Chapter 5

- $T : V \to V$ is **Diagonalizable**: $\exists$ an ordered basis β' for V such that
  - $D = [T]_{\beta'} = [ITI]_{\beta'} = [I]_{\beta}^{\beta'}[T]_{\beta}[I]_{\beta'}^{\beta} = Q^{-1}[T]_{\beta}Q$
- A matrix A is diagonalizable: L_A is diagonalizable.
  - Let β is the standard basis for F^n. Then $[L_A]_{\beta} = A$.
    $D = [L_A]_{\beta'} = [IL_AI]_{\beta'} = [I]_{\beta}^{\beta'}[L_A]_{\beta}[I]_{\beta'}^{\beta} = Q^{-1}AQ$
    The columns of $Q = [I]_{\beta'}^{\beta}$ are the vectors in β'.
- This diagonalizing basis β' is a set of "eigenvectors".
  For $v_j \in \beta'$,

$$T(v_j) = \Sigma_{i=1}^{n} D_{ij} v_i = D_{jj} v_j = \lambda_j v_j.$$

Eigenvalue and eigenvector

- For $T : V \to V$, if $T(v) = \lambda v$, $v \neq 0$, then λ is an **eigenvalue** and v is an **eigenvector** (corresponding to λ) of T.
  - For a matrix A, if $Av = \lambda v$, $v \neq 0$, then λ is an eigenvalue and v is an **eigenvector**(corresponding to λ) of A.
- Note:
  - v is an eigenvector of T
    $\Leftrightarrow T(v) = \lambda v = \lambda I(v) = (\lambda I)(v)$
    $\Leftrightarrow (T - \lambda I)(v) = 0$
    $\Leftrightarrow v \in N(T - \lambda I) - \{0\}$
  - v is an eigenvector of A
    $\Leftrightarrow Av = \lambda v = \lambda I_n v = (\lambda I_n)v$
    $\Leftrightarrow (A - \lambda I_n)v = (L_A - \lambda I)(v) = L_{A-\lambda I_n}(v) = 0$
    $\Leftrightarrow v \in N(L_A - \lambda I) - \{0\} = N(L_{A-\lambda I_n}) - \{0\}$

- Theorem 5.1: $T : V \to V$ is diagonalizable
  $\Leftrightarrow \exists$ a basis for V consisting of eigenvectors of T.

- example: $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, β is the standard basis for F^2.

  $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \lambda_1 = -2; v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \lambda_2 = 5$

  $\beta' = \{v1, v2\} \Rightarrow Q = [I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$

  $[L_A]_{\beta'} = Q^{-1}AQ = [I]_{\beta}^{\beta'}[L_A]_{\beta}[I]_{\beta'}^{\beta} = \begin{pmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$

  $= \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$

- Theorem 5.2: λ is an eigenvalue of a matrix A.
  $\Leftrightarrow$ det$(A-\lambda I)$=0.

- **characteristic polynomial** of a matrix A: $f(t)=$det$(A-tI)$

- **characteristic equation** of a matrix A: $f(t)=$det$(A-tI)$=0

- $f(t)=$det$(A-tI)$=det $\begin{pmatrix} A_{11}-t & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22}-t & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn}-t \end{pmatrix}$

- Theorem 5.3: $A \in M_{n\times n}(F)$; its characteristic polynomial is
  $f(t)=a_n t^n + a^{n-1}t^{n-1}+\cdots+a_1 t+a_0$. Then

  1. $a_n=(-1)^n$
  2. $a_{n-1}=(-1)^{n-1}tr(A)$=$(-1)^{n-1}(A_{11}+A_{22}+\cdots+Ann)$
  3. a_0=det(A) [set $t=0$]

- Theorem 5.3a: $A \in M_{n\times n}(F)$; is diagonalizable with eigenvalues $\lambda_1, \cdots, \lambda_n$(with possible repetitions). Then

  1. $tr(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \lambda_i$
  2. det(A) = $\prod_{i=1}^{n} \lambda_i$

- det($[T]_{\beta'} - tI$)=det($Q^{-1}[T]_\beta Q - Q^{-1}tIQ$)

  =det($Q^{-1}([T]_\beta - tI)Q$)=det($[T]_\beta - tI$)

- characteristic polynomial and characteristic equation for T.

  - **characteristic polynomial** of a linear operator $T : V \to V$:det($T - tI$)=det($[T]_\beta - tI$) for any basis β for V
  - **characteristic equation** of a linear operator $T : V \to V$:det($T - tI$)=det($[T]_\beta - tI$)$= 0$ for any basis β for V

[End of Review]

- example:
  - reflection: $T((a_1, a_2)) = (a_1, -a_2) \Rightarrow [T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, β std

    $\Rightarrow (1-t)(1+t) = 0 \Rightarrow$

λ	1	-1
v	(1,0)	(0,-1)

  - projection: $T((a_1, a_2)) = (0, a_2) \Rightarrow [T]_\beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, β std

    $\Rightarrow -t(1-t) = 0 \Rightarrow$

λ	0	1
v	(1,0)	(0,1)

  - rotation: $T((a_1, a_2)) = (a_1 cos\theta - a_2 sin\theta, a_1 sin\theta + a_2 cos\theta)$

    $\Rightarrow [T]_\beta = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \Rightarrow (cos\theta - t)^2 + sin^2\theta = 0$

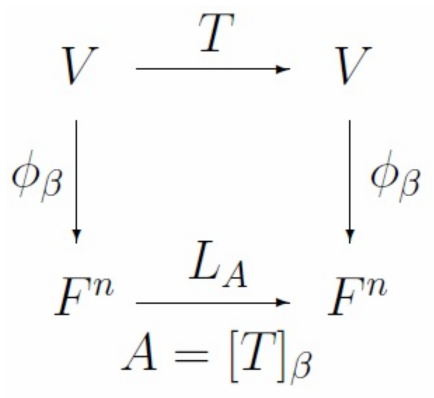
    $\Rightarrow t^2 - 2tcos\theta + 1 = 0$

    When the field is real, neither eigenvalues nor eigenvectors exist.

- Finding eigenvalues and eigenvectors of T,
  $T \to [T]_\beta \to \det([T]_\beta - \lambda I) = 0 \to \lambda$
  $\lambda \to ([T]_\beta - \lambda I)[v]_\beta = 0 \to [v]_\beta \to v$

- **eigenspace** E_λ: the set of all eigenvectors for the eigenvalue λ with 0 included.

  - $E_\lambda = N(T - \lambda I)$ for a linear operator $T : V \to V$.
    E_λ is a subspace of V.
  - $E_\lambda = N(T - \lambda I) = N(L_{A-\lambda I_n})$ for a matrix $A \in M_{n\times n}$
    E_λ is a subspace of F^n.



■■ example: $F = \mathbf{C}$ (The same results hold if $F = \mathbf{R}$.)

A= $\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$,det$(A - tI)$=$-(t+2)^2(t-4)$

For $\lambda = -2 \Rightarrow \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_2 - s_1 \end{pmatrix}$

$\Rightarrow E_{-2} = \{s_1(1,0,-1)^t + s_2(0,1,1)^t : s_1, s_2 \in \mathbf{C}\}$

For $\lambda = 4 \Rightarrow \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} s \\ s \\ 2s \end{pmatrix}$

$\Rightarrow E_4 = \{s(1,1,2)^t : s \in \mathbf{C}\}$

T-invariant set

- For a linear operator T and a set S, $T(S)$ is defined as $T(S) := \{T(v) : v \in S\}$.
- $T-$**invariant set** $S : T(S) \subseteq S$
- $T-$**invariant subspace** W :
  W is a subspace and $T(W) \subseteq W$.
- E_λ is a $T-$invariant subspace.
- $\{0\}, N(T), R(T)$ and V are $T-$ invariant subspaces.

Diagonalizability

- We now consider
  1. how to test whether T is diagonalizable;
  2. how to diagonalize T

- Theorem 5.5 : $T : V \to V$ is a linear operator; $\lambda_1, \cdots, \lambda_k$ are "distinct" eigenvalues of T; and $v_1, \cdots, v_k$ are respective eigenvectors. Then the eigenvectors are linearly independent.

  proof: induction in k : (i) $\{v_1\}$ is linearly independent.
  (ii) Assume $\{v_1, \cdots, v_{k-1}\}$ is linearly independent.
  (iii) Assume $\sum_{i=1}^{k} a_i v_i = 0$
  $\Rightarrow (T - \lambda_k I)(\sum_{i=1}^{k} a_i v_i) = 0$
  $\Rightarrow T(\sum_{i=1}^{k} a_i v_i) - \lambda_k(\sum_{i=1}^{k} a_i v_i) = 0$
  $\Rightarrow \sum_{i=1}^{k} a_i T(v_i) - \sum_{i=1}^{k} a_i \lambda_k v_i = 0$ [linear]
  $\Rightarrow \sum_{i=1}^{k} a_i \lambda_i v_i - \sum_{i=1}^{k} a_i \lambda_k v_i = 0$ [eigenvector]

$\Rightarrow \sum_{i=1}^{k-1} a_i(\lambda_i - \lambda_k)v_i = 0$ [kth terms cancel]
$\Rightarrow a_1 = \cdots = a_{k-1} = 0$[$\lambda_i$are distinct, (ii)]
$\Rightarrow a_k v_k = 0$ [$\sum_{i=1}^{k} a_i v_i = 0$]
$\Rightarrow a_k = 0[v_k \neq 0]$
$\Rightarrow a_1 = \cdots = a_k = 0$
$\Rightarrow \{v_1, \cdots, v_k\}$ is linearly independent.

- Corollary 5.5: $T : V \to V$ is a linear operator; dim(V)$= n$. Then if T is n distinct eigenvalues, T is diagonalizable.

- example : identity operator I
  - Every vector is an eigenvector of I because $I(v) = v$.
  - I has only one eigenvalue 1.
  - There are n linearly independent eigenvectors of I for the eigenvalue 1.
  - eigenspace $E_1 = V$
  - So the converses of Theorem 5.5 and corollary 5.5 are false.
- A polynomial $f(t)$ is said to **split** over the field F if there are $c, a_1, \cdots, a_n \in F$ such that $f(t) = c(t - a_1)(t - a_2)\cdots(t - a_n)$
  - We can also say that $f(t)$ factors into first-degree factors or into linear factors.
  - $a_1, \cdots, a_n$ may not all be distinct.
  - Over the complex field, all polynomials split.

- Theorem 5.6; The characteristic polynomial of a diagonalizable linear operator splits.
  proof: diagonalizable $\Rightarrow \exists\beta$ such that $[T]_\beta = D$ diagonal
  $\Rightarrow f(t)$ =det($D - tI$)=det $\begin{pmatrix} \lambda_1 - t & & O \\ & \ddots & \\ O & & \lambda_n - t \end{pmatrix}$ [indep of β]
  $= (-1)^n(t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$
  - Its converse is not true, ie, splitting characteristic polynomial does not guarantee diagonalizability.

- **algebraic multiplicity** of an eigenvalue λ :
  a-mul(λ)=k if $f(t) = (t-\lambda)^k g(t)$ and $g(\lambda) \neq 0$.
  - characteristic polynomial of I : $f(t) = (-1)^n(t-1)^n$
    a-mul(1)$= n$
  - characteristic polynomial $f(t) = -(t-3)^2(t-4)$
    a-mul(3)$= 2$, a-mul(4)$= 1$

- **geometric multiplicity** of an eigenvalue λ :
  g-mul(λ)=dim(E_λ)
  - It is the maximum number of linearly independent eigenvectors corresponding to λ.
  - g-mul(λ)=nullity($T-\lambda I$)
  - identity operator I : g-mul(1)=n
  - A=$\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$,$f(t)$=$-(t+2)^2(t-4)$

    $E_{-2}=\{s_1(1,0,-1)^t+s_2(0,1,1)^t : s_1,s_2\in\mathbf{C}\}$

    $E_4=\{s(1,1,2)^t : s\in\mathbf{C}\}$

    $\Rightarrow$ g-mul(-2)=2 , g-mul(4)=1

- Theorem 5.7; $1 \leq$ g-mul(λ) $\leq$ a-mul(λ) .
  proof: Let a-mul(λ)=m and g-mul(λ)=p.
  $\Rightarrow \exists$ basis $\{v_1, \cdots, v_p\}$ for E_λ.
  $\Rightarrow$ It can be extended to a basis $\beta = \{v_1, \cdots, v_p, v_{p+1}, \cdots, v_n\}$.
  $\Rightarrow [T]_\beta = A = \begin{pmatrix} \lambda I_p & B \\ O & C \end{pmatrix}$ $[T(v_i) = \lambda v_i, i = 1, \cdots, p]$
  $\Rightarrow f(t) =$det$(A - tI_n)$=det$\begin{pmatrix} \lambda I_p - tI_p & B \\ O & C - tI_{n-p} \end{pmatrix}$
  $= (-1)^n (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n)$
  = det$((\lambda - t)I_p)$ det$(C - tI_{n-p})$ [exercise 4.3.21]
  = $(\lambda - t)^p$det$(C - tI_{n-p}) \Rightarrow$ a-mul(λ) is at least $p \Rightarrow p \leq m$.

- For example, if $\lambda = 3$ and $p = 2$,

$$[T]_\beta = \begin{pmatrix} 3 & 0 & -1 & 9 & 2 \\ 0 & 3 & 1 & 0 & -1 \\ 0 & 0 & 4 & 5 & 1 \\ 0 & 0 & 7 & 1 & 7 \\ 0 & 0 & 0 & 4 & -1 \end{pmatrix}, \ f(t) = \text{det}\begin{pmatrix} 3-t & 0 & -1 & 9 & 2 \\ 0 & 3-t & 1 & 0 & -1 \\ 0 & 0 & 4-t & 5 & 1 \\ 0 & 0 & 7 & 1-t & 7 \\ 0 & 0 & 0 & 4 & -1-t \end{pmatrix}$$

- example: $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$, $T(f) = f', \beta = \{1, x, x^2\}$

$[T]_\beta$=$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$,$f(t)$=det($[T]_\beta - tI$)=det$\begin{pmatrix} -t & 1 & 0 \\ 0 & -t & 2 \\ 0 & 0 & -t \end{pmatrix} = -t^3$

$\lambda = 0 \Rightarrow$ a-mul(0)=3

eigenspace of $[T]_\beta$ in $\mathbf{R^3}$: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$=$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$E_0 = N([T]_\beta - 0I) = \{v : [T]_\beta v = 0\} = \left\{ s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : s \in \mathbf{R} \right\}$$

eigenspace of T in $P_2(\mathbf{R})$:

$E_0 = N(T - 0I) = \{s + 0x + 0x^2 : s \in \mathbf{R}\}$

g-mul(0)$= 1 \leq 3 =$a-mul(0)

- example: identity operator: $\lambda = 1$, a-mul(1)=g-mul(1)$= n$

- Theorem 5.9: dim(V)$< \infty$; $T : V \to V$ is a linear operator; its characteristic polynomial splits; and $\lambda_1, \cdots, \lambda_k$ are distinct eigenvalues of T. Then
  1. T is diagonalizable $\Leftrightarrow$ a-mul(λ_i)=g-mul(λ_i), $i = 1, \cdots, k$.
  2. T is diagonalizable; β_i is an ordered basis for E_{λ_i} $\Rightarrow \beta = \beta_1 \cup \cdots \cup \beta_k$ is an ordered basis for V.
  - characteristic polynomial splits $\Rightarrow \sum_{i=1}^{k}$ a-mul(λ_i)$= n =$dim(V) $\Rightarrow$ a-mul(λ_i)=g-mul(λ_i),$i = 1, \cdots, k$ is equivalent to existence of n linearly independent eigenvectors.
  - If we can find a basis β_i for E_{λ_i} with size a-mul(λ_i)=g-mul(λ_i) for each λ_i, the linear operator or matrix can be diagonalized using the basis $\Rightarrow \beta = \beta_1 \cup \cdots \cup \beta_k$ for V.
  - $[T]_\beta = \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_k \end{pmatrix}$ where λ_i is repeated as many times as a-mul(λ_i)=g-mul(λ_i)