- Theorem 6.10: V is an inner product space; $\dim(V) < \infty$; $\beta = \{v_1, \cdots, v_n\}$ is an orthonormal basis for V; T is a linear operator on V. Then $[T^*]_\beta = [T]_\beta^*$. $A = [T]_\beta$, $A_{ij} = \langle T(v_j), v_i \rangle$

- Corollary 6.10: A is an $n \times n$ matrix. Then $L_{A^*} = (L_A)^*$. $\leftarrow [T^*]_\beta = A^*$

- Theorem 6.11 and Corollary 6.11: V is an inner product space; T and U are linear operators on V; A and B are $n \times n$ matrices; c is a scalar. Then the following hold.

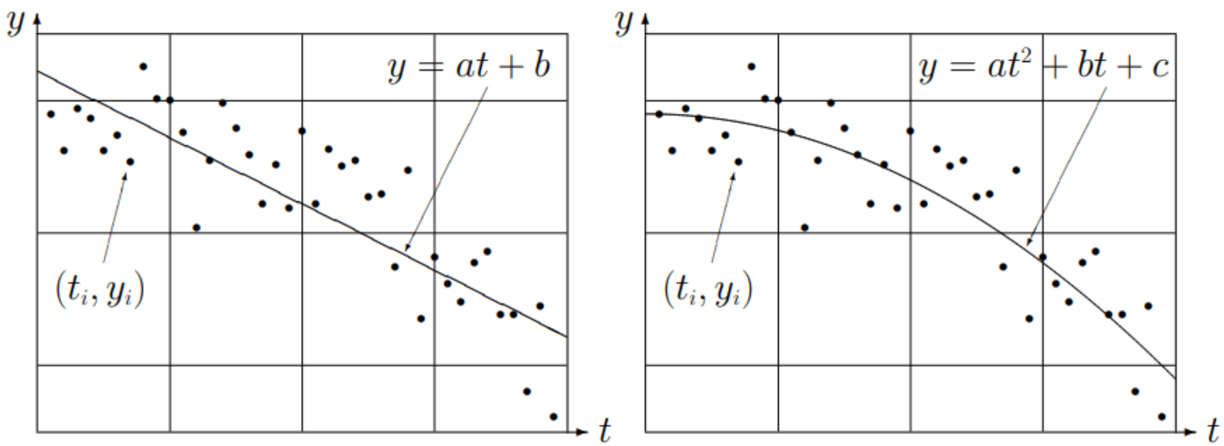
(a)	$(T+U)^* = T^* + U^*$	$(A+B)^* = A^* + B^*$
(b)	$(cT)^* = \bar{c}T^*$	$(cA)^* = \bar{c}A^*$
(c)	$(TU)^* = U^*T^*$	$(AB)^* = B^*A^*$
(d)	$(T^*)^* = T$	$(A^*)^* = A$
(e)	$I^* = I$	$I_n^* = I_n$

■ least square approximation:

measurements: $(t_1, y_1), (t_2, y_2), \cdots, (t_m, y_m)$

approximation: Find a and b such that $y = at + b$, or

find a, b and c such that $y = at^2 + bt + c$.



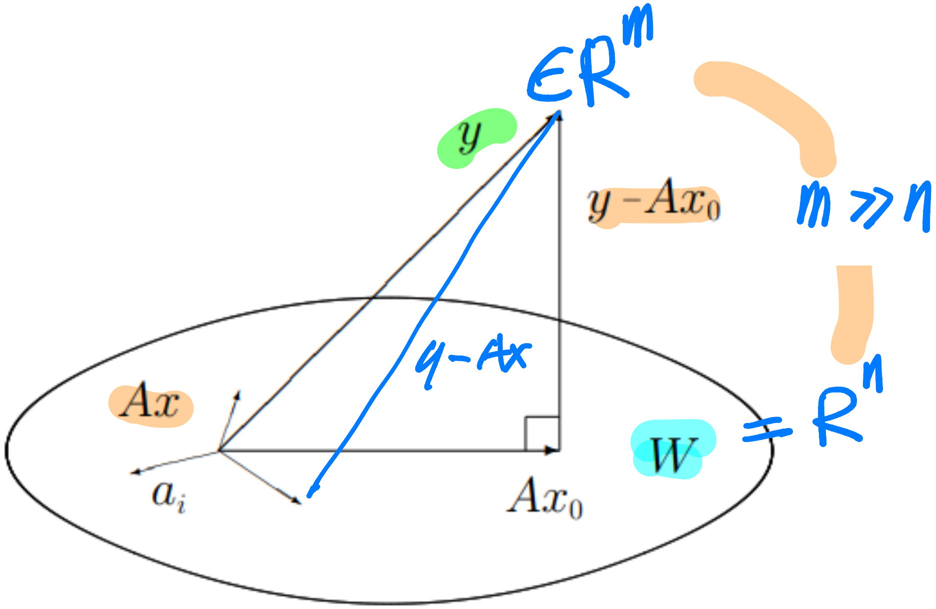
$a, b, c = ?$

$\in R^m$

$\Rightarrow$ minimize $E = \|y - Ax\|^2$, where $y = (y_1, \cdots, y_m)^t$, and

$$A = \begin{pmatrix} t_1 & 1 \\ \vdots & \vdots \\ t_m & 1 \end{pmatrix}, x = \begin{pmatrix} a \\ b \end{pmatrix}, \text{ or } A = \begin{pmatrix} t_1^2 & t_1 & 1 \\ \vdots & \vdots & \vdots \\ t_m^2 & t_m & 1 \end{pmatrix}, x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in R^n$$

- **orthogonality principle:**
  The minimizing x_0 satisfies
  $\forall x, \langle Ax, y - Ax_0 \rangle = 0$
  $W = R(L_A)$
  $= \{Ax : x \in F^n\}$
  $= \{\sum a_i x_i : x \in F^n\}$,
  a_i are the column vectors of A.
  So W is the column space of A.
  The principle is quite general.



- Theorem 6.12: $A \in M_{m\times n}(F)$; $y \in F^m$. Then $\exists x_0 \in F^n$ such that
  1. $A^*Ax_0 = A^*y$. $(\langle Ax, y\rangle_m = \langle x, A^*y\rangle_n.)$ $\Rightarrow \langle x, \underbrace{A^*(y - Ax_0)}_{=0} \rangle = 0$
  2. $\forall x \in F^n, \|Ax_0 - y\| \leq \|Ax, y\|$
  3. $\text{rank}(A) = n \Rightarrow x_0 = (A^*A)^{-1}A^*y$. $(\text{rank}(A^*A) = \text{rank}(A))$

- [End of Review]

→ $m \times n$, $m \le n$ (full rank가 아닐때)

- **minimal solution** of $Ax = b$: 부정 (많은 solution 중 최소 solution) Nullity $\neq 0$

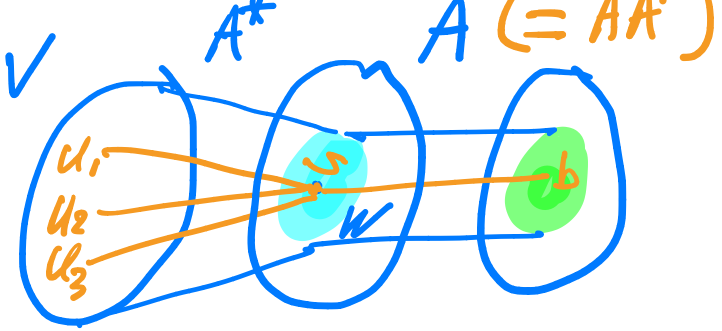
  solution with the smallest norm

  - least energy, least power, etc.

- Theorem 6.13: $A \in M_{m\times n}(F)$; $b \in F^m$; and s is a minimal solution of $Ax = b$.

  $\begin{cases} \min_x \|x\| \\ \text{s.t. } Ax = b \end{cases}$

  Then

  - $s \in R(L_{A^*})$. $\Leftarrow$ $s = A^*u$, $u \in V$
  - s is the only solution in $R(L_{A^*})$.

    That is, $AA^*u = b \Rightarrow s = A^*u$.

    $m\times n \cdot n \times m = m \times m \Rightarrow$ 부정

    Nullity $\neq 0$ ($\operatorname{rank}(AA^*) \le m < n$)

    u : not unique

  - s is unique.

    proof: Let $W = R(L_{A^*})$. (solution space)

    $v \in N(L_A) \Leftrightarrow Av = 0 \Leftrightarrow \forall u \in F^m, \langle u, Av\rangle = \langle A^*u, v\rangle = 0$

    $W \perp N(L_A)$

This shows that $N(L_A) = W^{\perp}$.

(In general, $N(T) = R(T^*)^{\perp}$ and $N(T^*) = R(T)^{\perp}$ are true.)

For any solution x of $Ax = b$, we have $x = s + y$, such that

$s \in W$ is unique and closest to x, and

$y \in W^{\perp}$ is unique and closest to x. [Thm 6.6]

This y is a homogeneous solution: $Ay = 0$. $s \perp y$

$\Rightarrow As = As + 0 = As + Ay = A(s + y) = Ax = b$ $(\because \langle s, y \rangle = 0)$

$\Rightarrow$ s is a solution in W. $\langle s+y, s+y \rangle = \langle s, s \rangle + \langle y, y \rangle$

$\|s\|^2 \leq \|s\|^2 + \|y\|^2 = \|s + y\|^2 = \|x\|^2$ [Pythagorean thm]

$\Rightarrow$ s is a minimal solution in W.

minimal sol $\Rightarrow x = s$

Since x is an arbitrary solution, if it is minimal, then

$\|x\|^2 = \|s\|^2 + \|y\|^2 = \|s\|^2$.

$\Rightarrow y = 0 \Rightarrow x = s \in W$

$R(L_{A^*})$?

So a minimal solution must be in W. : "1" $s = A^* u$

"uniqueness within W": Assume $s' \in W$ and $As' = b$

$AA^* u = b$

$\Rightarrow A(s - s') = As - As' = b - b = 0$

$\Rightarrow s - s' \in N(L_A) = W^\perp$

$\Rightarrow s = s'[s - s' \in W \cap W^\perp = \{0\}]$: "2"

"3" follows from "1" and "2".

- So all the solutions to $Ax = b$ are of the form $x = s + y$, where s is the unique minimal solution and y varies in $W^\perp$.
- Compare this argument with Theorem 3.9: $K = \{s\} + K_H$.

- example:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= -11, \quad A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 5 & 0 \end{pmatrix}, b = \begin{pmatrix} 4 \\ -11 \\ 19 \end{pmatrix} \\ x_1 + 5x_2 &= 19 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & -1 & 2 & -11 \\ 1 & 5 & 0 & 19 \end{array}\right) \overset{3,3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 1 & -15 \\ 0 & 3 & -1 & 15 \end{array}\right) \overset{2,3,3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & -6 \\ 0 & 1 & -\frac{1}{3} & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + \frac{5}{3}x_3 = -6$
$x_2 - \frac{1}{3}x_3 = 5$
$x_3 = t$
......

$\Rightarrow$ There are multiple solutions, eg, $x = (-6, 5, 0)^t$ or $(-11, 6, 3)^t$.

minimal sol?

$$AA^* = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 11 \\ 1 & 6 & -4 \\ 11 & -4 & 26 \end{pmatrix}$$

We solve $AA^*u = b$.

$$\left(\begin{array}{ccc|c} 6 & 1 & 11 & 4 \\ 1 & 6 & -4 & -11 \\ 11 & -4 & 26 & 19 \end{array}\right) \overset{1,3,3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 6 & -4 & -11 \\ 0 & -35 & 35 & 70 \\ 0 & -70 & 70 & 140 \end{array}\right) \overset{2,3,3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow u = (1, -2, 0)^t, u' = (-1, -1, 1)$, two of many solutions

$A^*u = A^*u' = (-1, 4, -3)^t$ is the minimal solution.

Note that $\|A^*u\| \leq \|x\|$ and also for any other solution x.

Normal and self-adjoint operator

- We now investigate diagonalizability of a linear operator on an inner product space.

- Lemma 6.14 V is an inner product space; $\dim(V) < \infty$; $T : V \to V$ is a linear operator. Then
T has an eigenvector $\Rightarrow$ T^* has an eigenvector.

  proof: Assume T has an eigenvector, β is an orthonormal basis for V, and $A = [T]_\beta$
$\Rightarrow \exists \lambda$such that $\det(A - \lambda I) = 0$
$\Rightarrow \det(A - \lambda I)^* = \det\left(A^* - (\overline{\lambda}) I\right) = 0$
$\Rightarrow A^*$ has an eigenvector with the corresponding eigenvalue $\overline{\lambda}$.
$\Rightarrow T^*$ has an eigenvector with the corresponding eigenvalue $\overline{\lambda}$.
$[A^* = [T^*]_\beta]$

- Theorem 6.14 (Schur): din$(V) < \infty; T : V \to V$ is a linear operator; and $f_T(t)$ splits. Then $\exists$ an orthonormal basis β such that $[T]_\beta$ is upper triangular. → $(t - \lambda_i)$로 인수분해 가능.

proof : induction in n = dim(V)

(i) If dim$(V) = 1$, all 1×1 matrices are triangular.

(ii) Assume that it holds for dim$(V) = n - 1$.

(iii) Let dim$(V) = n$, and assume $f_T(t)$ splits.

$\Rightarrow$ T has an eigenvalue and an eigenvector.
$\Rightarrow$ T^* has a normalized eigenvector z. [Lemma 6.14]

Let $T^*(z) = \lambda z$ and $W = span(\{z\})$. 1차원 공간

Now show that $W^\perp$ is T-invariant. n-1차원 공간

- Let $y \in W^\perp$

  $\Rightarrow \langle T(y), z\rangle = \langle y, T^*(z)\rangle = \langle y, \lambda z\rangle = (\overline{\lambda})\langle y, z\rangle = 0$

  $\Rightarrow T(y) \in W^\perp$

  $\Rightarrow W^\perp$ is T-invariant.

  $\Rightarrow f_{T_{W^\perp}}(t)$ divides $f_T(t)$. [Thm 5.21]

  $\Rightarrow f_{T_{W^\perp}}$ splits. [$f_T(t)$ splits]

  $\Rightarrow \exists \gamma = \{v_1, \cdots, v_{n-1}\} \in W^\perp$, orthonormal, such that $[T]_\beta =$ is $(n-1) \times (n-1)$ upper triangular. [(ii)]

  $\Rightarrow \beta = \gamma \cup \{z\}$ is an orthonormal basis for V.

  unit & 서로 orthogonal

$$\Rightarrow [T]_\beta = ([T(v_1)]_\beta, \cdots, [T(v_{n-1}]_\beta, [T(z)]_\beta)$$

$$= \begin{bmatrix} [T_{W^\perp}]_\gamma & [T(z)]_\beta \\ O & \downarrow \end{bmatrix}$$ is $n \times n$ upper triangular.

- Schur's theorem does not say that $[T]_\beta$ is invertible.
- Neither does it say that $[T]_\beta$ is diagonalizable.
- β in the theorem is not unique; nor $[T]_\beta$.
- T is diagonalizable $Rightarrow [T]_\beta$ is diagonal for some β.

In addition β ~~is~~ can be orthonormal.

$$\Rightarrow [T^*]_\beta = [T]^*_\beta \text{ is diagonal.}$$

$$\Rightarrow [TT^*]_\beta = [T]_\beta [T]^*_\beta = [T]^*_\beta [T]_\beta = [T^*T]_\beta \text{ [diagonal]}$$

$\Rightarrow TT^* = T^*T$ [rep is unique] $\Leftarrow$ T is diagonalizable

?

$\Rightarrow$

Does this commutativity imply diagonalizability?

- **normal operator** T on an inner product space: $TT^* = T^*T$
- **normal matrix** A : $AA^* = A^*A$
- $T(\cancel{A})$ is normal $\Leftrightarrow [T]_\beta$ is normal.
- Even If β is orthonormal, T is normal $\Leftrightarrow [T]_\beta$ is normal.
- example: $A = \begin{bmatrix} 1 & i \\ 1 & 2+i \end{bmatrix} \Rightarrow A^* = \begin{bmatrix} 1 & 1 \\ -i & 2-i \end{bmatrix}$

$$\Rightarrow AA^* = A^*A = \begin{bmatrix} 2 & 2+2i \\ 2-2i & 6 \end{bmatrix}$$

- example: $T : \mathbb{R}^2 \to \mathbb{R}^2$ is rotation by θ; β is the standard basis.

$$\Rightarrow [T]_\beta = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$\Rightarrow [T^*]_\beta = [T]_\beta^* = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

$$\Rightarrow [T]_\beta [T]_\beta^* = [T]_\beta^* [T]_\beta = I$$

(Normal operator)

$\Rightarrow TT^* = T^*T = I$: orthogonal operator [to be defined]

But since $F = \mathbb{R}$, no eigenvector, not diagonalizable.

T is normal $\not\Rightarrow$ diagonalizable

아닐 수 있음.

- Theorem 6.15 : V is an inner product space; $T : V \to V$ is normal.
  Then
  1. $\forall x \in V, ||T(x)|| = ||T^*(x)||$.

  2. $\forall c \in F, T - cI$ is normal.

  3. $T(x) = \lambda x \Leftrightarrow T^*(x) = \overline{\lambda} x$
  That is, T and T^* have the same eigenvector x with the respective eigenvalues λ and $\overline{\lambda}$.

  4. $T(x_1) = \lambda_1 x_1; T(x_2) = \lambda_2 x_2; \lambda_1 \neq \lambda_2 \Rightarrow \langle x_1, x_2 \rangle = 0$.

  proof:
  "1": $||T(x)||^2 = \langle T(x), T(x) \rangle = \langle T^*T(x), x \rangle = \langle TT^*(x), x \rangle$
  $= \langle T^*(x), T^*(x) \rangle = ||T^*(x)||^2$

"2" : $(T - cI)(T - cI)^* = (T - cI)(T^* - \overline{c}I)$

$= TT^* - \overline{c}T - cT^* + |c|^2 I$

$(T - cI)^*(T - cI) = (T^* - \overline{c}I)(T - cI)$

$= T^*T - cT^* - \overline{c}T + |c|^2 I$

"3" : $T(x) = \lambda x$

$\Leftrightarrow 0 = ||(T - \lambda I)(x)||$

$= ||(T - \lambda I)^*(x)||$ [1,2]

$= ||(T^* - \overline{\lambda} I)(x)||$

$= ||T^*(x) - \overline{\lambda} x||$

$\Leftrightarrow ||T^*(x) = \overline{\lambda} x||$

"4": $\lambda_1 \langle x_1, x_2 \rangle = \langle T(x_1), x_2 \rangle = \langle x_1, T^*(x_2) \rangle$

$= \langle x_1, \overline{\lambda}_2 x_2 \rangle = \lambda_2 \langle x_1, x_2 \rangle$ [3] $\Rightarrow \langle x_1, x_2 \rangle = 0$

- Theorem 6.16 : V is an inner product space over the "complex" field; $\dim(V) < \infty; T : V \to V$ is linear. Then T is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.

proof: "Only if": Assume T is normal.
$f_T(t)$ splits. [All polynomials split over the complex field.]
$\Rightarrow \exists$ orthonormal $\beta = \{v_1, \cdots, v_n\}$ such that
$A = [T]_\beta$ is upper triangular. [Schur's Thm]
Show $v_1, \cdots, v_n$ are eigenvectors by induction in $i = 1, \cdots, n$.

$\Rightarrow A = [T]_\beta$ is diagonal,

$A_{ij} = \langle T(v_j), v_i \rangle$
$= 0$

(i) 1st column of $A : [T(v_1)]_\beta = (A_{11}, 0, \cdots, 0)^t$ [upper Δ]

diagonal

$\Rightarrow T(v_1) = A_{11} v_1$
$\Rightarrow v_1$ is an eigenvector of T.

(ii) Assume $v_1, \cdots, v_{k-1}$ are eigenvectors of T.

(iii) $T(v_k) = A_{1k}v_1 + \cdots + A_{kk}v_k$ [upper triangular]
For $j = 1, \cdots, k-1$,
$A_{jk} = \langle T(v_k), v_j \rangle = \langle v_k, \overline{\lambda}_j v_j \rangle$ [T is normal; (ii); Thm 6.15]
$= \lambda_j \langle v_k, v_j \rangle = 0$ [orthonormal basis]
$\Rightarrow T(v_k) = A_{kk}v_k$
$\Rightarrow v_k$ is an eigenvector.

"if": Assume that β is an orthonormal basis of eigenvectors.
$\Rightarrow [T^*]_\beta = [T]_\beta^*$ is diagonal.
$[TT^*]_\beta = [T]_\beta [T]_\beta^* = [T]_\beta^* [T]_\beta = [T^*T]_\beta$ [diagonal]
$\Rightarrow TT^* = T^*T$. [rep is unique]

- For the real field, normality is not enough for diagonalizability.