- **minimal solution** of $Ax = b$:

  solution with the smallest norm:least energy, least power, etc.

- Theorem 6.13: $A \in M_{m\times n}(F); b \in F^m$; and s is a minimal solution of $Ax = b$.

  Then

  - $s \in R(L_{A^*})$.
  - s is the only solution in $R(L_{A^*})$.
    That is, $AA^*u = b \Rightarrow s = A^*u$.
  - s is unique.

- Compare $x = s + y, s \in R(L_{A^*}), y \in N(L_A)$ vs Theorem 3.9: $K = \{s\} + K_H$.

Normal and self-adjoint operator

- We now investigate diagonalizability of a linear operator on an inner product space.

- Lemma 6.14 V is an inner product space; $\dim(V) < \infty; T : V \to V$ is a linear operator. Then
  T has an eigenvector $\Rightarrow T^*$ has an eigenvector.

- Theorem 6.14 (Schur): din$(V) < \infty; T : V \to V$ is a linear operator; and $f_T(t)$ splits. Then $\exists$ an orthonormal basis β such that $[T]_\beta$ is upper triangular.

  - Schur's theorem does not say that $[T]_\beta$ is invertible.
  - Neither does it say that $[T]_\beta$ is diagonalizable.
  - β in the theorem is not unique; nor $[T]_\beta$.

- T is diagonalizable $\Rightarrow [T]_\beta$ is diagonal for some β.

  $\Rightarrow [T^*]_\beta = [T]_\beta^*$ is diagonal.

  $\Rightarrow [TT^*]_\beta = [T]_\beta [T]_\beta^* = [T]_\beta^* [T]_\beta = [T^*T]_\beta$ [diagonal]

  $\Rightarrow TT^* = T^*T$ [rep is unique]

  Does this commutativity imply diagonalizability? NO

- **normal operator** T on an inner product space: $TT^* = T^*T$
- **normal matrix** $A : AA^* = A^*A$
- $T(A)$ is normal $\Leftrightarrow [T]_\beta$ is normal.

- Theorem 6.15 : V is an inner product space; $T : V \to V$ is normal. Then

$T^*T = TT^*$

1. $\forall x \in V, ||T(x)|| = ||T^*(x)||$.
2. $\forall c \in F, T - cI$ is normal.
3. $T(x) = \lambda x \Leftrightarrow T^*(x) = \overline{\lambda} x$. That is, T and T^* have the same eigenvector x with the respective eigenvalues λ and $\overline{\lambda}$.
4. $T(x_1) = \lambda_1 x_1; T(x_2) = \lambda_2 x_2; \lambda_1 \neq \lambda_2 \Rightarrow \langle x_1, x_2 \rangle = 0$.

- Theorem 6.16 : V is an inner product space over the "complex" field; $\dim(V) < \infty; T : V \to V$ is linear. Then T is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.

(diagonalizable)

- [End of Review]

- **self-adjoint (Hermitian) operator** T on an inner product space: $T = T^*$.
- **self-adjoint (Hermitian) matrix** A: $A = A^*$
- these definitions apply to both the real and complex field.
- For complex matrices, self-adjoint means conjugate symmetric.
- For real matrices, self-adjoint means symmetric.
- For orthonormal $\beta, T = T^* \Leftrightarrow [T]_\beta = [T]_\beta^*$
- self-adjoint $\Rightarrow$ normal; normal $\not\Rightarrow$ self-adjoint

- Lemma 6.17: V is an inner product space; $\dim(V) < \infty$; T is Self-adjoint,

  1. Every eigenvalue of T is real.
  2. $f_T(t)$ splits over the real field.

  proof: "1": Let $T(x) = \lambda x, x \neq 0$.
  $\Rightarrow \lambda\langle x, x\rangle = \langle T(x), x\rangle = \langle x, T^*(x)\rangle = \langle x, T(x)\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle$
  2 follows from 1.

- Theorem 6.17: V is an inner product space over the "real" field; $\dim(V) < \infty$; $T : V \to V$ is linear. Then T is self-adjoint if and only if there exists an orthonormal basis β for V consisting of eigenvectors of T.

proof:

"only if": Assume $T = T^*$.

$\Rightarrow f_T(t)$ splits (over the real field). [Lemma 6.17]

$\Rightarrow \exists \beta$, orthonormal, such that $[T]_\beta$ is upper triangular. [Schur's Thm]

$\Rightarrow [T]_\beta^* = [T^*]_\beta = [T]_\beta$ [self-adjoint]

$\Rightarrow [T]_\beta$ is (real) diagonal.

$\Rightarrow \beta$ consists of eigenvectors of T.

"if" : Assume an orthonormal basis β of eigenvectors of T.

$\Rightarrow [T]_\beta$ is (real) diagonal.

$\Rightarrow [T^*]_\beta = [T]_\beta^* = [T]_\beta$

$\Rightarrow T^* = T$ [rep is unique]

$T^*T = TT^* = I$, $\Rightarrow T^{-1} = T^*$

Unitary and orthogonal operator

$F = \mathbb{C}$ $F = \mathbb{R}$

- V is an inner product space; $\dim(V) < \infty$; $T : V \to V$ is linear.
  - **unitary operator**: $\forall x \in V(\mathbb{C}), ||T(x)|| = ||x||$
  - **orthogonal operator**: $\forall x \in V(\mathbb{R}), ||T(x)|| = ||x||$
- Lemma 6.18: V is an inner product space; $\dim(V) < \infty$; $U : V \to V$ is self-adjoint. Then $\forall x \in V, \langle x, U(x)\rangle = 0 \Rightarrow U = T_0$. $\because U(x) \neq 0 ?$ $U(x) \perp x$

  proof.
  Assume U is self-adjoint and $\forall x \in V, \langle x, U(x)\rangle = 0$.
  $\Rightarrow \exists \beta = \{v_1, \cdots, v_n\}$ orthonormal and consisting of eigenvectors of U. [Thm 6.16, 6.17]
  $\Rightarrow \overline{\lambda}_i \langle v_i, v_i\rangle = \langle v_i, \lambda_i v_i\rangle = \langle v_i, U(v_i)\rangle = 0, i = 1, \cdots, n$
  $\Rightarrow \lambda_i = 0, i = 1, \cdots, n$
  $\Rightarrow \forall x \in V,$
  $U(x) = U(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i U(v_i) = \sum_{i=1}^n a_i \lambda_i v_i = 0$

▪▪ Theorem 6.18: V is an inner product space; $\dim(V) < \infty; T : V \to V$ is linear. Then these are all equivalent.

1. $TT^* = T^*T = I$ unitary. orthogonal.
2. $\forall x, y \in V, \langle T(x), T(y)\rangle = \langle x, y\rangle$
3. β is an orthonormal basis $\Rightarrow T(\beta)$ is an orthonormal basis.
4. There exists an orthonormal basis β such that $T(\beta)$ is an orthonormal basis.
5. $\forall x \in V, ||T(x)|| = ||x||$

- 5 means length-preserving; 2 means sort of angle-preserving.
- rotation and reflection in $\mathbb{R}^2$ are orthogonal.
- unitary or orthogonal $\Rightarrow$ normal; not conversely.

proof: ”1⇒2”: Assume $TT^* = T^*T = I$
$\Rightarrow \langle T(x), T(y)\rangle = \langle x, T^*T(y)\rangle = \langle x, y\rangle$

"2⇒3" : Assume "2" and an orthonormal basis $\{v_1, \cdots, v_n\}$.

$\Rightarrow \langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle = \delta_{ij}$

$\Rightarrow \{T(v_i), \cdots, T(v_n)\}$ is an orthonormal basis.

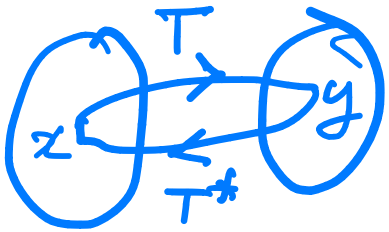
"3⇒4" : Obvious.

"4⇒5" : Assume $\beta = \{v_1, \cdots, v_n\}$ and $T(\beta)$ are orthonormal.

$\Rightarrow ||T(x)||^2 = ||T(\sum_{i=1}^n a_i v_i)||^2 = ||\sum_{i=1}^n a_i T(v_i)||^2$

$= \langle \sum_{i=1}^n a_i T(v_i), \sum_{j=1}^n a_j T(v_j)\rangle$

$= \sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} \langle T(v_i), T(v_j)\rangle = \sum_{i=1}^n |a_i|^2 = ||x||^2$



"5⇒1": Assume $\forall x \in V, ||T(x)|| = ||x||$.

$\Rightarrow \langle x, x\rangle = \langle T(x), T(x)\rangle = \langle x, T^*T(x)\rangle$

$\Rightarrow \forall x \in V, \langle x, (I - T^*T)(x)\rangle = 0$

$\Rightarrow (I - T^*T) = T_0$, [$(I - T^*T)$ is self-adjoint; Lemma6.18]

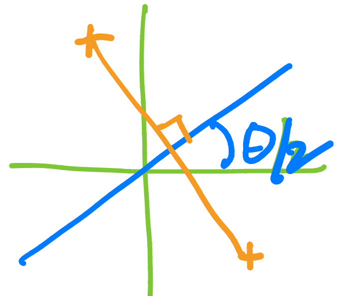
$\Rightarrow T^*T = I \Rightarrow TT^* = I$ [Thm 2.17c]

$$[T_\theta]_\beta = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$
$$[T_\theta^*]_\beta = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} = T_{-\theta}$$

- T is unitary or orthogonal; v is an eigenvector.
  $\Rightarrow ||v|| = ||T(v)|| = ||\lambda v|| = |\lambda|||v||$
  $\Rightarrow |\lambda| = 1$

- example: linear operators on $\mathbb{R}^2$
  - T_θ: rotation by θ; $T_\theta^* = T_{-\theta} \Rightarrow T_\theta T_\theta^* = T_\theta T_{-\theta} = I$
  - T: reflection about a line or the origin; $T^* = T \rightarrow TT^* = T^*T = I$

- **unitary matrix**: $AA^* = A^*A = I$ $\rightarrow A^{-1} = A^*$
  - **orthogonal matrix** over the real field : $AA^t = A^tA = I$
  - real unitary matrix = orthogonal matrix $\hookrightarrow A^{-1} = A^t$
  - columns ~~from~~ form an orthonormal basis for F^n.
  - rows form an orthonormal basis for F^n.
  - T is unitary $\Leftrightarrow [T]_\beta$ is unitary for an orthonormal β.
  - T is orthogonal $\Leftrightarrow [T]_\beta$is orthogonal for an orthonormal β.



$$[T]_\beta = \begin{bmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{bmatrix} = [T]_\beta^*$$

- A matrix A is **unitarily equivalent** to B : $\exists$ a unitary matrix Q such that $A = Q^*BQ$.
- A real matrix A is orthogonally equivalent to real B : $\exists$ an orthogonal matrix Q such that $A = Q^tBQ$.
- A and B are unitarily equivalent $\Rightarrow$ they are similar, but not conversely.
- Theorem 6.19: A complex $n \times n$ matrix A is normal.
  $\Leftrightarrow$ A is unitarily equivalent to a diagonal matrix.
- This is the matrix version of Theorem 6.16.

proof: "$\Rightarrow$": Assume A is normal and γ is the std basis for F^n.
$\Rightarrow$ L_A is normal and $[L_A]_\gamma = A$.
$\Rightarrow$ $\exists$ an orthonormal basis β of eigenvectors of L_A. [Thm 6.16]
$\Rightarrow$ $[L_A]_\beta = D$, diagonal.

$\Rightarrow A = [L_A]_\gamma = [I]_\beta^\gamma [L_A]_\beta [I]_\gamma^\beta = Q^{-1}DQ.$

We show that $Q = [I]_\gamma^\beta$ is unitary if β and γ are orthonormal.
Let $\beta = (u_1, \cdots, u_n)$ and $\gamma = (v_1, \cdots, v_n)$, both orthonormal.
$\Rightarrow v_j = \sum_{i=1}^n \langle v_j, u_i \rangle u_i \Rightarrow Q_{ij} = \langle v_j, u_i \rangle$
$\Rightarrow Q^*_{ij} = \overline{Q}_{ji} = \overline{\langle v_i, u_j \rangle} = \langle u_j, v_i \rangle = ([I]_\beta^\gamma)_{ij}$
$\Rightarrow QQ^* = [I]_\gamma^\beta [I]_\beta^\gamma = I$, and similarly, $Q^*Q = I$.

"$\Leftarrow$": Assume Q, unitary, and D, diagonal, are such that $A = Q^*DQ$.
$\Rightarrow AA^* = Q^*DQ(Q^*DQ)^* = Q^*DQQ^*D^*Q$
$= Q^*DD^*Q = Q^*D^*DQ = Q^*D^*QQ^*DQ = (Q^*DQ)^*Q^*DQ =$
$= A^*A$ [diagonals commute]

⊛ Thm 6.17

T is self-adjoint

- Theorem 6.20: A real $n \times n$ matrix A is symmetric.
  $\Leftrightarrow$ A is orthogonally equivalent to a real diagonal matrix.

- This is the matrix version of Theorem 6.17.

- Theorem 6.21 (Schur) : $A \in M_{n\times n}(F)$; $f_A(t)$ splits over F. Then

  1. If $F = \mathbb{C}$, then A is unitarily equivalent to a complex upper triangular matrix.
  2. If $F = \mathbb{R}$, then A is orthogonally equivalent to a real upper triangular matrix.

- This is the matrix version of Schur's Theorem 6.14.

Orthogonal projection and spectral theorem

- Let$V = W_1 \bigoplus W_2$ for subspaces W_1 and W_2.
  Then, $\forall x \in V, x = x_1 + x_2$ for some $\cancel{x_1} \in W_1, x_2 \in W_2$.
  - **projection** T on W_1 along $W_2 : T(x_1 + x_2) = x_1$
  - $R(T) = W_1 = \{x \in V : T(x) = x\}$
  - $N(T) = W_2 = \{x \in V : T(x) = 0\}$
  - For a projection T on W_1,we can choose various W_2.
  - If T is an ”orthogonal” projection on W_1, then W_2 is unique.
- T is a projection $\Leftrightarrow T = T^2$ [alt def]

  proof: $" \Rightarrow " : \forall x \in V, T^2(x) = T(x_1) = T(x_1 + 0) = x_1$
  $" \Leftarrow "$: Assume that $T = T^2$.

1

First we show that $R(T) \cap N(T) = \{0\}$.

Assume $x \in R(T) \cap N(T)$.

$x \in R(T) \Rightarrow \exists u$ such that $T(u) = x \Rightarrow T^2(u) = T(u) = x$

$x \in N(T) \Rightarrow T(x) = 0 \Rightarrow T^2(x) = T(x) = 0$ $\quad \mapsto T^2(u) = T(x)$

$\Rightarrow x = 0, (\because T^2(u) = T(x) = x = 0)$

2

Now we show that $V = R(T) + N(T)$.

$\forall x \in V$, let $x_1 = T(x)$ and $x_2 = x - x_1$. $\Rightarrow x_1 \in R(T)$

$T(x_2) = T(x) - T(x_1) = T(x) - T^2(x) = 0 \Rightarrow x_2 \in N(T)$

3

$V = R(T) \bigoplus N(T)$

- **orthogonal projection** T:
  $R(T)^{\perp} = N(T)$ and $N(T)^{\perp} = R(T)$
- If $\dim(V) < \infty$, $(R(T)^{\perp} = N(T) \Leftrightarrow N(T)^{\perp} = R(T))$.
- Given a subspace W, $T(y) = u$, [Thm 6.6]
  where $y = u + z, u \in W, z \in W^{\perp}$,
  defines an orthogonal projection on W.
- A truncated Fourier series, for $k < n$, $u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$ is an orthogonal projection on span$(\{v_1, \cdots, v_k\})$.
- There is only one orthogonal projection on W.
- The orthogonal projection on W provides the best approximation.

■ Theorem 6.24: V is an inner product space: $T : V \to V$ is linear. Then T is an orthogonal projection $\Leftrightarrow \underbrace{T^2 = T}_{\text{projection}} = T^*$. orthogonal proj.

Proof:

"$\Rightarrow$": Assume T is an orthogonal projection.
$\Rightarrow T^2 = T$ [projection]
$\Rightarrow V = R(T) \bigoplus N(T)$; $R(T)^\perp = N(T)$
$\Rightarrow \forall x, y \in V$, $x = x_1 + x_2$ and $y = y_1 + y_2$, for some $x_1, y_1 \in R(T)$ and $x_2, y_2 \in N(T)$.
$\Rightarrow \langle x, T(y) \rangle = \langle x_1 + x_2, y_1 \rangle = \langle x_1, y_1 \rangle + \overbrace{\langle x_2, y_1 \rangle}^{=0} = \langle x_1, y_1 \rangle = \langle x_1, y_1 \rangle + \underbrace{\langle x_1, y_2 \rangle}_{=0} = \langle x_1, y_1 + y_2 \rangle = \langle T(x), y \rangle = \langle x, T^*(y) \rangle$

$\Rightarrow T = T^*$

"$\Leftarrow$": Assume $T^2 = T = T^*$
$\Rightarrow T$ is a projection. orthogonal. (증명 필요)

Show $R(T)^\perp = N(T)$, $N(T)^\perp = R(T)$

Let $x \in R(T)$ and $y \in N(T)$. $\Rightarrow T(x) = x, T(y) = 0.$

$\Rightarrow \langle x, y\rangle = \langle T(x), y\rangle = \langle x, T^*(y)\rangle = \langle x, T(y)\rangle = \langle x, 0\rangle = 0$

$\Rightarrow x \in N(T)^\perp$ and $y \in R(T)^\perp$ (서로 수직)

1 $\Rightarrow R(T) \subseteq N(T)^\perp$ and $N(T) \subseteq R(T)^\perp$ (1)

Let $x \in N(T)^\perp$

$\subseteq N(T)^\perp$

$\Rightarrow x = x_1 + x_2, x_1 \in R(T), x_2 \in N(T)$ [projection]

$\Rightarrow 0 = \langle x, x_2\rangle = \underbrace{\langle x_1, x_2\rangle}_{=0} + \langle x_2, x_2\rangle = ||x_2||^2$ [$x_1 \in N(T)^\perp$ (1)]

$\Rightarrow x_2 = 0 \Rightarrow x = x_1 \in R(T) \overset{2}{\Rightarrow} N(T)^\perp \subseteq R(T) \Rightarrow N(T)^\perp = R(T)$ [(1), cf textbook]

Let $y \in R(T)^\perp$.

$\subseteq R(T)^\perp$

$\Rightarrow y = y_1 + y_2, y_1 \in R(T), y_2 \in N(T)$ [projection]

$\Rightarrow 0 = \langle y, y_1\rangle = \langle y_1, y_1\rangle + \underbrace{\langle y_2, y_1\rangle}_{=0} = ||y_1||^2 [y_2 \in R(T)^\perp$ (1)]

$\Rightarrow y_1 = 0 \Rightarrow y = y_2 \in N(T) \Rightarrow R(T)^\perp \subseteq N(T)$

$\Rightarrow R(T)^\perp = N(T)$ [(1), cf textbook]

- **self-adjoint (Hermitian) operator** T: $T = T^*$.
- **self-adjoint (Hermitian) matrix** A: $A = A^*$
- A: Conjugate symmetric.
- Theorem 6.17: For finite V over the "real" field, T is self-adjoint **iff** $\exists$ an orthonormal basis β for V consisting of eigenvectors of T.

proof:
"only if": Assume $T = T^*$. (real eigenvalues)
$\Rightarrow f_T(t)$ splits (over the real field). [Lemma 6.17]
$\Rightarrow \exists\beta$, orthonormal, s. t. $[T]_\beta$ is upper triangular. [Schur's Thm]
$\Rightarrow [T]^*_\beta = [T^*]_\beta = [T]_\beta$ [self-adjoint]
$\Rightarrow [T]_\beta$ is (real) diagonal.
$\Rightarrow \beta$ consists of eigenvectors of T.

T: Normal . $TT^* = T^*T$

Unitary and orthogonal operator $\Rightarrow TT^* = T^*T = I$

- V is an inner product space; $\dim(V) < \infty$; $T : V \to V$ is linear.
  - **unitary operator**: $\forall x \in V(\mathbb{C}), ||T(x)|| = ||x||$
  - **orthogonal operator**: $\forall x \in V(\mathbb{R}), ||T(x)|| = ||x||$
- Lemma 6.18: V is an inner product space; $\dim(V) < \infty$; $U : V \to V$ is self-adjoint. Then $\forall x \in V, \langle x, U(x)\rangle = 0 \Rightarrow U = T_0$.

  proof.
  Assume U is self-adjoint and $\forall x \in V, \langle x, U(x)\rangle = 0$.
  $\Rightarrow \exists \beta = \{v_1, \cdots, v_n\}$ orthonormal and consisting of eigenvectors of U. [Thm 6.16, 6.17]
  $\Rightarrow \overline{\lambda}_i \langle v_i, v_i\rangle = \langle v_i, \lambda_i v_i\rangle = \langle v_i, U(v_i)\rangle = 0, i = 1, \cdots, n$
  $\Rightarrow \lambda_i = 0, i = 1, \cdots, n$
  $\Rightarrow \forall x \in V,$
  $U(x) = U(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i U(v_i) = \sum_{i=1}^n a_i \lambda_i v_i = 0$

- Theorem 6.18: The followings are all equivalent.
  1. $TT^* = T^*T = I$
  2. $\forall x, y \in V, \langle T(x), T(y)\rangle = \langle x, y\rangle$
  3. $\exists$ an orthonormal basis β s. t. $T(\beta)$ is an orthonormal basis.
  4. $\forall x \in V, ||T(x)|| = ||x||$

proof: "1": $TT^* = T^*T = I$

$\Rightarrow$ "2": $\langle T(x), T(y)\rangle = \langle x, T^*T(y)\rangle = \langle x, y\rangle$

$\Rightarrow$"3" for $\beta = \{v_1, \cdots, v_n\}$, $\langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle = \delta_{ij}$

$\Rightarrow \{T(v_i), \cdots, T(v_n)\}$ is an orthonormal basis.

$\Rightarrow$"4" $: ||T(x)||^2 = ||T(\sum_{i=1}^n a_i v_i)||^2 = \langle \sum_{i=1}^n a_i T(v_i), \sum_{j=1}^n a_j T(v_j)\rangle$

$= \sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} \langle T(v_i), T(v_j)\rangle = \sum_{i=1}^n |a_i|^2 = ||x||^2$

$\Rightarrow$"1": Assume $\forall x \in V, ||T(x)|| = ||x||$.

$\Rightarrow \langle x, x\rangle = \langle T(x), T(x)\rangle = \langle x, T^*T(x)\rangle$

$\Rightarrow T^*T = I \Rightarrow TT^* = I$ [Thm 2.17c]

- **unitary matrix**: $AA^* = A^*A = I$
  - **orthogonal matrix** over the real field : $AA^t = A^tA = I$
  - T is unitary $\Leftrightarrow [T]_\beta$ is unitary for an orthonormal β.
  - T is orthogonal $\Leftrightarrow [T]_\beta$is orthogonal for an orthonormal β.

- A matrix A is **unitarily (or orthogonally) equivalent** to B : $\exists$ a unitary matrix Q such that $A = Q^*BQ(or \ \ Q^tBQ)$.

- Theorem 6.19: A is a complex and normal matrix.
  $\Leftrightarrow$ A is unitarily equivalent to a diagonal matrix.

  proof: "$\Rightarrow$": L_A is normal, where $[L_A]_\gamma = A$.
  $\Rightarrow \exists$ an orthonormal basis β of eigenvectors of L_A. [Thm 6.16]
  $\Rightarrow [L_A]_\beta = D \Rightarrow A = [L_A]_\gamma = [I]_\beta^\gamma [L_A]_\beta [I]_\gamma^\beta = Q^{-1}DQ$
  $\Rightarrow A = Q^*DQ$ since $Q = [I]_\gamma^\beta$ is $\underline{\text{unitary}}$ for orthonormal β and γ.
  $(\because \quad \mathbf{Q}_{ij} = \langle v_j, u_i\rangle, Q^*_{ij} = \overline{Q}_{ji} = \overline{\langle v_i, u_j\rangle} = \langle u_j, v_i\rangle = ([I]_\beta^\gamma)_{ij})$

- Theorem 6.20: A real $n \times n$ matrix A is symmetric(self-adjoint). $\Leftrightarrow A = Q^t DQ$, where D is a real diagonal matrix.

- Theorem 6.21 (Schur) : $A \in M_{n\times n}(F)$; $f_A(t)$ splits over F. Then

  1. If $F = \mathbb{C}$, then $A = Q^* UQ$, where U is a complex upper triangular matrix.
  2. If $F = \mathbb{R}$, then $A = Q^t UQ$, where U is a real upper triangular matrix.

Orthogonal projection and spectral theorem

- **projection** T on W_1 along $W_2 : T(x_1 + x_2) = x_1$

- $R(T) = W_1 = \{x \in V : T(x) = x\}$

- $N(T) = W_2 = \{x \in V : T(x) = 0\}$

$\Rightarrow V = R(T) \bigoplus N(T)$.

- For a projection T on W_1,we can choose various W_2.
- If T is an "orthogonal" projection on W_1, then W_2 is unique.
- T is a projection $\Leftrightarrow T = T^2$ [alt def]
- **orthogonal projection** T:
  $R(T)^\perp = N(T)$ and $N(T)^\perp = R(T)$
- If $\dim(V) < \infty$, $(R(T)^\perp = N(T) \Leftrightarrow N(T)^\perp = R(T))$.
- Given a subspace $W, T(y) = u$, where $y = u + z, u \in W, z \in W^\perp$, is an orthogonal projection on W.
- A truncated Fourier series, for $k < n$, $u = \sum_{i=1}^{k} \langle y, v_i \rangle$ is an orthogonal projection on span($\{v_1, \cdots, v_k\}$).
- The orthogonal projection on W provides the best approximation.
- [End of Review]

- Theorem 6.24: V is an inner product space:$T : V \to V$ is linear. Then T is an orthogonal projection $\Leftrightarrow$ $\underbrace{T^2 = T}_{\text{projection}} = T^*$. orthogonal

Proof:

" $\Rightarrow$ ": Assume T is an orthogonal projection.
$\Rightarrow T^2 = T$ [projection]
$\Rightarrow V = R(T) \bigoplus N(T)$; $R(T)^{\perp} = N(T)$
$\Rightarrow \forall x, y \in V, x = x_1 + x_2$ and $y = y_1 + y_2$, for some $x_1, y_1 \in R(T)$ and $x_2, y_2 \in N(T)$.
$\Rightarrow \langle x, T(y) \rangle = \langle x_1 + x_2, y_1 \rangle = \langle x_1, y_1 \rangle + \overset{=0}{\langle x_2, y_1 \rangle} = \langle x_1, y_1 \rangle = \langle x_1, y_1 \rangle + \underset{=0}{\langle x_1, y_2 \rangle} = \langle x_1, y_1 + y_2 \rangle = \langle T(x), y \rangle = \langle x, T^*(y) \rangle$

$\Rightarrow T = T^*$

" $\Leftarrow$ ": Assume $T^2 = T = T^*$
$\Rightarrow$ T is a projection.

$\Rightarrow$ orthogonal 임을 증명
$R(T)^{\perp} = N(T)$

Let $x \in R(T)$ and $y \in N(T)$. $\Rightarrow T(x) = x, T(y) = 0.$

$\Rightarrow \langle x, y\rangle = \langle T(x), y\rangle = \langle x, T^*(y)\rangle = \langle x, T(y)\rangle = \langle x, 0\rangle = 0$

$\Rightarrow x \in N(T)^\perp$ and $y \in R(T)^\perp$

$\Rightarrow R(T) \subseteq N(T)^\perp$ and $N(T) \subseteq R(T)^\perp$ (1)

Let $x \in N(T)^\perp$

$\Rightarrow x = x_1 + x_2, x_1 \in R(T), x_2 \in N(T)$ [projection]

$\Rightarrow 0 = \langle x, x_2\rangle = \overset{=0}{\langle x_1, x_2\rangle} + \langle x_2, x_2\rangle = ||x_2||^2$, [$\because x_1 \in N(T)^\perp$ (1)]

$\Rightarrow x_2 = 0 \Rightarrow x = x_1 \in R(T) \Rightarrow N(T)^\perp \subseteq R(T) \Rightarrow N(T)^\perp = R(T)$ [(1)]

Let $y \in R(T)^\perp$.

$\Rightarrow y = y_1 + y_2, y_1 \in R(T), y_2 \in N(T)$ [projection]

$\Rightarrow 0 = \langle y, y_1\rangle = \langle y_1, y_1\rangle + \overset{=0}{\langle y_2, y_1\rangle} = ||y_1||^2$, [$\because y_2 \in R(T)^\perp$ (1)]

$\Rightarrow y_1 = 0 \Rightarrow y = y_2 \in N(T) \Rightarrow R(T)^\perp \subseteq N(T)$

$\Rightarrow R(T)^\perp = N(T)$ [(1)]

■ **Theorem 6.25 (The spectral theorem):** V is an inner product space over F; $\dim(V) < \infty$; $T : V \to V$ is a linear operator with

distinct eigenvalues: **spectrum**	$\lambda_1 \cdots \lambda_k$
corresponding eigenspaces	$W_1 \cdots W_k$
orthogonal projection on W_i	$T_1 \cdots T_k$

and T is normal if $F = \mathbb{C}$ and self-adjoint if $F = \mathbb{R}$. Then the following statements are true.

1. $V = W_1 \oplus \cdots \oplus W_k$.
2. $W_i^{\perp} = \oplus_{j=1, j \neq i}^{k} W_j$.
3. $T_i T_j = T_0, i \neq j$.
4. $T_1 + \cdots + T_k = I$.
5. $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$: **spectral decomposition**

$$[T]_\beta = \begin{bmatrix} \lambda_1 & & & & & & & 0 \\ & \lambda_1 & & & & & & \\ & & \lambda_2 & & & & & \\ & & & \lambda_2 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_n & & \\ 0 & & & & & & \lambda_n \end{bmatrix}$$

proof of 5: $x \in V$

$\Rightarrow x = x_1 + \cdots + x_k, x_i \in W_i$ [1]

$$
\begin{aligned}
T(x) &= T(x_1) + \cdot + T(x_k) \\
&= \lambda_1 x_1 + \cdots + \lambda_k x_k \\
&= \lambda_1 T_1(x) + \cdots + \lambda_k T_k(x) \\
&= (\lambda_1 T_1 + \cdots + \lambda_k T_k)(x)
\end{aligned}
$$

- example: consider $T = L_A$,

  where $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

  $\lambda_1 = 2, W_1 = \{(a_1, 0, 0, 0)^t : a_1 \in \mathbb{R}\}, T_1 = L_{A_1}$,

  $\lambda_2 = 3, W_2 = \{(0, a_2, a_3, 0)^t : a_2, a_3 \in \mathbb{R}\}, T_2 = L_{A_2}$,

  $\lambda_3 = 5, W_3 = \{(0, 0, 0, a_4)^t : a_4 \in \mathbb{R}\}, T_3 = L_{A_3}$,

  where $A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and $A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

  $T = \lambda_1 T_1 + \lambda_2 T_2 + \lambda_3 T_3 : L_A = 2L_{A_1} + 3L_{A_2} + 5L_{A_3}$

- example: consider $T = L_A$, where $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

  $(\lambda_1 I - A) v = 0$

  $\lambda_1 = 3, W_1 = \{(a_1, a_2)^t \in \mathbb{R}^2 : a_1 = a_2\}, T_1 = L_{A_1}$,

  $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} v = 0$

  $\lambda_2 = -1, W_2 = \{(a_1, a_2)^t \in \mathbb{R}^2 : a_1 + a_2 = 0\}, T_2 = L_{A_2}$,

  $v_1 = v_2$

  $\vdots$

  where $A_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $A_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

  $T = \lambda_1 T_1 + \lambda_2 T_2 : L_A = 3L_{A_1} - L_{A_2}$

- **Corollary 6.25.2:** V is an inner product space over $\mathbb{C}$; $\dim(V) < \infty; T : V \to V$ is unitary $\Leftrightarrow$ T is normal and $|\lambda| = 1$ for every eigenvalue λ of T.

Proof.

"$\Rightarrow$" If T is unitary, then T is normal and every eigenvalue of T has absolute value 1 ($\because ||T(x)|| = ||x||$).

"$\Leftarrow$" Let $T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ be the spectral decomposition of T. If $|\lambda| = 1$ for every eigenvalue λ of T, then by 3. of the spectral theorem,

$$\begin{aligned} TT^* &= (\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k)(\overline{\lambda}_1 T_1^* + \overline{\lambda}_2 T_2^* + \cdots + \overline{\lambda}_k T_k^*) \\ &= |\lambda_1|^2 T_1 T_1^* + |\lambda_2|^2 T_2 T_2^* + \cdots + |\lambda_k|^2 T_k T_k^* \\ &= (T_1 + T_2 + \cdots + T_k)(T_1 + T_2 + \cdots + T_k)^* \\ &= I \end{aligned}$$

Hence T is unitary.

- **Corollary 6.25.3:** V is an inner product space over $\mathbb{C}$; $\dim(V) < \infty; T : V \to V$ is normal. Then T is self-adjoint $\Leftrightarrow$ every eigenvalue of T is real.

Proof.

"$\Rightarrow$" $T^* = T \Rightarrow \overline{\lambda}_1 T_1 + \overline{\lambda}_2 T_2 + \cdots + \overline{\lambda}_k T_k = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k \Rightarrow \lambda_i$ is real.

"$\Leftarrow$" Let $T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ be the spectral decomposition of T. Suppose that every eigenvalue of T is real. Then $\underline{T_i^*(v_i)} = \lambda_i v_i = \underline{T_i(v_i)}$, so $T^* = \lambda_1 T_1^* + \lambda_2 T_2^* + \cdots + \lambda_k T_k^* = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k) = T$.

- Reflection is an example that is both self adjoint and unitary.

$$[T]_\beta = \begin{bmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{bmatrix}$$

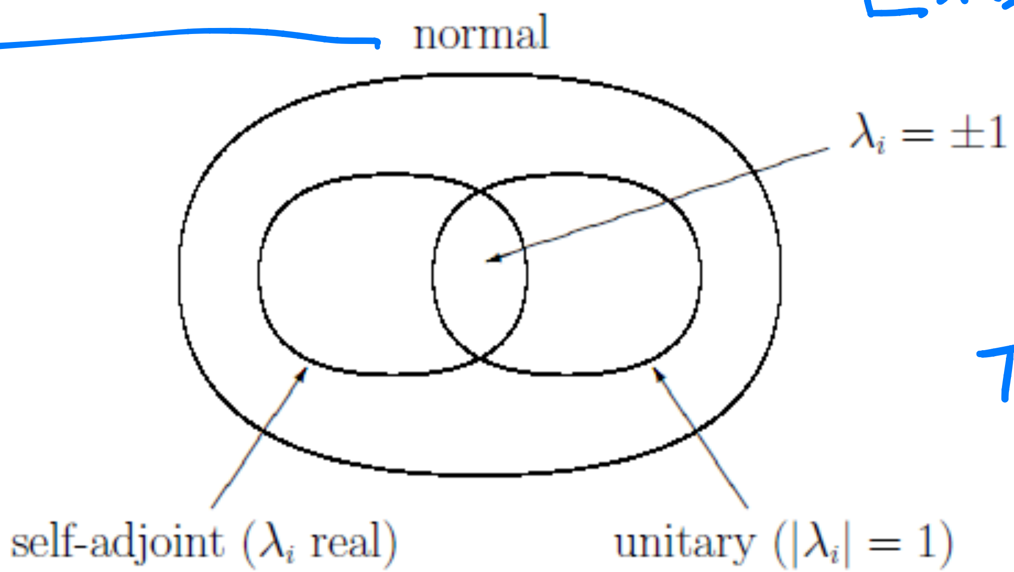
$T^* = T$

$T^*T = I = TT$

$$* \begin{bmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = TT = T^*T$$

T^*T

$= TT^*$



$T^* = T$

λ_i real

$T^*T = TT^* = I$

$\|T(x)\| = \|x\|$

$|\lambda_i| = 1$