## Composition of linear transformations

- The **composition** $UT$ of $T\colon\ V \to W$ and $U\colon\ W \to Z$ is a linear transformation such that $\forall x \in V,\ (UT)(x) = U(T(x))$
- Theorem 2.9 and 2.10: $S$, $T$, $T_1$, $T_2$, $U$, $U_1$, and $U_2$ are linear transformations with appropriate domains and co-domains; $I$ is an identity transformation; $a \in F$. Then we have as belows:

  $I(x) = x$

  1. $UT$ is a linear transform.
  2. distributivity(1): $U(T_1 + T_2) = UT_1 + UT_2$
  3. distributivity(2): $(U_1 + U_2)T = U_1T + U_2T$
  4. associativity: $S(UT) = (SU)T$
  5. identity: $IT = TI = T$ (Note that the two $I$'s are different.)
  6. $a(UT) = (aU)T = U(aT)$

## matrix representation of composition

$$
\begin{array}{cccccc}
 & V & \xrightarrow{T} & W & \xrightarrow{U} & Z \\
\text{basis} & \alpha & & \beta & & \gamma \\
 & \{v_j\} & B & \{w_k\} & A & \{z_i\} \\
\text{dim} & n & & m & & p
\end{array}
$$

$$B = [T]_\alpha^\beta,\ A = [U]_\beta^\gamma$$

- Theorem 2.11: $[UT]_\alpha^\gamma = C = AB = [U]_\beta^\gamma [T]_\alpha^\beta$

- example:

  $U\colon\ P_3(\mathbb{R}) \to P_2(\mathbb{R}),\ U(f) = f'$

  $T\colon\ P_2(\mathbb{R}) \to P_3(\mathbb{R}),\ T(f) = \int_0^x f(t)\,dt$

$$\boxed{\begin{array}{cccc} & P_2 \xrightarrow{T} & P_3 \xrightarrow{U} & P_2 \\ \text{std basis} & \alpha & \beta & \alpha \end{array}} \qquad \boxed{\begin{array}{ccc} P_3 \xrightarrow{U} & P_2 \xrightarrow{T} & P_3 \\ \beta & \alpha & \beta \end{array}}$$

$$[UT]_\alpha = [U]_\beta^\alpha [T]_\alpha^\beta = [I_{P_2}]_\alpha$$

$$[TU]_\beta = [T]_\alpha^\beta [U]_\beta^\alpha = \neq [I_{P_3}]_\beta$$

- **identity matrix**: $I\colon\ I_{ij} = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{else } (i \neq j) \end{cases},$

  where $\delta_{ij}$ is the **Kronecker Delta**.

**left-multiplication transformation** $L_A$ for a matrix $A$:

$L_A : F^n \to F^m$ such that $L_A(x) = Ax$

$m\times 1$ $m\times n \cdot n\times 1$

Theorem 2.15 and 2.16: $A$ is an $m \times n$ matrix; $a \in F$. Then the followings hold:

1. $L_A$ is a linear transformation.

$$\begin{array}{|llcl|} & F^n & \xrightarrow{L_A} & F^m \\ \text{std basis} & \beta & & \gamma \end{array}$$

2. $[L_A]_\beta^\gamma = A$
3. $L_A = L_B \Leftrightarrow A = B$: uniqueness of $L_A$ given $A$
4. $L_{A+B} = L_A + L_B$; $L_{aA} = aL_A$ $L_{A+B}(x) = (A+B)x$
5. $T$: $F^n \to F^m$ is linear $\Rightarrow [T]_\beta^\gamma = ?$

$\Rightarrow \exists$ a unique $m \times n$ matrix $C$ such that
$T = L_C$ and $C = [T]_\beta^\gamma$

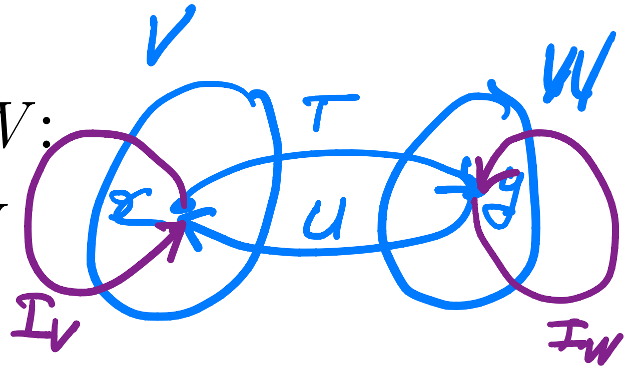
6. $G$ is an $n \times p$ matrix $\Rightarrow L_{AG} = L_A L_G$

7. $G$ is an $n \times p$ matrix; $H$ is an $p \times q$ matrix
$\Rightarrow L_{A(GH)} = L_A L_{GH} = L_A L_G L_H = L_{AG} L_H = L_{(AG)H}$
$\Rightarrow A(GH) = (AG)H$
: associativity of composition $\Rightarrow$ associativity of matrix multiplication

8. $m = n \Rightarrow L_{I_n} = I_{F^n}$ $\left( \because I_{F^n}(x) = x = I_n x = L_{I_n}(x) \right)$

proof: Once we have $[L_A]_\beta^\gamma = A$, all these can be proven through the properties of linear transformations and their matrix representations.

# Invertibility and isomorphism




- **inverse** $U$ of a linear transformation $T\colon\ V \to W$: $U\colon\ W \to V$ such that $TU = I_W$ and $UT = I_V$
  - notation: $U = T^{-1}, T = U^{-1}$

- Theorem 2.17a including Theorem 2.17: $T\colon\ V \to W$ is a linear transformation. Then the followings hold.
  1. $T$ is invertible $\Leftrightarrow$ $T$ is one-to-one and onto
  2. $T^{-1}$ is unique.
  3. $T^{-1}$ is linear. $\because$ $T^{-1}(cy_1 + y_2) = T^{-1}(cT(x_1) + T(x_2)) = T^{-1}T(cx_1 + x_2) = cx_1 + x_2 = cT^{-1}(y_1) + T^{-1}(y_2)$.
  4. $(T^{-1})^{-1} = T$. $\because (T^{-1})^{-1}(x_1) = U^{-1}(x_1) = y_1 = T(x_1),$
  5. $(UT)^{-1} = T^{-1}U^{-1}$.$\because ((UT)^{-1})(x_1) = x_1 = T^{-1}(y_1) = T^{-1}(U^{-1}(x_1)) = T^{-1}U^{-1}(x_1)$

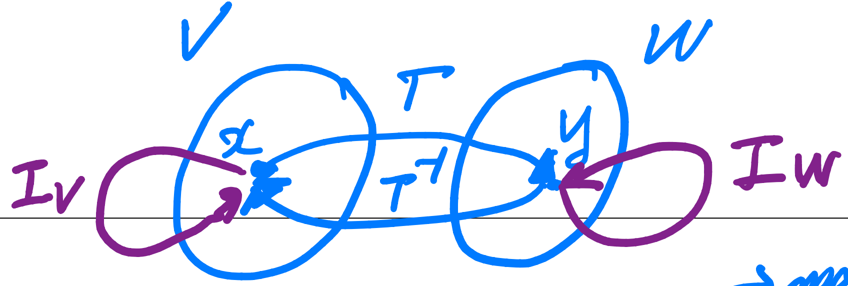
- **inverse** $B$ of a matrix $A$: $AB = I_n$ and $BA = I_n$
  - notation: $B = A^{-1}$ $TU = I_W$ $UT = I_V$

- Theorem 2.17b: $A$ and $B$ are $n \times n$ invertible matrices. Then the followings hold.
  1. $A^{-1}$ is unique.
  2. $(A^{-1})^{-1} = A$
  3. $(AB)^{-1} = B^{-1}A^{-1}$

- Theorem 2.17c: $\dim(V) = \dim(W) < \infty$;
  $T\colon V \to W$ and $U\colon W \to V$ are linear transformations.
  Then $UT = I_V \Leftrightarrow TU = I_W$.

- Theorem 2.17d: Let $A, B \in M_{n\times n}$, Then $AB = I_n \Leftrightarrow BA = I_n$.
- Lemma 2.18: $T$: $V \to W$ is a linear transformation; $\dim(V)$, $\dim(W) < \infty$. Then

  $T$ is invertible $\Rightarrow \dim(V) = \dim(W)$.




## Theorem 2.18:

$$\begin{array}{cccc} & V & \xrightarrow{T} & W \\ \text{basis} & \beta & \xleftarrow{T^{-1}} & \gamma \\ & \{v_i\} & & \{w_j\} \end{array}$$

$T$ is invertible $\Leftrightarrow$ $\underbrace{[T]_\beta^\gamma}_{\text{matrix}}$ is invertible; and $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$. □

proof: "$\Rightarrow$": Assume $T$ is invertible.

$\Rightarrow \dim(V) = \dim(W) = n$ [Lemma 2.18] $\Rightarrow [T]_\beta^\gamma$ is $n \times n$

$$I_n = [I_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma \quad \Leftarrow \text{Thm 2.11}$$

$$I_n = [I_W]_\gamma = [TT^{-1}]_\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta$$

$\Rightarrow [T]_\beta^\gamma$ is invertible and $\underbrace{[T^{-1}]_\gamma^\beta}_{B} = (\underbrace{[T]_\beta^\gamma}_{A})^{-1}$

"$\Leftarrow$": Assume $A = [T]_\beta^\gamma$ is invertible and $B = A^{-1}$.

$$[U]_\beta = \begin{bmatrix} B_{1j} & \cdots \\ B_{2j} & \cdots \\ \vdots & \vdots \\ B_{nj} & \cdots \end{bmatrix} \triangleq B$$

Let $U$ be the linear transformation such that $U(w_j) = \sum_{i=1}^n B_{ij} v_i$.

$= T^{-1}(w_j)$

$\Rightarrow [U]_\gamma^\beta = B$

$\Rightarrow U$ is unique. [Theorem 2.6]

$[UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_V]_\beta$ $= \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots 1 \end{bmatrix}$

$[TU]_\gamma = [T]_\beta^\gamma [U]_\gamma^\beta = AB = I_n = [I_W]_\gamma$

$\Rightarrow UT = I_V,\ TU = I_W$ [matrix representation is unique]

$\Rightarrow T$ is invertible and $U = T^{-1}$

■ Corollary 2.18.2: Let $A \in M_{n\times n}$, then the following is true:
$A$ is invertible $\Leftrightarrow L_A$ is invertible; and $(L_A)^{-1} = L_{A^{-1}}$.

$y = L_A(x) = Ax$

$\underline{(L_A)^{-1}}(y) = x = A^{-1}y = \underline{L_{A^{-1}}}(y)$

- **isomorphism**: invertible linear transformation
  - $V$ and $W$ are **isomorphic**:
    $\exists$ an invertible linear transformation $T\colon\ V \to W$.

- example:

  $\mathbb{R}^4$ and $P_3(\mathbb{R})$: $(a_1, a_2, a_3, a_4) \leftrightarrow a_1 + a_2x + a_3x^2 + a_4x^3$

  $P_3(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$: $a_0 + a_1x + a_2x^2 + a_3x^3 \leftrightarrow \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}$

- Theorem 2.19: $\dim(V)$, $\dim(W) < \infty$. Then

  $V$ and $W$ are isomorphic $\Leftrightarrow \dim(V) = \dim(W)$

- Corollary 2.19: $V$ and $F^n$ are isomorphic $\Leftrightarrow \dim(V) = n$

1:1 대응 → Rep. Theorem

* n-tuple 로 표현 가능.

L.T.

$\Phi$ : L.T. → Matrix

■ Theorem 2.20:

$$\begin{array}{lccc} & V & \xrightarrow{T} & W \\ \text{basis} & \beta & \xrightarrow{A} & \gamma \\ \text{dim} & n & & m \end{array}$$

$\Phi : \mathcal{L}(V, W) \rightarrow M_{m\times n}$ defined by $\Phi(T) = \underbrace{[T]_\beta^\gamma}_{A}$ is an isomorphism. □

proof: $\Phi$: $L(V, W) \to M_{m\times n}$ is linear. [Theorem 2.8]

The matrix representation is unique.

■ Given a basis $\beta$ for $V$,
$\phi_\beta$: $V \to F^n$ defined by
$\phi_\beta(x) = [x]_\beta$ is also an isomorphism.

$$\begin{array}{ccccc} V & \xrightarrow{T} & W & \mathcal{L}(V,W) & T \\ \downarrow \phi_\beta & & \downarrow \phi_\gamma & \downarrow \Phi & \downarrow \Phi \\ F^n & \xrightarrow[A = [T]_\beta^\gamma]{L_A} & F^m & M_{m\times n} & A \end{array}$$

- example: $T \quad P_2(\mathbb{R}) \to P_1(\mathbb{R})$ such that $T(f) = f'$

$$\beta = \{1, x, x^2\},\ \gamma = \{1, x\} \Rightarrow A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Leftarrow \begin{cases} T(1) = 0 & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ T(x) = 1 & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T(x^2) = 2x & \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{cases}$$

$$\begin{tikzcd}
a_0 + a_1 x + a_2 x^2 \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & a_1 + 2a_2 x \arrow[d, "\phi_\gamma"] \\
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \arrow[r, "L_A", "A = {[T]_\beta^\gamma}"'] & \begin{pmatrix} a_1 \\ 2a_2 \end{pmatrix}
\end{tikzcd}$$

$$\begin{bmatrix} a_1 \\ 2a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$\beta = \{1, 1+x, 1+x+x^2\},\ \gamma = \{1, 1+x\} \Rightarrow A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{tikzcd}
a_0 + a_1 x + a_2 x^2 \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & a_1 + 2a_2 x \arrow[d, "\phi_\gamma"] \\
\begin{pmatrix} a_0 - a_1 \\ a_1 - a_2 \\ a_2 \end{pmatrix} \arrow[r, "L_A", "A = [T]_\beta^\gamma"'] & \begin{pmatrix} a_1 - 2a_2 \\ 2a_2 \end{pmatrix}
\end{tikzcd}$$

$= a_1 - 2a_2 + 2a_2(1+x)$

$T(1) = 0$

$T(1+x) = 1$

$T(1+x+x^2) = 1+2x$

$= -1 + 2(1+x)$

- This example illustrates that the representations depend on the bases. If we change the given set of bases to a new one, how do the representations change?

?

## Change of Basis

- Consider two bases $\beta = \{v_1, \cdots, v_n\}$ and $\beta' = \{u_1, \cdots, u_n\}$ for the same vector space $V$ and the matrix representations of the identity transformation $I$.

  The $i$-th column of $[I]_{\beta}^{\beta'}$ is $[v_i]_{\beta'}$ ; the $i$-th column of $[I]_{\beta'}^{\beta}$ is $[u_i]_{\beta}$.

  For $x \in V, x = \sum a_i v_i$ for $\beta \to I_{\beta}^{\beta'}(x) = x, x = \sum b_i u_i$ for $\beta'$.

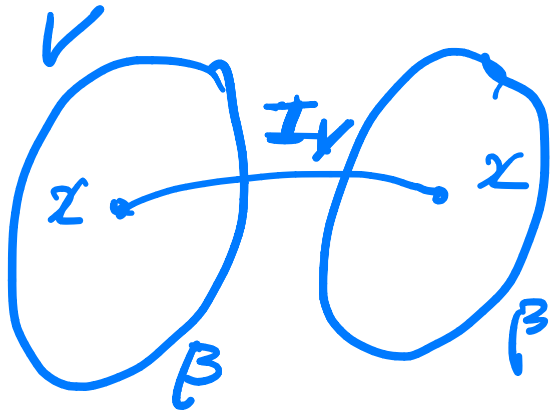
  $\Rightarrow [x]_{\beta} = (a_1, ..., a_n)^t, [x]_{\beta'} = (b_1, ..., b_n)^t$.

  $\Rightarrow (b_1, ..., b_n)^t = [I]_{\beta}^{\beta'} (a_1, ..., a_n)^t$.

$[I_V]_{\beta}^{\beta'} = ?$

$[x]_{\beta'} = [I_V]_{\beta}^{\beta'} [x]_{\beta}$

$[x]_{\beta} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$




$[x]_{\beta'} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

- example:

  for $I : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, let bases for domain and co-domain be

  $\beta = \{1, x, x^2\}$ and $\beta' = \{1, 1+x, 1+x+x^2\}$.

  $0 \cdot 1 \cdot -1(1+x) + 1 \cdot (1+x+x^2)$

  $\mathbf{x} = a_0 + a_1 x + a_2 x^2$

  $\Rightarrow [\mathbf{x}]_\beta = (a_0, a_1, a_2)^t, \ [\mathbf{x}]_{\beta'} = (a_0 - a_1, a_1 - a_2, a_2)^t$

  $a_0 \ a_1 \ a_2$

  $\mathbf{x} = I(\mathbf{x})$

  $\Rightarrow [\mathbf{x}]_{\beta'} = [I]_\beta^{\beta'} [\mathbf{x}]_\beta, \ [I]_\beta^{\beta'} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

  $I(1) = 1$ $\quad 1 \ 0 \ 0$

  $I(x) = x$ $\quad 0 \ 1 \ 0$

  $I(x^2) = x^2$ $\quad 0 \ 0 \ 1$

  $$\Rightarrow \begin{pmatrix} a_0 - a_1 \\ a_1 - a_2 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

- example: Consider two bases for $\mathbb{R}^2$

$\beta = \{(1,0),(0,1)\}$ and $\beta' = \{(\cos\lambda, \sin\lambda), (-\sin\lambda, \cos\lambda)\}$ ,

where $\beta'$ is the rotated version of $\beta$ by the angle $\lambda$.

$x = (a,b)^t$

$$\Rightarrow [x]_\beta = (a,b)^t,\ [I]_\beta^{\beta'} = \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix}: \text{ rotation by } -\lambda$$

$$\Rightarrow [x]_{\beta'} = [I]_\beta^{\beta'}[x]_\beta = \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

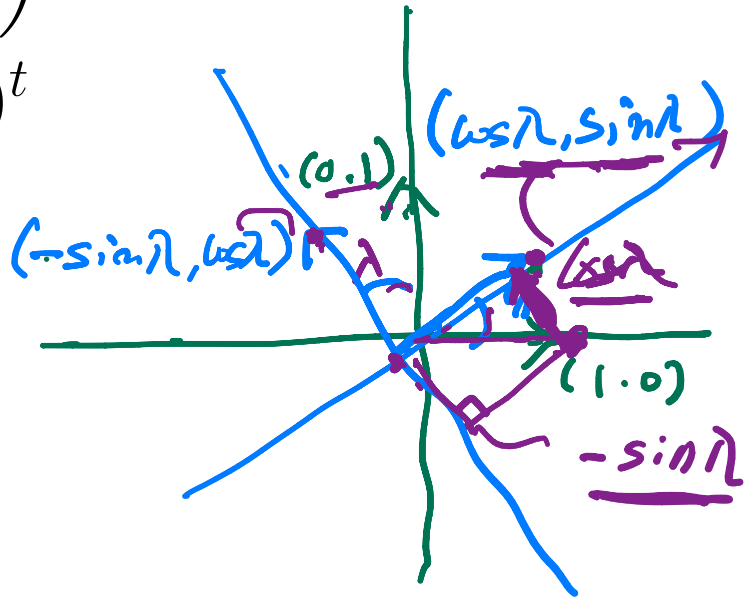
$$\therefore [x]_{\beta'} = (a\cos\lambda + b\sin\lambda, -a\sin\lambda + b\cos\lambda)^t$$

$I(1.0)]_\beta = (1.0)^T$, $[(1.0)]_{\beta'} = \begin{bmatrix} \cos\lambda \\ -\sin\lambda \end{bmatrix}$

$\Rightarrow \cos\lambda(\cos\lambda, \sin\lambda)$

$+ -\sin\lambda(-\sin\lambda, \cos\lambda)$

$= (\cos^2\lambda + \sin^2\lambda, 0)$



- $[I]_{\beta}^{\beta'}$ and $[I]_{\beta'}^{\beta}$ are called **change-of-coordinate matrices**.
  - $[I]_\beta = [II]_\beta = [I]^{\beta\cancel{'}}_{\beta'}[I]^{\beta'}_{\beta\cancel{'}} = I_n$

    $\Rightarrow [I]_{\beta}^{\beta'} = ([I]_{\beta'}^{\beta})^{-1}$,

    $[I]_{\beta'}^{\beta} = ([I]_{\beta}^{\beta'})^{-1}$
  - $[x]_{\beta'} = [I]_{\beta}^{\beta'}[x]_\beta$

    $[x]_\beta = [I]_{\beta'}^{\beta}[x]_{\beta'}$

$$
\begin{tikzcd}
x \quad V \arrow[r, "T"] \arrow[d, "\phi_\beta"] \arrow[dd, bend right=60, "\phi_{\beta'}"'] & V \arrow[d, "\phi_\beta"'] \arrow[dd, bend left=60, "\phi_{\beta'}"] \\
{[x]_\beta} \quad F^n \arrow[r, "L_A", "A=[T]_\beta"'] \arrow[d, "{[I]_\beta^{\beta'}}"'] \arrow[dr, phantom, "*"] & F^n \arrow[d, "{[I]_\beta^{\beta'}}"] \\
{[x]_{\beta'}} \quad F^n \arrow[r, "L_B", "B=[T]_{\beta'}"'] & F^n
\end{tikzcd}
$$

  - We can play with these matrices as follows.

    $[I]_\alpha = [III]_\alpha = I_\gamma^\alpha I_\beta^\gamma I_\alpha^\beta = I_n$

    $[x]_\gamma = [I]_\alpha^\gamma [x]_\alpha = [I]_\beta^\gamma [I]_\alpha^\beta [x]_\alpha$

- What is the relationship between $A = [T]_\beta$ and $B = [T]_{\beta'}$ ?

$$B = [T]_{\beta'} = [ITI]_{\beta'} = [I]_\beta^{\beta'}[T]_\beta[I]_{\beta'}^\beta ; = Q^{-1}AQ$$

$$B = Q^{-1}AQ, \text{ where } Q = [I]_{\beta'}^\beta \quad [\star]$$

- Matrix $A, B \in M_{n\times n}$ are **similar**.
  $\exists$ invertible $Q$ such that $B = Q^{-1}AQ$.
  - So similar matrices can be considered representations of the same linear transformation.

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\beta"] \arrow[dd, bend right, "\beta'"'] & V \arrow[d, "\beta"] \\
F^n \arrow[r, "L_A"] \arrow[d, "{[I]_\beta^{\beta'}}"] & F^n \arrow[d, "{[I]_\beta^{\beta'} = Q^{-1}}"] \\
F^n \arrow[r, "L_B"] & F^n
\end{tikzcd}$$

$$B = Q^{-1}AQ$$

- example: $T$: $P_2(\mathbb{R}) \to P_2(\mathbb{R})$ such that $T(f) = f'$

$\beta = \{1, x, x^2\}$, $\beta' = \{1, 1+x, 1+x+x^2\}$

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad [T]_{\beta'} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} T(1) = 0 & (0,0,0) \\ T(1+x) = 1 & (1,0,0) \\ T(1+x+x^2) = 1+2x \end{cases}$$
$$= -1 + 2(1+x) \Rightarrow (-1, 2, 0)$$

$$Q = [I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q^{-1} = [I]_{\beta}^{\beta'} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} x = -1 \cdot 1 + 1(1+x) \\ x^2 = 0 \cdot 1 + (-1)(1+x) + 1 \cdot (1+x+x^2) \end{cases}$$

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q$$

$$\Leftrightarrow \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = Q^{-1} \quad A \quad Q$$

## Review



- Theorem 2.18:

$$\boxed{\begin{array}{cccc} & V & \xrightarrow{T} & W \\ \text{basis} & \beta & & \gamma \\ & \{v_i\} & & \{w_j\} \end{array}}$$

$T$ is invertible $\Leftrightarrow$ $[T]_\beta^\gamma$ is invertible; and $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$. $\square$

- Corollary 2.18.2: Let $A \in M_{n\times n}$, then the following is true: $A$ is invertible $\Leftrightarrow$ $L_A$ is invertible; and $(L_A)^{-1} = L_{A^{-1}}$.

- **isomorphism**: invertible linear transformation

- Theorem 2.19: $\dim(V)$, $\dim(W) < \infty$. Then $V$ and $W$ are isomorphic $\Leftrightarrow$ $\dim(V) = \dim(W)$

- Corollary 2.19: $V$ and $F^n$ are isomorphic $\Leftrightarrow$ $\dim(V) = n$

- Theorem 2.20:

  Given a basis $\beta$ for $V$,
  $\phi_\beta$: $V \to F^n$ defined by
  $\phi_\beta(x) = [x]_\beta$ is also an isomorphism.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] & \mathcal{L}(V,W) \arrow[d, "\Phi"] \\ F^n \arrow[r, "L_A", "A = [T]_\beta^\gamma"'] & F^m & M_{m\times n} \end{tikzcd}$$

- example:

  $\beta = \{1, x, x^2\}$,
  $\gamma = \{1, x\}$
  $\Rightarrow A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$\begin{tikzcd} a_0 + a_1 x + a_2 x^2 \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & a_1 + 2a_2 x \arrow[d, "\phi_\gamma"] \\ \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \arrow[r, "L_A", "A = [T]_\beta^\gamma"'] & \begin{pmatrix} a_1 \\ 2a_2 \end{pmatrix} \end{tikzcd}$$

# Change of Basis

- matrix representations of the identity transformation $I$.
- for $I : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, let bases for domain and co-domain be $\beta = \{1, x, x^2\}$ and $\beta' = \{1, 1+x, 1+x+x^2\}$.

  $\mathbf{x} = a_0 + a_1 x + a_2 x^2$

  $\Rightarrow [\mathbf{x}]_\beta = (a_0, a_1, a_2)^t,\ [\mathbf{x}]_{\beta'} = (a_0 - a_1, a_1 - a_2, a_2)^t$

  $\mathbf{x} = I(\mathbf{x})$

  $\Rightarrow [\mathbf{x}]_{\beta'} = [I]_\beta^{\beta'}[\mathbf{x}]_\beta,\ [I]_\beta^{\beta'} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

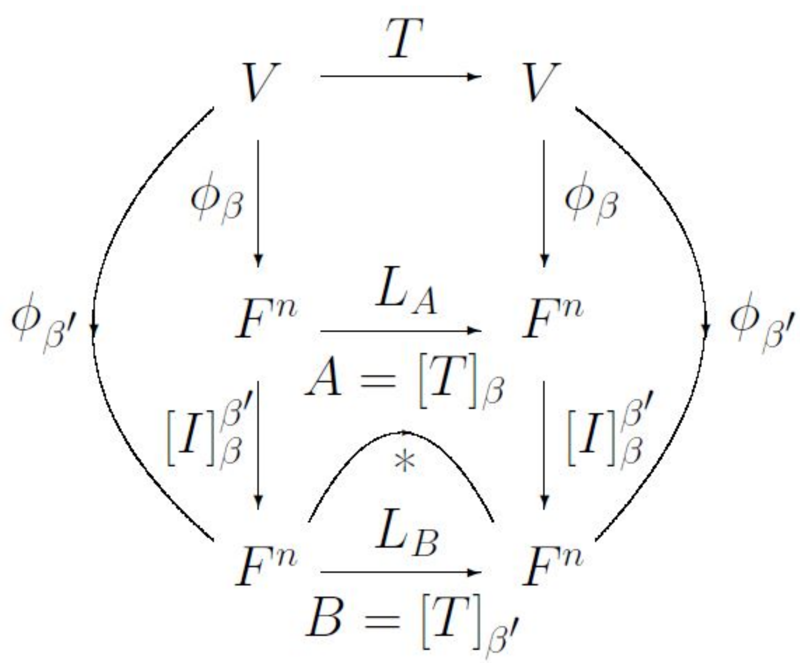
  $\Rightarrow \begin{pmatrix} a_0 - a_1 \\ a_1 - a_2 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$

- $[I]_{\beta}^{\beta'}$ and $[I]_{\beta'}^{\beta}$ are called **change-of-coordinate matrices**.
  - $[I]_{\beta} = [II]_{\beta} = [I]_{\beta'}^{\beta}[I]_{\beta}^{\beta'} = I_n$

    $\Rightarrow [I]_{\beta}^{\beta'} = ([I]_{\beta'}^{\beta})^{-1},$

    $[I]_{\beta'}^{\beta} = ([I]_{\beta}^{\beta'})^{-1}$
  - $[x]_{\beta'} = [I]_{\beta}^{\beta'}[x]_{\beta}$

    $[x]_{\beta} = [I]_{\beta'}^{\beta}[x]_{\beta'}$




- $[T]_{\beta'} = [ITI]_{\beta'} = [I]_{\beta}^{\beta'}[T]_{\beta}[I]_{\beta'}^{\beta}$ ;

  $B = Q^{-1}AQ$, where $Q = [I]_{\beta'}^{\beta}$

$\rightarrow$ Matrix $A, B \in M_{n\times n}$ are **similar**.

# Chapter 3
# Elementary matrix operation and system of linear equations

$$
\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\
F^n \arrow[r, "L_A"] & F^m
\end{tikzcd}
$$

$$A = [T]_\beta^\gamma$$

- Given bases, we can work on any kinds of vectors and any kinds of linear transformations using only $n$-tuple vectors in $F^n$ and matrices in $M_{m\times n}(F)$.

Vector

Matrix

- $w = T(v)$ turns into $y = Ax$, a system of linear equations.

ex) 미적분 ⇒ 선형대수 ( 미분방정식 → 상태방정식 )

- We now investigate $n$-tuples, matrices, and linear equations.

( 실세계 문제 → 최적화 formulation

# Elementary matrix operation and elementary matrix

- **elementary row operation, ero,** on a matrix:

  **type 1**, ero1: interchange two rows
  **type 2**, ero2: multiply a row by a non-zero scalar
  **type 3**, ero3: add to a row a scalar multiple of another

$$\begin{pmatrix} \mathbf{a}_{31} & \mathbf{a}_{32} & \mathbf{a}_{33} \\ a_{21} & a_{22} & a_{23} \\ \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{b}a_{21} & \mathbf{b}a_{22} & \mathbf{b}a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} - \mathbf{ca}_{11} & a_{32} - \mathbf{ca}_{12} & a_{33} - \mathbf{ca}_{13} \end{pmatrix}$$

- **Elementary column operations, ecos,** are similar.

- You may recall these having being used in solving a system of linear systems.

▪▪ example:

$$\begin{pmatrix} 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \\ 9 & 10 & 11 & 12 \end{pmatrix} \text{[ero1(r1,r2)]}$$

$5 \times$

+) $r_2$

5 10 15 20

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ -\mathbf{5} & -\mathbf{6} & -\mathbf{7} & -\mathbf{8} \\ 9 & 10 & 11 & 12 \end{pmatrix} \text{[ero2(r2}\times\text{(-1))]}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ \mathbf{0} & \mathbf{4} & \mathbf{8} & \mathbf{12} \\ 9 & 10 & 11 & 12 \end{pmatrix} \text{[ero3(r2+r1}\times\text{5)]}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 8 & 12 \\ \mathbf{0} & -\mathbf{8} & -\mathbf{16} & -\mathbf{24} \end{pmatrix} \text{[ero3(r3+r1}\times\text{(-9))]}$$

▪▪ **elementary matrix, em**: matrix obtained by an ero on $I_n$ □

**type 1**: $\begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}$ : em1(r2, r3) $\times$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

**type 2**: $\begin{pmatrix} \mathbf{-2} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ : em2(r1×(-2)) $\times$ $\begin{pmatrix} & '' & \end{pmatrix} = \begin{pmatrix} -2a_{11} & -2a_{12} & -2a_{13} \\ & '' & \end{pmatrix}$

**type 3**: $\begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} \\ 0 & 0 & 1 \end{pmatrix}$ : em1(r2+r3×(-2)) $\times$ $\begin{pmatrix} & '' & \end{pmatrix} = \begin{pmatrix} & '' & \\ a_{21}-2a_{31}, & \circ, & \circ \\ & '' & \end{pmatrix}$

▪▪ Theorem 3.1: An elementary row operation on a matrix $A$ is the same as pre-multiplying an elementary matrix of the same type to $A$; and an elementary column operation is the same as post-multiplying and elementary matrix of the same type. □

$E^rA = \tilde{A}$, $AE^c = \bar{A}$

▪▪ example:

$$\begin{pmatrix}5&6&7&8\\1&2&3&4\\9&10&11&12\end{pmatrix}\to\begin{pmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}5&6&7&8\\1&2&3&4\\9&10&11&12\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&3&4\\-5&-6&-7&-8\\9&10&11&12\end{pmatrix}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&3&4\\0&4&8&12\\9&10&11&12\end{pmatrix}=\begin{pmatrix}1&0&0\\5&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\-5&-6&-7&-8\\9&10&11&12\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&3&4\\0&4&8&12\\0&-8&-16&-24\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\-9&0&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\0&4&8&12\\9&10&11&12\end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & \overset{2}{\cancel{3}} & \overset{3}{\cancel{2}} & 1 \\ 8 & 6 & 7 & 5 \\ 12 & 10 & 11 & 9 \end{pmatrix} \text{[eco1(c1,c4)]}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & \overset{0}{\cancel{1}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 5 & -4 & 7 & 8 \\ 9 & -8 & 11 & 12 \end{pmatrix} \text{[eco3(c2+c1}\times\text{(-2))]}$$

- Theorem 3.2: Elementary matrices are invertible, and the inverse of an elementary matrix is an elementary matrix of the same type. □
- Also the transpose of an elementary matrix is an elementary matrix of the same type.

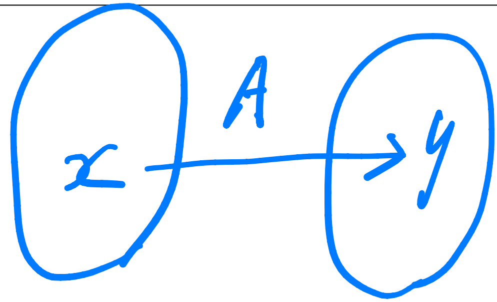
▪▪ example:

em1(r2,r3) $\leftrightarrow$
em1(r2,r3)
em1(c2,c3) $\leftrightarrow$ : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
em1(c2,c3)

em2(r1×(-2)) $\leftrightarrow$
em2(r1×($-\frac{1}{2}$))
em2(c1×(-2)) $\leftrightarrow$ : $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
em2(c1×($-\frac{1}{2}$))

em3(r2+r3×(-2)) $\leftrightarrow$
em3(r2+r3×2)
em3(c3+c2×(-2)) $\leftrightarrow$ : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$
em3(c3+c2×2)

# Rank and inverse of matrix




- **rank** of a matrix $A$, rank($A$): rank($L_A$) $\square$
  - $A \in M_{n\times n}$; rank($A$) = $n$ [**full rank**]
    $\Rightarrow$ nullity($L_A$) = 0 [dim thm]
    $\Rightarrow$ $L_A$ is one-to-one. [Thm 2.4]
    $\Rightarrow$ $L_A$ is onto. [Thm 2.5] $(\because \dim(V)=\dim(W)=n)$
    $\Rightarrow$ $L_A$ is invertible.
    $\Rightarrow$ $A = [L_A]_\beta$, in standard basis $\beta$, is invertible. [Thm 2.18]

$y = Ax$
$= [a_1, a_2 \cdots a_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
$= \sum x_i a_i$

$R(A) = \text{span}\{a_i\}$

$\text{rank}(A) = \dim(R(A))$
$= \#$ of LI vectors in $\{a_i\}$
$= \#$ of LI columns of $A$

- Theorem 3.3:

$$\begin{array}{lccc} & V & \xrightarrow{T} & W \\ \text{basis} & \beta & & \gamma \end{array}$$

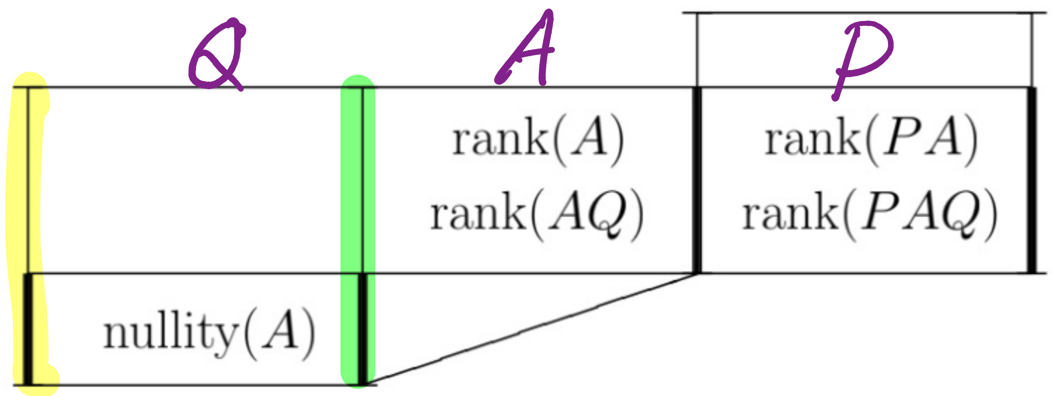
rank($T$) = rank($[T]_\beta^\gamma$) $\square$ $= \#$ of LI columns

- ▪▪ Theorem 3.4: $A \in M_{m\times n}$; $P \in M_{m\times m}$; $Q \in M_{n\times n}$; $P$ and $Q$ are invertible. Then

if $m \leq n$

$m =$ rank$(A)$ = rank$(PA)$ = rank$(AQ)$ = rank$(PAQ)$ □

$$F^n \ni x \xrightarrow{L_Q} F^n \xrightarrow{L_A} F^m \xrightarrow{L_P} F^m \ni y$$




$\Rightarrow y = PAQx$

- ▪ nullity$(A)$ = nullity$(PA)$ = nullity$(AQ)$ = nullity$(PAQ)$

full rank

- ▪▪ Corollary 3.4: Elementary row and column operations on a matrix are rank-preserving. □

- How do we practically find the rank of a matrix?
- column space and row space (Range space)
  - **column space** of a matrix: span of the columns □
  - **row space** of a matrix: span of the rows □

column vector

row vector

$$x \xrightarrow{A} y$$

$$y = Ax = \sum x_i a_i$$

$$y^T = x^T A = \sum x_i a_i^T$$

$$[x_1 \; \dots \; x_n] \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}$$

- Theorem 3.5: rank($A$) = dim(column space of $A$) $\square$
  - In other words, the rank of a matrix $A$ is the maximum number of linearly independent columns of $A$.

    proof: $\beta$ = $\{e_1, \cdots, e_n\}$ is the standard basis of $F^n$

    such that $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)^t$ with 1 in the $j$-th place.

    rank($A$) = rank($L_A$) = dim($R(L_A)$) [def]

    $R(L_A)$ = span($\{L_A(e_1), \cdots, L_A(e_n)\}$ [Thm 2.2] $\to \{a_1, a_2, \cdots a_n\}$

    $L_A(e_j) = \boxed{Ae_j}$ is the $j$-th column. $\square$

- example:

$$\text{rank}\left(\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}\right) = 2 \text{ [c1×3-c2=c3], rank}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 0$$

- Theorem 3.6: An $m \times n$ matrix $A$ of rank $r$ can be transformed into $D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$ by elementary row and column operations, where $O_i$ are zero matrices. $\square$

- example:

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 3 & -1 \\ 1 & 1 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & 2 & -2 \\ 0 & -1 & 1 & -1 \end{pmatrix}$$ [ero3(r2+r1×(-1))],

[ero3(r3+r1×(-1))]

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix}$$ [ero2(r2×$(-\frac{1}{2})$)] $\to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ [ero3(r3+r2)]}$$

$$\to \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ [ero3(r1+r2}\times\text{(-2))]}$$

$$\to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to D = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \text{ [eco3(c3+c1}\times\text{(-3))], [eco3(c4+c1)],}$$

$$\text{[eco3(c3+c2)], [eco3(c4+c2}\times\text{(-1))]}$$

$$\Rightarrow \text{rank}(A) = 2$$

- Corollary 3.6.1: $D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix} = BAC$ for some invertible matrices $B$ and $C$. $\square$
  proof: $D = E_p E_{p-1} \cdots E_1 A G_1 G_2 \cdots G_p$,
  where $E_i$ and $G_j$ are elementary matrices. $\square$

- These two equations of elementary matrices are not unique.

- Corollary 3.6.2: rank($A$) = dim(column space of $A$)
  = dim(row space of $A$) = rank($A^t$) $\square$

- In other words, the rank of a matrix $A$ is not only the maximum number of linearly independent columns but also the maximum number of linearly independent rows of $A$.

- ▪▪ Corollary 3.6.3: Every invertible matrix is a product of elementary matrices. $\square$ $[D = I]$

proof: $A \in M_{n\times n}$ is invertible $\Rightarrow$ rank($A$) = $n$
$\Rightarrow D = I_n = BAC = E_pE_{p-1}\cdots E_1AG_1G_2\cdots G_p$, [Thm 3.6]
where $E_i$ and $G_j$ are elementary matrices.
$\Rightarrow A = E_1^{-1}E_2^{-1}\cdots E_p^{-1}I_nG_q^{-1}G_{q-1}^{-1}\cdots G_1^{-1}$,
where $E_i^{-1}$ and $G_j^{-1}$ are elementary matrices. [Thm 3.2] $\square$

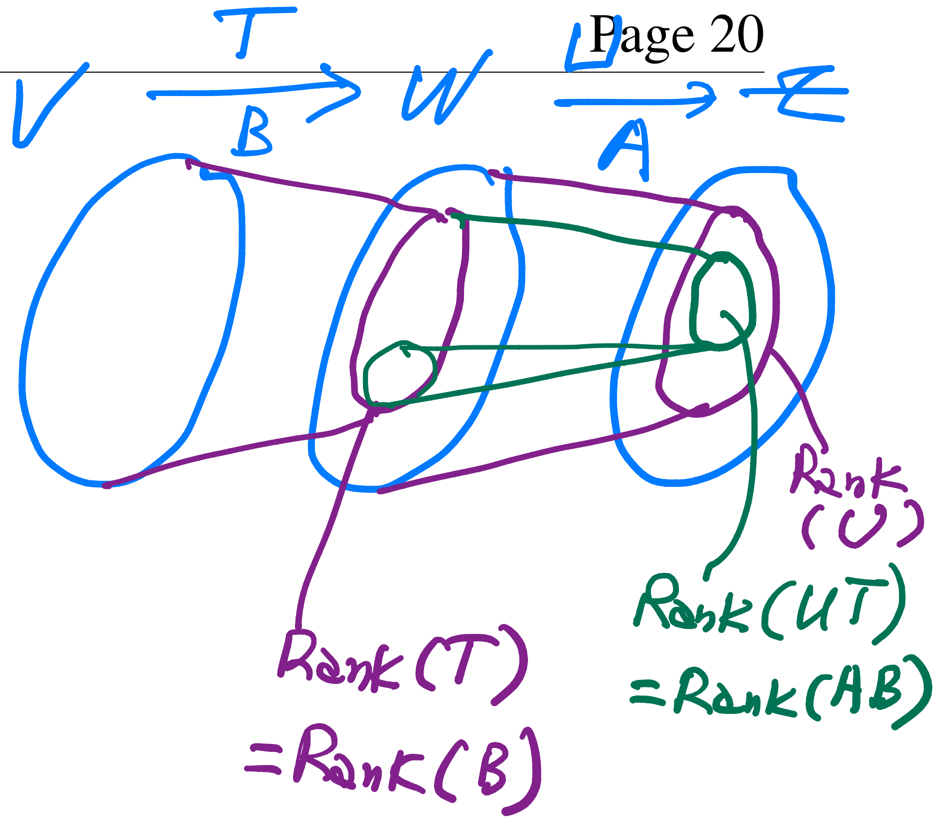
▪▪ Theorem 3.7:

$T : V \to W$ and $U : W \to Z$

are linear transformations;

$A$ and $B$ are matrices. Then

1. rank$(UT) \leq$ rank$(U)$
2. rank$(UT) \leq$ rank$(T)$
3. rank$(AB) \leq$ rank$(A)$
3. rank$(AB) \leq$ rank$(B)$ $\Box$



■■ **augmented matrix**: $(A|B)$ □

■■ matrix inversion by **Gaussian elimination**:

$(A|I_n) \to (I_n|B)$ by eros [not by ecos]

$\Rightarrow (I_n|B) = E_p \cdots E_1(A|I_n)$

$\Rightarrow I_n = E_p \cdots E_1 A,\ B = E_p \cdots E_1 I_n$

$\Rightarrow I_n = BA$

$\Rightarrow I_n = AB$ [Thm 2.17d] $(\because \dim(A) = \dim(B))$

■■ example:

$$(A|I) = \left(\begin{array}{ccc|ccc} 0 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \text{[ero1(r1,r2)], [ero2(r1}\times(\tfrac{1}{2})\text{))]}$$

$r_3 + (-3r_1) \to 0 \quad -3 \quad -2 \quad \cdots$

×3

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & -3 & -2 & 0 & -\frac{3}{2} & 1 \end{array}\right) \text{[ero3(r3+r1}\times\text{(-3))],[ero2(r2}\times(\tfrac{1}{2}))]$$

+ $r_3$

$r_1$

− ×2

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 4 & \frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right) \text{[ero3(r3+r2}\times\text{3)]}$$

$r_1$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{array}\right) \text{[ero3(r1+r2}\times\text{(-2))], [ero2(r3}\times(\tfrac{1}{4}))]$$

+ ×3

$r_2$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{array}\right) \begin{pmatrix} \text{[ero3(r1+r2} \times \text{3)]} \\ \text{[ero2(r2+r3} \times \text{(-2))]} \end{pmatrix}$$

+ ×−2

$A^{-1}$

$$\Rightarrow \underset{A}{\begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}^{-1}} = \underset{A^{-1}}{\begin{pmatrix} \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{pmatrix}}$$

▪▪ We can find the inverse of a linear transformation $T : V \to V$ through computing the inverse of its matrix representation in a basis $\beta$.
$T \to [T]_\beta \to [T]_\beta^{-1} \to [T^{-1}]_\beta \to T^{-1}$

▪▪ example: $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f) = f + f' + f''$, and $\beta = \{1, x, x^2\}$.

$= g \Rightarrow f = T^{-1}(g)$

$= ?$

$$[T]_\beta = \underbrace{\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \Rightarrow [T]_\beta^{-1} = [T^{-1}]_\beta = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$[A\ I] \longrightarrow [I\ A^{-1}]$

$$g = a_0 + a_1 x + a_2 x^2 \Rightarrow [g]_\beta = (a_0, a_1, a_2)^t$$

$$[T^{-1}(g)]_\beta = [T]_\beta^{-1}[g]_\beta = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

$$= (a_0 - a_1, a_1 - 2a_2, a_2)^t$$

$$\Rightarrow T^{-1}(g) = (a_0 - a_1) + (a_1 - 2a_2)x + a_2 x^2$$