(t)|^2 dt}$

  Cauchy-Schwarz inequality :

  l_2 space : $|\sum_{i=1}^{n} a_i \overline{b_i}| \leq \sqrt{(\sum_{i=1}^{n} |a_i|^2)(\sum_{i=1}^{n} |b_i|^2)}$

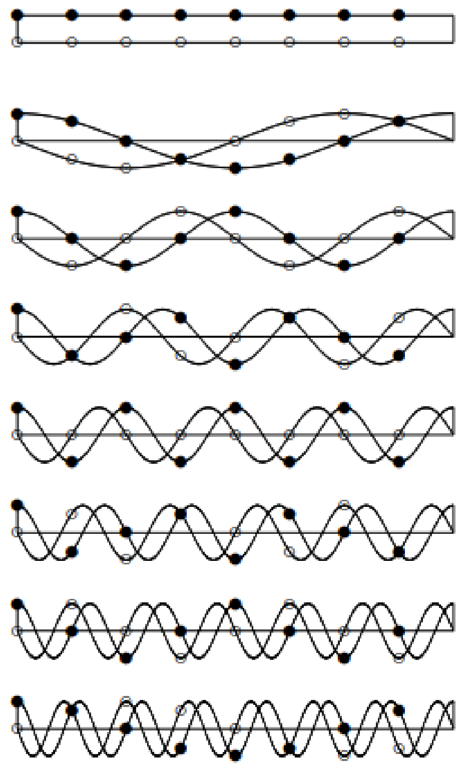
  L_2 space : $|\int_a^b f(t)\overline{g(t)} dt| \leq \sqrt{(\int_a^b |f(t)|^2 dt)(\int_a^b |g(t)|^2 dt)}$

- x and y are **orthogonal**: $\langle x, y \rangle = 0$ for "nonzero" vectors

  - $\{u_1, \cdots, u_k\}$ is orthogonal: $\langle u_i, u_j \rangle = 0, i \neq j$

- $\{u_1, \cdots, u_k\}$ is **orthonormal**: orthogonal and $\|u_i\| = 1$
  - orthonormal $\Leftrightarrow$ $\langle u_i, u_j \rangle = \delta_{ij} = \begin{Bmatrix} 1, & i = j \\ 0, & i \neq j \end{Bmatrix}$, Kronecker delta

- example: orthonormal set
  - $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subseteq \mathbb{R}^3$
  - $\{\frac{1}{\sqrt{2}}(1, 1, 0), \frac{1}{\sqrt{3}}(1, -1, 1), \frac{1}{\sqrt{6}}(-1, 1, 2)\}$ $\mathbb{R}^3$
  - $\{f_k(t) = e^{ikt} : k \in \mathbb{Z}\} \subseteq C([0, 2\pi), \mathbb{C})$, where $\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt$



- We are interested in an "orthonormal basis" $\beta = \{v_1, \cdots, v_n\}$ because for $x = a_1 v_1 + \cdots + a_n v_n$, $\langle x, v_i \rangle = \langle a_1 v_1 + \cdots + a_n v_n, v_i \rangle$
$$= a_1 \langle v_1, v_i \rangle + \cdots + a_i \langle v_i, v_i \rangle + \cdots + a_n \langle v_n, v_i \rangle = a_i,$$
so that for any x we can write $x = \langle x, v_1 \rangle v_1 + \cdots + \langle x, v_n \rangle v_n$.
  - This is basically one of the reasons that many people can, through the same medium, air, have wireless communication without interference. [modulation, multiple access]

Inner product Spaces

- **norm** on V : $\|\cdot\| : V \to \mathbb{R}^+$ such that $\forall x, y \in V$ and $\forall a \in F$,

  1. $\|x\| \geq 0$
  2. $\|ax\| = |a| \cdot \|x\|$
  3. $\|x + y\| \leq \|x\| + \|y\|$: triangle inequality
  4. $\|x\| = 0 \Leftrightarrow x = \mathbf{0}$

- $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$, $\sum_{i=1}^{\infty} |x_i|^p < \infty$
  - l_1 norm: $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ [diamond]
  - l_2 norm: $\|x\|_2 = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$ [circle]
  - l_∞ norm: $\|x\|_\infty = \max_i |x_i|$ [square]

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- $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$, $\sum_{i=1}^{\infty} |x_i|^p < \infty$
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- $\|f\|_p = (\int_a^b |f(t)|^p dt)^{\frac{1}{p}}$, $\int_a^b |f(t)|^p dt < \infty$
  - L_1 norm: $\|f\|_1 = \int_a^b |f(t)| dt$
  - L_2 norm: $\|f\|_2 = \sqrt{\int_a^b |f(t)|^2 dt}$
  - L_∞ norm: $\|f\|_\infty =$max$_t |f(t)|$
- **inner product** on $V : \langle \cdot, \cdot \rangle : V \times V \to F$ such that $\forall x, y, z \in V$ and $\forall a \in F$,

  1. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$
  2. $\langle ax, y \rangle = a \langle x, y \rangle$
  3. $\overline{\langle x, y \rangle} = \langle y, x \rangle$, where $\overline{a}$ is the complex conjugate of a.
  4. $x \neq \mathit{0} \Rightarrow \langle x, x \rangle > 0$

- example:
  - $V = F^n \Rightarrow \langle (a_1, \cdots, a_n), (b_1, \cdots, b_n) \rangle = \sum_{i=1}^n a_i \overline{b_i} = b^* a$
  - $V = C([a, b], \mathbb{R}) \Rightarrow \langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt$
  - $V = M_{n \times n}(F) \Rightarrow \langle A, B \rangle =tr(B^* A) \leftarrow$ (Frobenius inner product),

- Theorem 6.1: V is an inner product space.

  Then $\forall x, y, z \in V$ and $c \in F$,

  1. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
  2. $\langle x, cy \rangle = \overline{c} \langle x, y \rangle$
  3. $\langle x, \mathit{0} \rangle = \langle \mathit{0}, x \rangle = 0$
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  5. $\forall x \in V, \langle x, y \rangle = \langle x, z \rangle \Rightarrow y = z$

- Theorem 6.2a: $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm.
- Theorem 6.2b (Cauchy-Schwarz inequality): $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$, and equality holds iff $x = cy$ (or $y = cx$) for some scalar c.
- x and y are **orthogonal**: $\langle x, y\rangle = 0$ for "nonzero" vectors
  - $\{u_1, \cdots, u_k\}$ is orthogonal: $\langle u_i, u_j\rangle = 0, i \neq j$
- $\{u_1, \cdots, u_k\}$ is **orthonormal**: orthogonal and $\|u_i\| = 1$
  - orthonormal $\Leftrightarrow \langle u_i, u_j\rangle = \delta_{ij} = \begin{Bmatrix} 1, & i = j \\ 0, & i \neq j \end{Bmatrix}$, Kronecker delta
  - For $x = a_1 u_1 + \cdots + a_n u_n$, $\langle x, u_i\rangle = \langle a_1 u_1 + \cdots + a_n u_n, u_i\rangle$
    $= a_1\langle u_1, u_i\rangle + \cdots + a_i\langle u_i, u_i\rangle + \cdots + a_n\langle u_n, u_i\rangle = a_i$,
    For any x we can write $x = \langle x, v_1\rangle v_1 + \cdots + \langle x, v_n\rangle v_n$.
  - $\{f_k(t) = e^{ikt} : k \in \mathbb{Z}\}$. $\langle g(t), f_k(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi} g(t)\overline{f_k(t)}dt$.
- [End of Review]

Gram-Schmidt orthogonalization and orthogonal complement

- Theorem 6.3: V is an inner product space; $S = \{v_1, \cdots, v_k\} \subseteq V$ is orthogonal. Then

  $y = \sum_{i=1}^{k} a_i v_i \Rightarrow a_j = \frac{\langle y, v_j \rangle}{\|v_j\|^2}, j = 1, \cdots, k.$

  proof: $\langle y, v_j \rangle = \langle \sum_{i=1}^{k} a_i v_i, v_j \rangle = \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle = a_i \|v_j\|^2$

- Corollary 6.3.1: If S is orthonormal,

  $y = \sum_{i=1}^{k} a_i v_i \Rightarrow a_i = \langle y, v_i \rangle \Rightarrow y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i.$

- Corollary 6.3.2: Orthogonality implies linear independence.

  proof: Replace y with *0* in Theorem 6.3.

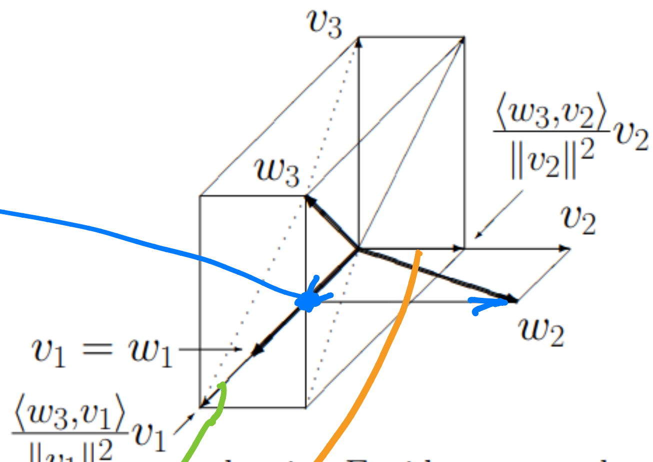
Gram-Schmidt orthogonalization:

$$v_1 = w_1$$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$

$\dots$



showing Eucidean example

$$v_k = w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_i\rangle}{\|v_i\|^2} v_i$$

$$\langle v_2, v_1\rangle = \langle w_2, v_1\rangle - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2}\langle v_1, v_1\rangle = 0 \qquad v_2 \perp v_1$$

$$\langle v_3, v_1\rangle = \langle w_3, v_1\rangle - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2}\langle v_1, v_1\rangle = 0 \Rightarrow v_3 \perp v_1$$

$$\langle v_3, v_2\rangle = \langle w_3, v_2\rangle - \langle w_3, v_2\rangle = 0 \Rightarrow v_3 \perp v_2$$

- Theorem 6.4: V is an inner product space; $S = \{w_1, \cdots, w_n\} \subseteq V$ is linearly independent. Then letting $v_1 = w_1$ and $v_i = w_i - \sum_{j=1}^{i-1} \frac{\langle w_i, v_j \rangle}{\|v_j\|^2} v_j, i = 2, \cdots, n,$

  makes $S' = \{v_1, \cdots, v_n\}$ orthogonal and span(S') = span(S).

  proof: by induction in n. First, $v_i \neq 0$ because $v_i = 0$ implies $w_i = \sum_{j=1}^{i-1} \frac{\langle w_i, v_j \rangle}{\|v_j\|^2} v_j$, where $v_j \in$ span$(\{w_1, \cdots, w_j\})$, and therefore contradicts linear independence of S.

  1. The theorem is obviously true for $n = 1$.
  2. Assume it holds for $|S| = n - 1$. That is, $\{v_1, \cdots, v_{n-1}\}$ is orthogonal, and span$(\{v_1, \cdots, v_{n-1}\})$ =span$(\{w_1, \cdots, w_{n-1}\})$.
  3. For $i = 1, \cdots, n-1$,

     $\langle v_n, v_i \rangle = \langle w_n - \sum_{j=1}^{n-1} \frac{\langle w_n, v_j \rangle}{\|v_j\|^2} v_j, v_i \rangle$

$$= \langle w_n, v_i \rangle - \sum_{j=1}^{n-1} \frac{\langle w_n, v_j \rangle}{\|v_j|^2} \langle v_j, v_i \rangle$$

$$= \langle w_n, v_i \rangle - \frac{\langle w_n, v_i \rangle}{\|v_i\|^2} \langle v_i, v_i \rangle = 0. \text{ [(ii)]}$$

$\Rightarrow \{v_1, \cdots, v_n\}$ is orthogonal.

Now show that span(S') =span(S).

$v_j \in$ span$(S), j = 1, \cdots, n \Rightarrow$span$(S') \subseteq$ span(S)

S' is orthogonal $\Rightarrow S'$ is linearly independent $\Rightarrow$ dim(span(S')) = dim(span(S)) =n

- example: $S = \{(1,1,1),(0,1,2),(2,1,0),(0,0,3)\}$

$v_1 = (1,1,1)$

$v_2 = (0,1,2) - \frac{\langle (0,1,2),(1,1,1)\rangle}{\|(1,1,1)\|^2}(1,1,1) = (-1,0,1)$

$v_3 = (2,1,0) - \frac{\langle (2,1,0),(1,1,1)\rangle}{\|(1,1,1)\|^2}(1,1,1) - \frac{\langle (2,1,0),(-1,0,1)\rangle}{\|(-1,0,1)\|^2}(-1,0,1) =$
$(0,0,0)$ [lin dep]

$v_4 = (0,0,3) - \frac{\langle (0,0,3),(1,1,1)\rangle}{\|(1,1,1)\|^2}(1,1,1) - \frac{\langle (0,0,3),(-1,0,1)\rangle}{\|(-1,0,1)\|^2}(-1,0,1) =$
$\frac{1}{2}(1,-2,1)$

By normalizing, we obtain an orthonormal set,

$\left\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(-1,0,1), \frac{1}{\sqrt{6}}(1,-2,1)\right\}$, which is an orthonormal basis for $\mathbb{R}^3$.

- example: Consider the inner product space $P_2(\mathbb{R})$ with $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$.

  $\beta = \{1, x, x^2\}$ is the standard basis but not orthonormal.

  $v_1 = 1$

  $v_2 = x - \frac{\langle x,1 \rangle}{\|1\|^2} \cdot 1 = x$

  $(\because \|v_1\|^2 = \int_{-1}^{1} 1^2 dt = 2, \langle x, v_1 \rangle = \int_{-1}^{1} t \cdot 1 dt = 0)$

  $v_3 = x^2 - \frac{\langle x^2,1 \rangle}{\|1\|^2} \cdot 1 - \frac{\langle x^2,x \rangle}{\|x\|^2} \cdot x = x^2 - \frac{1}{3}$

  By normalizing, we obtain an orthonormal basis,

  $\left\{ \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}} x, \sqrt{\frac{5}{8}} (3x^2 - 1) \right\}.$

  Why insist on an orthonormal basis?

- Theorem 6.5: V is an inner product space; $\dim(V) < \infty$. Then

  1. V has an orthonormal basis $\beta = \{v_1, \cdots, v_n\}$.

  2. $\forall x \in V, x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$

  - That is, $[x]_\beta = (\langle x, v_1 \rangle, \cdots, \langle x, v_n \rangle)^t$.
  - 2 is called a **Fourier series expansion.**
  - $\langle x, v_i \rangle$ are called **Fourier coefficients.**
  - $\|x\|^2 = \langle \sum_{i=1}^{n} \langle x, v_i \rangle v_i, \sum_{j=1}^{n} \langle x, v_j \rangle v_j \rangle$
    $= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, v_i \rangle \overline{\langle x, v_j \rangle} \langle v_i, v_j \rangle = \sum_{i=1}^{n} |\langle x, v_i \rangle|^2$
    : **Parseval's relation**
  - example: $C([0, 2\pi), \mathbb{C})$ with $\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt$
    $\beta = \{f_k(t) = e^{ikt} : k \in \mathbb{Z}\}$ is an orthonormal basis.
    $\langle f, f_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-ikt}dt$: k-th Fourier coefficient of f $= a_k$
    Fourier series? $f(t) = \sum_{-\infty}^{\infty} a_k e^{ikt}$

- Corollary 6.5: V is an inner product space; $\dim(V) < \infty$;

  $\beta = \{v_1, \cdots, v_n\}$ is an orthonormal basis for V; T is a linear operator on

  V; $A = [T]_\beta$. Then $A_{ij} = \langle T(v_j), v_i \rangle$.

j-th column: $[T(v_j)]_\beta = ?$

$$\Leftrightarrow T(v_j) = A_{1j} v_1 + A_{2j} v_2 + \cdots + A_{nj} v_n$$

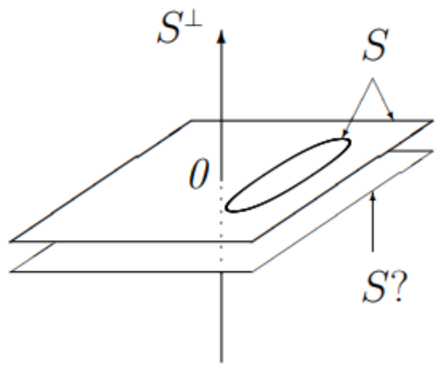
$$\Leftrightarrow [T(v_j)]_\beta = \begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{nj} \end{bmatrix}$$

$$A_{ij} = \langle T(v_j), v_i \rangle$$

$$\Rightarrow A = \left[[T(v_1)]_\beta, [T(v_2)]_\beta, \cdots, [T(v_n)]_\beta \right]$$

$$= \begin{bmatrix} \ddots & & \\ & A_{ij} = \langle T(v_j), v_i \rangle & \\ & & \ddots \end{bmatrix}$$

- **orthogonal complement** $S^\perp$ of S :
  $S^\perp = \{x \in V : \langle x, y \rangle = 0, \forall y \in S\}$



  - Do not confuse this with the set complement
  - $S^\perp$ is a subspace for any set S.
  - $S \subseteq (S^\perp)^\perp$, but if S is a subspace, then $S = (S^\perp)^\perp$.
  - $\{0\}^\perp = V, V^\perp = \{0\}$