Intro. to Electro-physics

Drude's model (1st)

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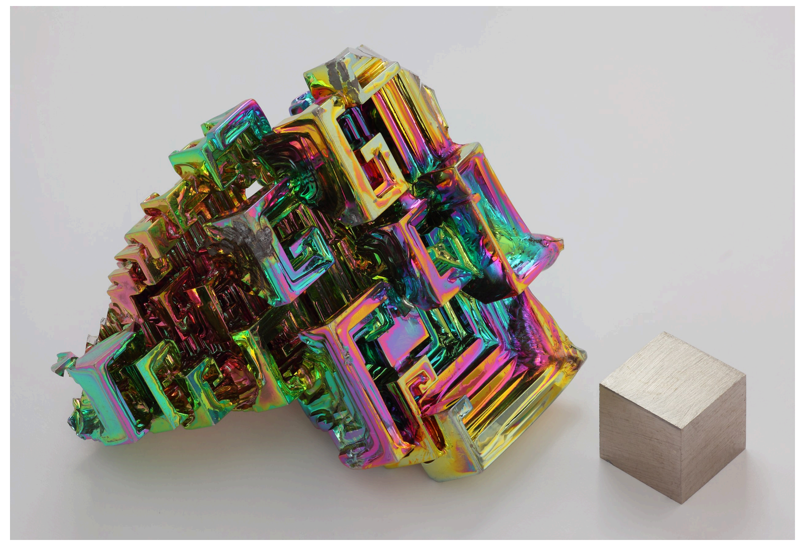
Metal


Crystalline gold


Crystalline silver


Crystalline Bismuth

- General characteristics
  - Excellent conductors of heat & electricity
  - Ductile and malleable
  - Shiny surface that reflects visible light (λ = 380~780 nm)

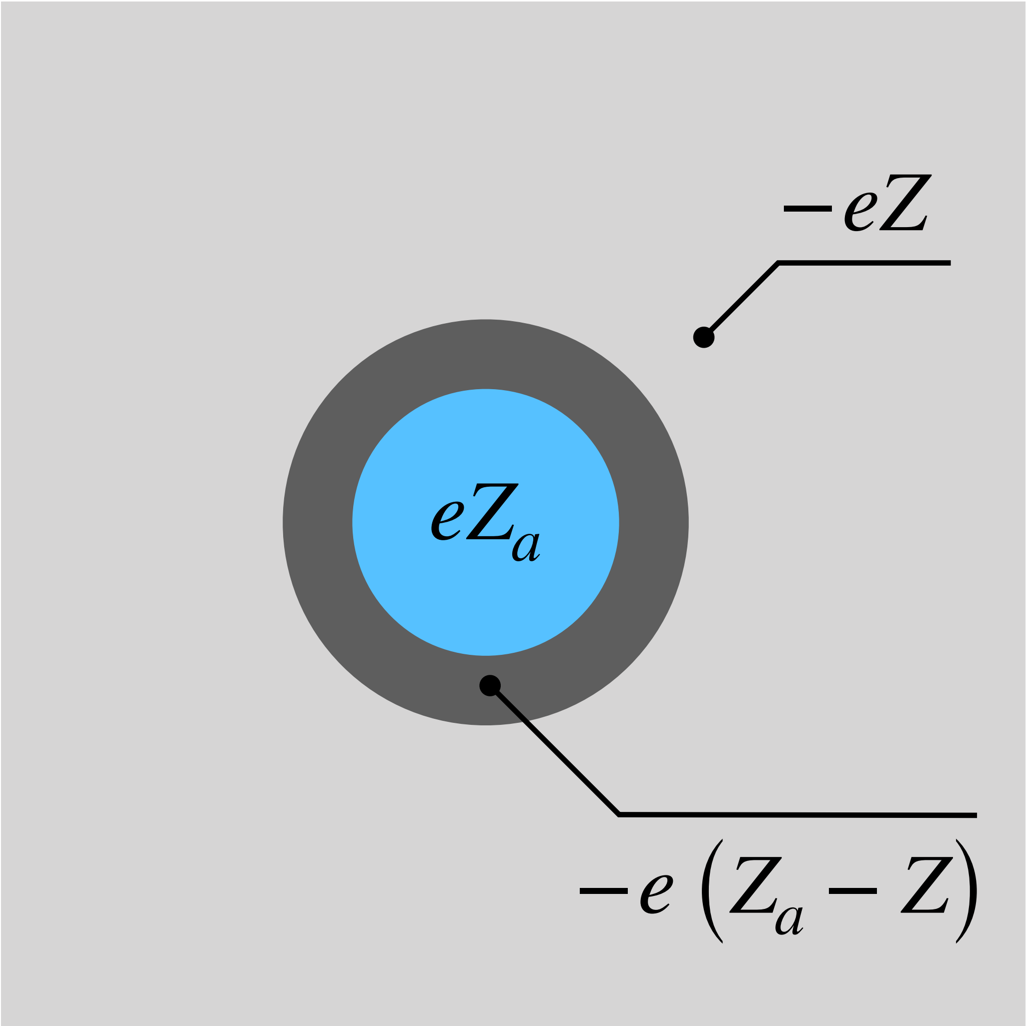
- Explaining metallic characteristics = A starting point of modern **solid-state physics**!

- Earliest model by Paul Drude (1900)
  - Simple and practical model that has been used even today for rough estimation (e.g. electrical conductivity)
  - Fails to explain a majority of experimental data of metals
  - Requires quantum mechanics to improve its validity


Paul K. L. Drude
(Germany)
1863-1906

Drude model

- Tried to explain electrical and thermal conduction of a metal
- Applied a kinetic theory of gases to electrons in a metal

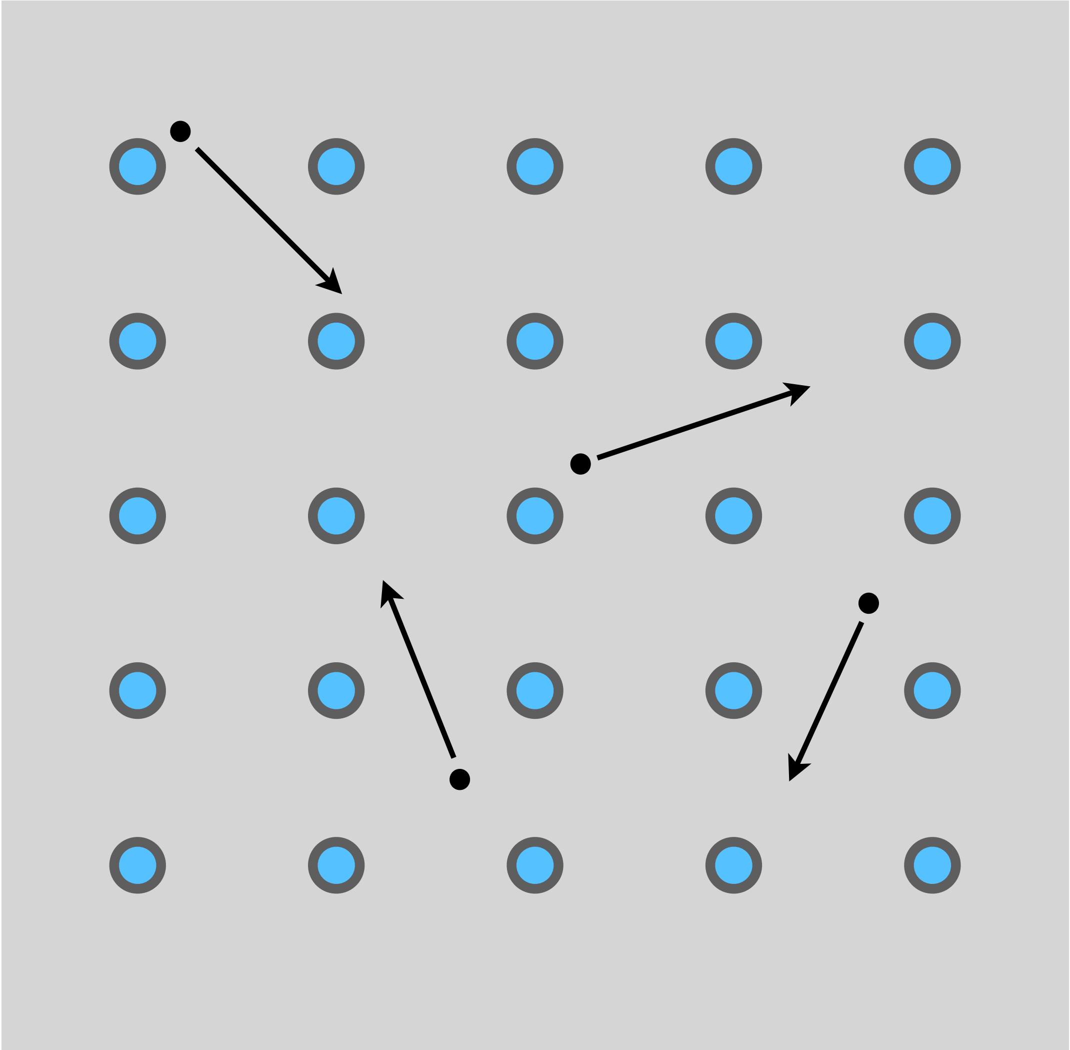


A single isolated atom
of the metallic element

Z_a : Atomic number

Z : Weakly-bound valence electrons (mobile)

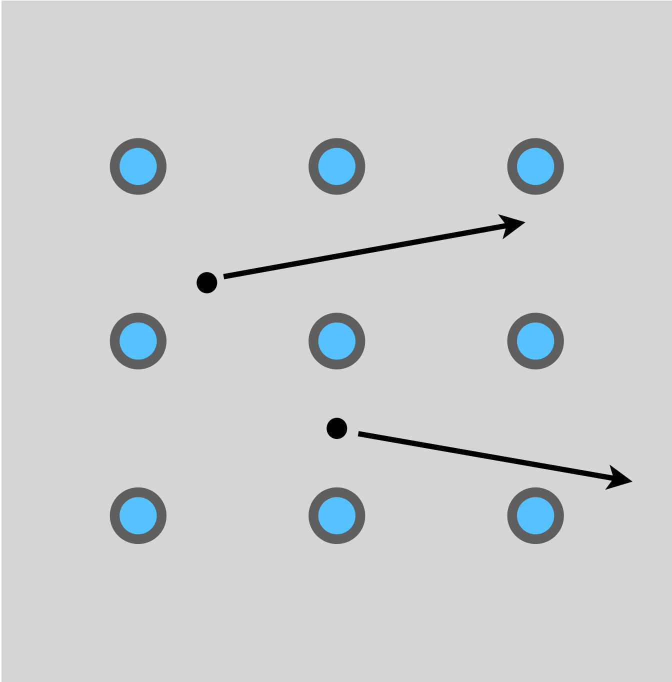
$Z_a - Z$: Tightly-bound core electrons (immobile)



An electron gas
moving against ions

- Conduction electrons move far away from their parent atoms
- A gas of conduction electrons move against a background of heavy immobile ions

Key assumptions in the Drude's model (1/2)



① Electrons move in straight lines until they collide with immobile ions

(i.e., no other forces during their travel)

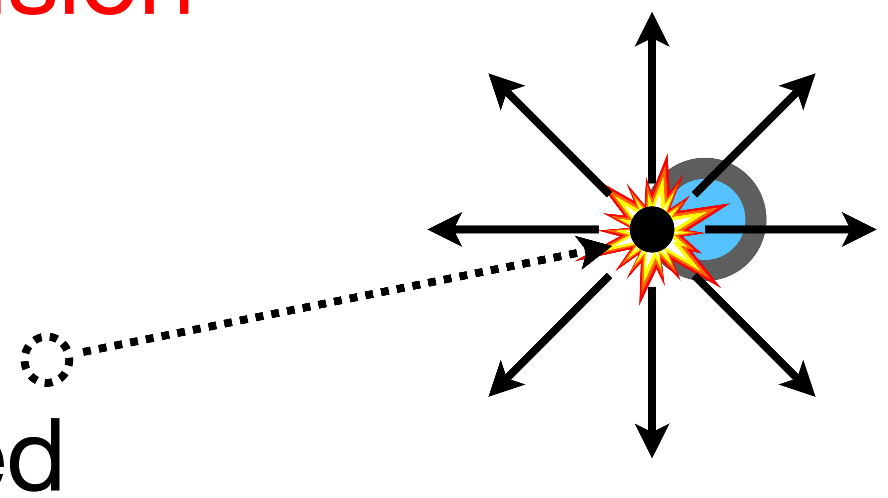
- No electron-electron interaction (Independent electron approx.)
- No electron-ion interaction (Free-electron approx.) (* except collisions)
- Under external ***E*** & ***H***-fields, electrons move according to Newton's law of motion

② Electrons bounce off ion cores instantaneously so that their velocities are abruptly changed.

(i.e., no time delay due to the collision)

③ Electrons achieve thermal equilibrium with their surroundings only through collision

- After each collision, the velocity $\left(\boldsymbol{v}_0\right)$ of the electron randomly oriented
- Speed $\left(|\boldsymbol{v}_0| = v_0\right)$ determined by the temperature where the collision occurred



Key assumptions in the Drude's model (2/2)

④ An electron undergoes a collision with a probability per unit time $\left(P \triangleq 1/\tau\right)$

τ = Mean-free time or collision time

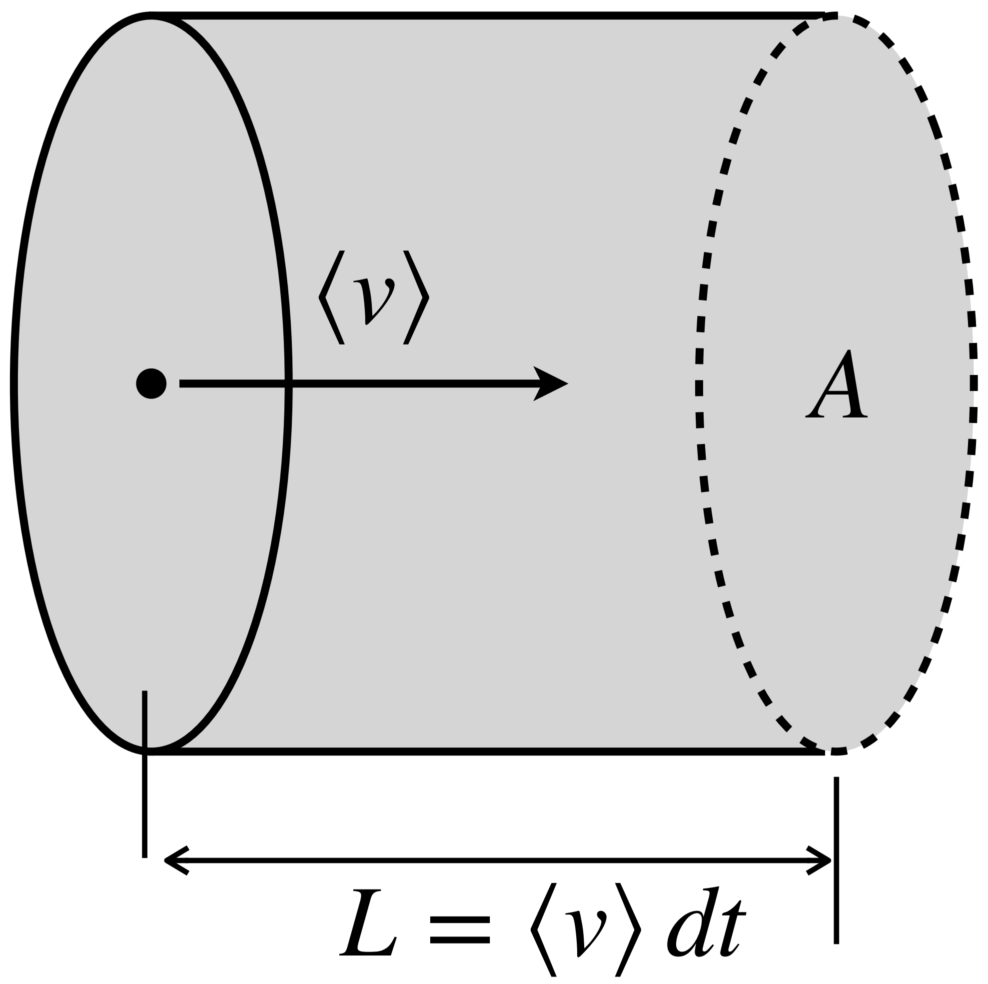
- Right after a collision, an electron travels for a time τ on the average before the next collision
- τ = An average survival time for an electron between collisions
- τ is independent of electron's position and velocity (i.e., the unique property of a metal)
- $\dfrac{dt}{\tau}$ = The probability that an electron undergoes a collision during dt, where $0 \leq dt \leq \tau$

Electrical conductivity by the Drude model (1/2)

- Ohm's law

$\boldsymbol{J} = \sigma \boldsymbol{E}$, where σ : electrical conductivity

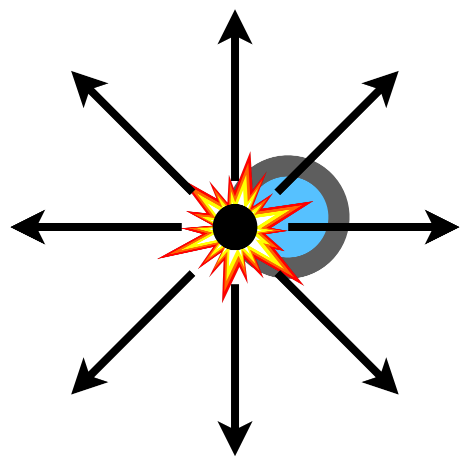
- Electrical current by electrons moving with an average speed $\langle v \rangle$ with a volume density (n)



- Average speed → $\langle v \rangle = \frac{1}{N} \sum_j v_j$
- Volume → $V = A \langle v \rangle dt$
- # of electrons within the volume → $N = n \cdot V = nA \langle v \rangle dt$
- Charge carried by these electrons → $dQ = -e \cdot N = -enA \langle v \rangle dt$
- Current density [A/m^2] → $J = \frac{I}{A} = \frac{1}{A} \cdot \frac{dQ}{dt} = -en \langle v \rangle$
- Current density in 3D → $\boldsymbol{J} = -ne \langle \boldsymbol{v} \rangle$, $\langle \boldsymbol{v} \rangle$: average electronic velocity

Electrical conductivity by the Drude model (2/2)

- Average electronic velocity
  - In the absence of E-field : $\sum_j \boldsymbol{v}_j = 0 \;\rightarrow\; \langle \boldsymbol{v} \rangle = 0 \;\rightarrow\; \boldsymbol{J} = 0$
  - In the presence of E-field :
    - Velocity after some time t : $\boldsymbol{v} = \boldsymbol{v}_0 - \frac{e\boldsymbol{E}t}{m}$, where $0 \leq t \leq \tau$
    - Average velocity : $\langle \boldsymbol{v} \rangle = \langle \boldsymbol{v}_0 \rangle - \frac{e\boldsymbol{E}\langle t \rangle}{m} = -\frac{e\boldsymbol{E}\tau}{m}$



$\boldsymbol{v}_0$: velocity of the electron right after a collision

- Current density (revisited)

$$\boldsymbol{J} = -ne\langle \boldsymbol{v} \rangle = \left(\frac{ne^2\tau}{m}\right)\boldsymbol{E} \triangleq \sigma \boldsymbol{E}$$

Electrical conductivity

$$\therefore \sigma = \frac{1}{\rho} = \frac{ne^2\tau}{m}$$

The density of electrons in a metal (1/2)

- Needed parameters for the metallic atom
  - Avogadro's number : $N_A = 6.022 \times 10^{23}$ (atoms/mol)
  - Mass density : ρ_m (g/cm^3)
  - Atomic mass : A (g/mol)
  - Number of conduction electrons : Z

- Electron density of a metal

$$n = \frac{N}{V} = N_A \left[\frac{\text{atoms}}{\text{mol}}\right] \times \frac{\rho_m}{A} \left[\frac{\text{mol}}{\text{cm}^3}\right] \times Z \left[\frac{\text{electrons}}{\text{atom}}\right] = N_A \frac{Z\rho_m}{A} \left[\frac{\text{electrons}}{\text{cm}^3}\right]$$

  - Typically, $n \sim 10^{22}$ [electrons/cm^3] for metals

The density of electrons in a metal (2/2)

- A widely used measure of electron density
  - r_s : A radius of a sphere whose volume is the volume per conduction electron as

$$\frac{V}{N} = \frac{1}{n} = \frac{4}{3}\pi r_s^3 \longrightarrow \boxed{r_s = \left(\frac{3}{4\pi n}\right)^{\frac{1}{3}}}$$

  - A typical range of r_s : $[\text{Å}]$ for metals

$$\frac{r_s}{a_0} : 2 \sim 6, \text{ where } a_0 : \text{Bohr radius}$$

  - Bohr radius (a_0)
    - A radius of a hydrogen atom in its ground state
    - A scale for measuring atomic distances

Element	Z	n[1] (10^{22} cm^{-3})	r_s (Å)	r_s/a_0
Li (78 K)	1	4.70	1.72	3.25
Na (5 K)	1	2.65	2.08	3.93
K (5 K)	1	1.40	2.57	4.86
Rb (5 K)	1	1.15	2.75	5.20
Cs (5 K)	1	0.91	2.98	5.62
Cu	1	8.47	1.41	2.67
Ag	1	5.86	1.60	3.02
Au	1	5.90	1.59	3.01
Be	2	24.7	0.99	1.87
Mg	2	8.61	1.41	2.66
Ca	2	4.61	1.73	3.27
Sr	2	3.55	1.89	3.57
Ba	2	3.15	1.96	3.71
Nb	1	5.56	1.63	3.07
Fe	2	17.0	1.12	2.12
Mn (α)	2	16.5	1.13	2.14
Zn	2	13.2	1.22	2.30
Cd	2	9.27	1.37	2.59
Hg (78 K)	2	8.65	1.40	2.65
Al	3	18.1	1.10	2.07
Ga	3	15.4	1.16	2.19
In	3	11.5	1.27	2.41
Tl	3	10.5	1.31	2.48
Sn	4	14.8	1.17	2.22
Pb	4	13.2	1.22	2.30
Bi	5	14.1	1.19	2.25
Sb	5	16.5	1.13	2.14

R. W. G. Wyckoff, Crystal Structures, 2nd ed., Interscience, New York, 1963.

Estimation of collision time (τ) (1/2)

- Collision time from the Drude model

$$\sigma = \frac{1}{\rho} = \frac{ne^2\tau}{m} \longrightarrow \tau = \frac{m}{\rho n e^2}$$

  - At $T = 300\,K,\ \tau : 10^{-15} \sim 10^{-14}\,[s]$. (What is the physical significance of this τ value?)

- Mean free path

$$\boxed{l = v_0 \tau}$$

  - The average distance an electron travels between collisions
  - v_0 : Average electronic speed. From classical thermodynamics,

$$\frac{1}{2}mv_0^2 = \frac{3}{2}k_B T \longrightarrow v_0 = \sqrt{\frac{3k_B T}{m}} \sim 10^7\ [\text{cm/s}] \text{ at } T = 300\,K$$

$$\therefore l = 1 \sim 10\ \left[\text{Å}\right]$$ ~ Interatomic distance

Estimation of collision time (τ) (2/2)

Formula from previous slides

$$\tau = \frac{m}{\rho n e^2},\ v_0 = \sqrt{\frac{3k_B T}{m}}$$

- Drude's conclusion from the mean-free path
  - $l = 1 \sim 10\ [\text{Å}]$ ~ Interatomic distance
  - Collision process = an electron bumps into the large heavy ion!
- Limitation of Drude's model (What is missing?)
  - The classical estimate of v_0 is way too small (> x100!)
  - τ → temp-dependent, v_0 → temp-independent
  - At very low T, $l \gg 10^3 \sim 10^8\ [\text{Å}]$ (Electrons do not simply bump off the ions!)
- In the absence of theory of collision and thus of τ,

  finding τ-independent quantities has remained of fundamental interest even now!

The equation of motion for Electron momentum (1/3)

Electrons subject to an external force $\boldsymbol{F}(t)$ (e.g., an electric field)

- Average electronic velocity at any time t : $\boldsymbol{v}(t) = \dfrac{\boldsymbol{p}(t)}{m}$ ← Momentum per electron at a time t
- Current density : $\boldsymbol{J} = -ne\boldsymbol{v} = -\dfrac{ne\boldsymbol{p}}{m}$
- Electron momentum **after a time** dt : $\boldsymbol{p}(t + dt)$
  - Two types of electrons
    - Electrons undergoes collisions during $[t, t + dt]$ with a probability $\dfrac{dt}{\tau}$
    - Electrons survive to $t + dt$ without collisions with a probability $1 - \dfrac{dt}{\tau}$
  - An additional momentum acquired by these electrons : $\boldsymbol{F}(t)\, dt + O\left(dt\right)^2$

External force

The equation of motion for Electron momentum (2/3)

- Momentum acquired by the "un-collided" electron

$$\boldsymbol{p}(t+dt) = \underbrace{\left(1-\frac{dt}{\tau}\right)}_{\text{Probability}} \left[\underbrace{\boldsymbol{p}(t)}_{\text{Previous}} + \underbrace{\boldsymbol{F}(t)\,dt + O(dt)^2}_{\text{Additional momentum}}\right]$$

$$= \boldsymbol{p}(t) - \frac{dt}{\tau}\boldsymbol{p}(t) + \boldsymbol{F}(t)\,dt + O(dt)^2$$

Absorbed!

- Momentum acquired by the "collided" electron

$$\boldsymbol{p}(t+dt) = \frac{dt}{\tau}\boldsymbol{F}(t)\,T \le \frac{dt}{\tau}\boldsymbol{F}(t)\,dt = \frac{\boldsymbol{F}(t)}{\tau}(dt)^2$$

T : an average time "after a collision at t" $(T \le dt)$

The equation of motion for Electron momentum (3/3)

- Overall electron momentum

$$\boldsymbol{p}\left(t+dt\right)=\boldsymbol{p}\left(t\right)-\frac{dt}{\tau}\boldsymbol{p}\left(t\right)+\boldsymbol{F}\left(t\right)dt+O\left(dt\right)^{2}$$

$$\rightarrow\lim_{dt\to 0}\frac{\boldsymbol{p}\left(t+dt\right)-\boldsymbol{p}\left(t\right)}{dt}=\lim_{dt\to 0}\left(-\frac{\boldsymbol{p}\left(t\right)}{\tau}+\boldsymbol{F}\left(t\right)\right)+O\left(dt\right)$$

$$\therefore\frac{d\boldsymbol{p}\left(t\right)}{dt}=-\frac{\boldsymbol{p}\left(t\right)}{\tau}+\boldsymbol{F}\left(t\right)$$

Frictional damping term due to collisions

Intro. to Electro-physics

Drude’s model (2^{nd})

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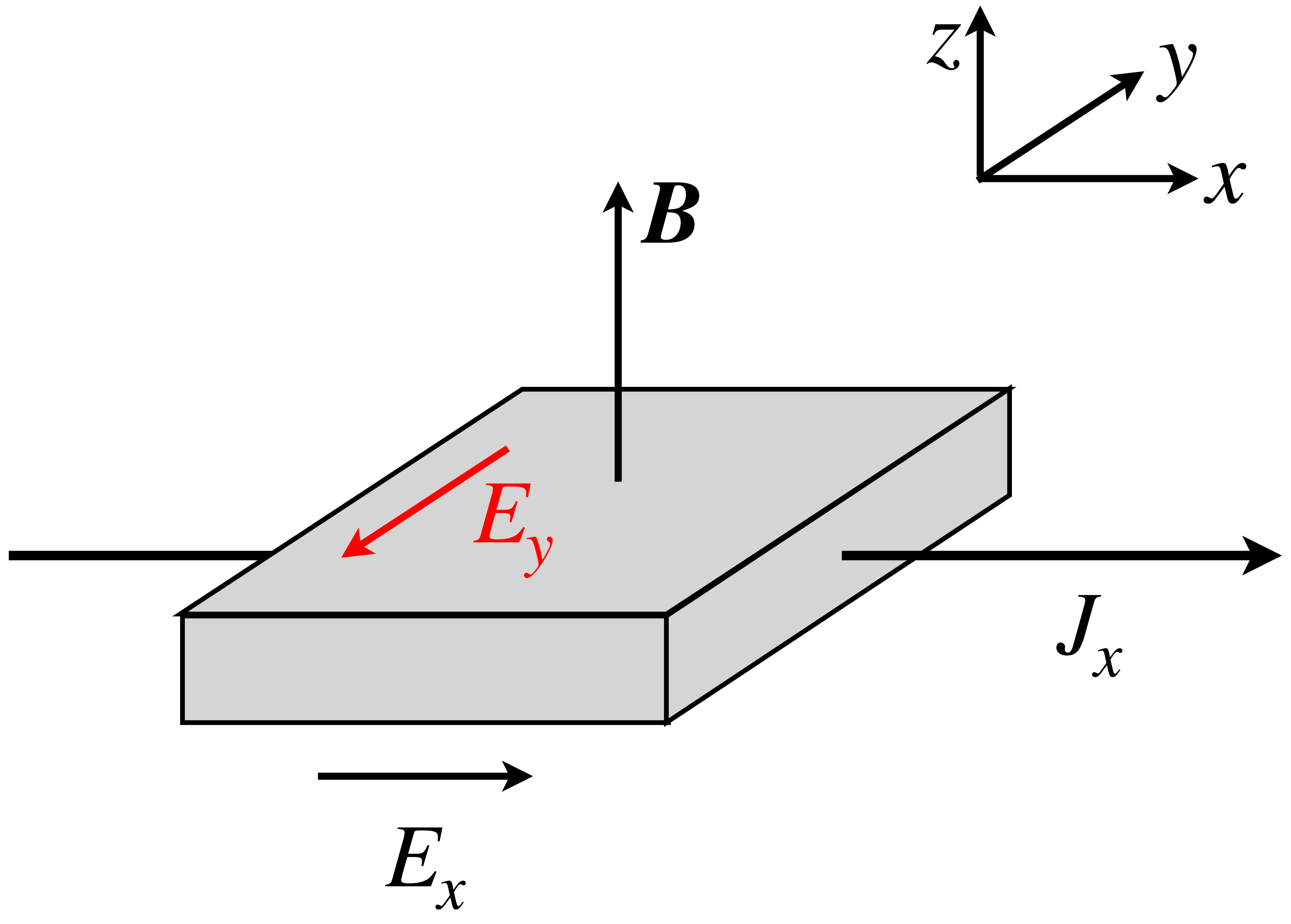
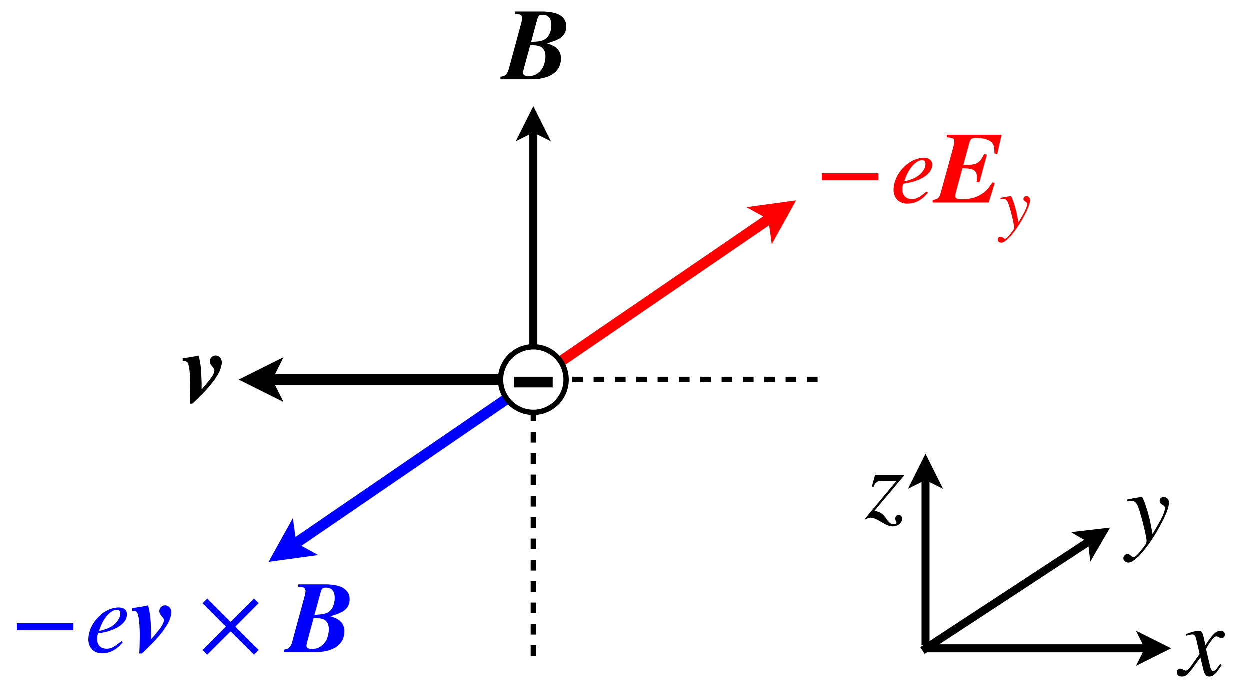
Hall effects

- Conditions
  - An electrical conductor carries an electric current (J_x) in $+x$ direction
  - A magnetic field $(\boldsymbol{B})$ applied transverse to the electric current in $+z$ direction
  - Due to the Lorentz force $(\boldsymbol{F} = -e\boldsymbol{v} \times \boldsymbol{B})$, electrons are deflected in $-y$ direction and are accumulated at that side of the conductor

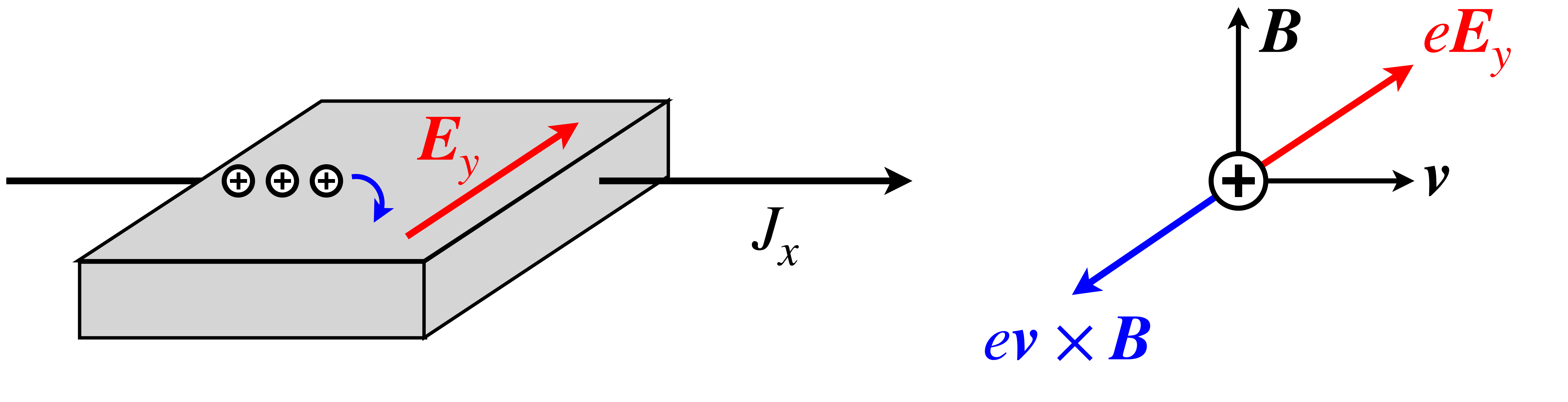


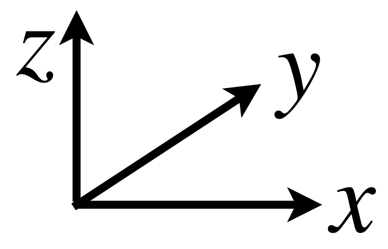
- Hall fields
  - To oppose the further accumulation of the electrons, **a Hall field in $-y$ direction $\left(E_y\right)$** is induced!
  - In equilibrium the Hall field $\left(E_y\right)$ counterbalances the Lorentz force and an electric current only flows in $+x$ direction



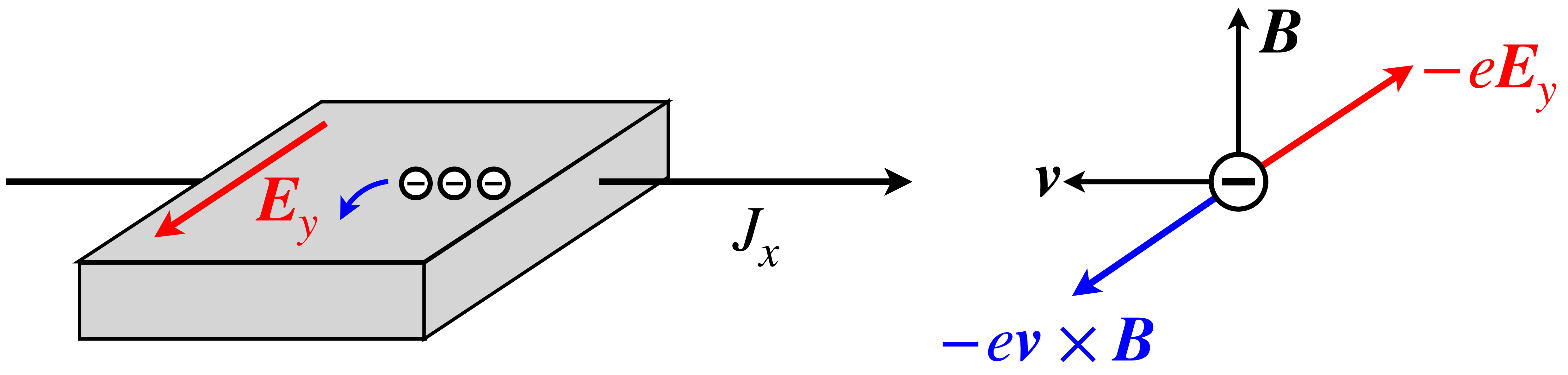
Hall coefficient

- Two quantities of interest
  - Magnetoresistance $\left(\rho = \dfrac{E_x}{J_x}\right)$ and Hall field $\left(E_y\right)$
- A Hall coefficient: $R_H \triangleq \dfrac{E_y}{J_x B}$ ($\because E_y$ may depend on B since it balances Lorenz force)
  - Positive charge : $R_H > 0$





  - Negative charge : $R_H < 0$



Drude's analysis of the Hall effect (1/3)

- Equation of motion for electron momentum

$$\frac{d\boldsymbol{p}(t)}{dt} = -\frac{\boldsymbol{p}(t)}{\tau} + \boldsymbol{F}(t), \text{ where } \boldsymbol{F} = -e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = -e\left(\boldsymbol{E} + \frac{1}{m}\boldsymbol{p} \times \boldsymbol{B}\right)$$

$$\text{Here, } \boldsymbol{p} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ p_x & p_y & p_z \\ 0 & 0 & B \end{vmatrix} = \boldsymbol{a}_x p_y B - \boldsymbol{a}_y p_x B.$$

- In the steady-state $(d\boldsymbol{p}/dt \to 0)$

$$\begin{cases} \boldsymbol{a}_x : -\frac{p_x}{\tau} - e\left(E_x + \frac{p_y B}{m}\right) = 0 \\ \boldsymbol{a}_y : -\frac{p_y}{\tau} - e\left(E_y + \frac{p_x B}{m}\right) = 0 \end{cases} \xrightarrow[p_y = m v_y]{p_x = m v_x} \begin{cases} -\frac{v_x}{m\tau} - e\left(E_x + v_y B\right) = 0 \\ -\frac{v_y}{m\tau} - e\left(E_y - v_x B\right) = 0 \end{cases}$$

Drude's analysis of the Hall effect (2/3)

- In equilibrium
  - The Hall field $\left(E_y\right)$ prevents the electron moving in $+y$ direction $\left(\text{i.e., } v_y = 0\right)$

$$\begin{cases} -\dfrac{v_x}{m\tau} - e\left(E_x + v_y B\right) = 0 \\ -\dfrac{v_y}{m\tau} - e\left(E_y - v_x B\right) = 0 \quad \longrightarrow \quad E_y = v_x B \end{cases}$$

  - The Hall coefficient :

$$R_H = \frac{E_y}{J_x B} = \frac{v_x B}{-nev_x B} = -\frac{1}{ne}$$

$$\therefore R_H = -\frac{1}{ne}$$

Drude's analysis of the Hall effect (3/3)

- Is R_H (the Hall coefficient) reliable?
  - $R_H = -\frac{1}{ne} \rightarrow n = -\frac{1}{eR_H}$ by Drude's model.

    Here, $R_H = \frac{E_y}{J_x B}$ by definition.
  - Drude's analysis only roughly valid at very low *T* and Strong $\boldsymbol{B}$ in very pure samples
- Also in reality,
  - R_H depends on both $\boldsymbol{B}$ and *T*

    (although temp-dependent τ uninvolved)
  - But, $R_H > 0$ in some metals

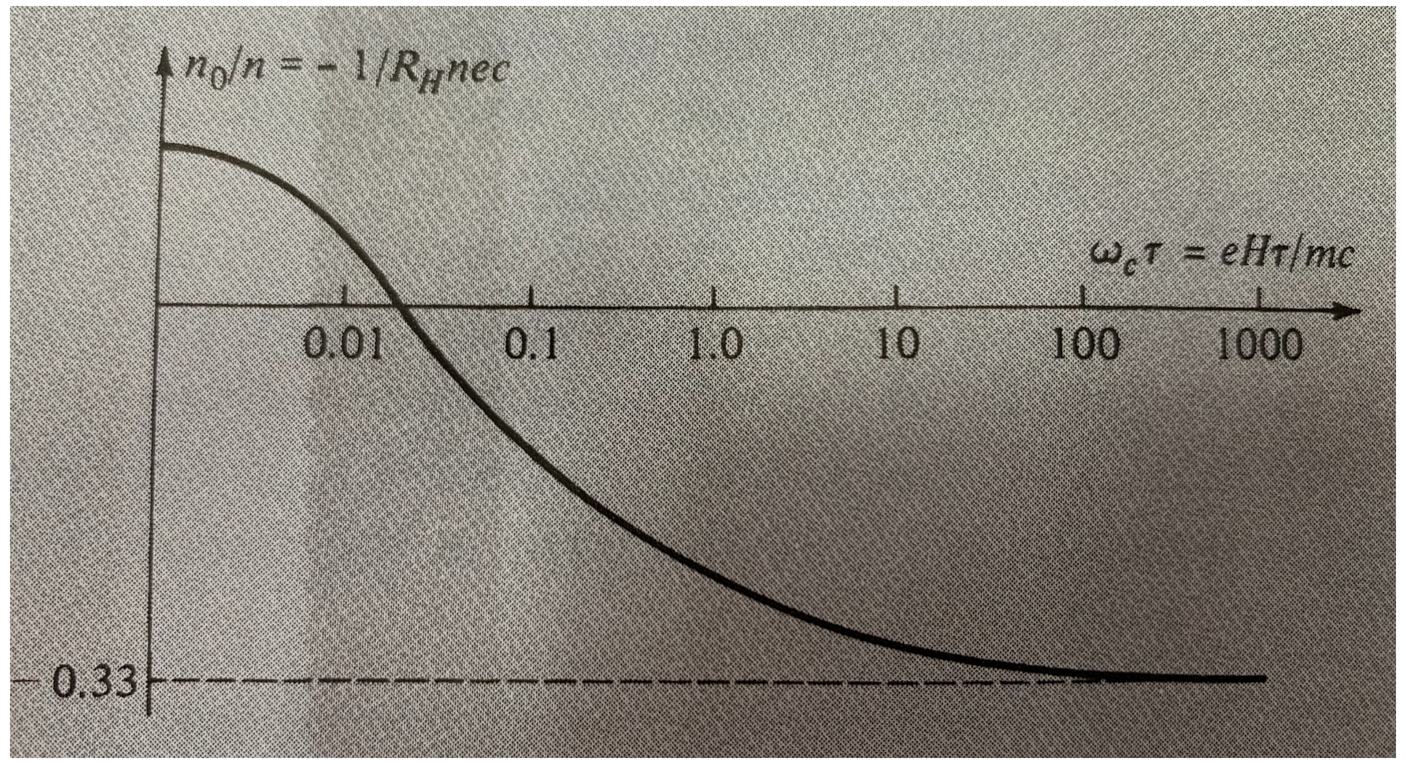
    (only quantum theory of solids can explain these)

Metal	Valence	n_D/n
Li	1	0.8
Na	1	1.2
K	1	1.1
Rb	1	1.0
Cs	1	0.9
Cu	1	1.5
Ag	1	1.3
Au	1	1.5
Be	2	-0.2
Mg	2	-0.4
In	3	-0.3
Al	3	-0.3

Negative R_H (Li–Au)

Positive R_H (Be–Al)

$$n_D = -\frac{1}{eR_H}, \quad n = N_A \frac{Z\rho_m}{A}$$



AC electrical conductivity of a metal (1/2)

- The current induced in a metal by a time-varying **E**-field
  - **E**-field : $\boldsymbol{E}(t) = \Re\left(\tilde{\boldsymbol{E}}(\omega)\, e^{-j\omega t}\right)$
  - The equation of motion for the electron momentum :

$$\frac{d\boldsymbol{p}(t)}{dt} = -\frac{\boldsymbol{p}(t)}{\tau} + \boldsymbol{F}(t), \text{ where } \boldsymbol{F}(t) = -e \cdot \Re\left(\tilde{\boldsymbol{E}}(\omega)\, e^{-j\omega t}\right)$$

$$\rightarrow -j\omega\tilde{\boldsymbol{p}}(\omega) = -\frac{\tilde{\boldsymbol{p}}(\omega)}{\tau} - e\tilde{\boldsymbol{E}}(\omega) \quad \rightarrow \quad \tilde{\boldsymbol{p}}(\omega)\left(j\omega - \frac{1}{\tau}\right) = e\tilde{\boldsymbol{E}}(\omega) \quad \cdots(1)$$

  - The current density by the **E**-field

$$\boldsymbol{J}(t) = \Re\left(\tilde{\boldsymbol{J}}(\omega)\, e^{-j\omega t}\right), \text{ where } \tilde{\boldsymbol{J}}(\omega) = -ne\tilde{\boldsymbol{v}}(\omega) = -ne\frac{\tilde{\boldsymbol{p}}(\omega)}{m} \quad \cdots(2)$$

$$\tilde{\boldsymbol{J}}(\omega) = -\frac{ne}{m}\frac{e\tilde{\boldsymbol{E}}(\omega)}{j\omega - \frac{1}{\tau}} = \sigma(\omega)\tilde{\boldsymbol{E}}(\omega), \text{ where } \boxed{\sigma(\omega) \triangleq \frac{\frac{ne^2\tau}{m}}{1 - j\omega\tau} = \frac{\sigma_0}{1 - j\omega\tau}}$$

Frequency-dependent AC conductivity

AC electrical conductivity of a metal (2/2)

- Two puzzles in the derivation

(1) Time-varying **E**-field accompanies **H**-field (i.e., electromagnetic wave)

→ The Lorenz force : $\boldsymbol{F} = -e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

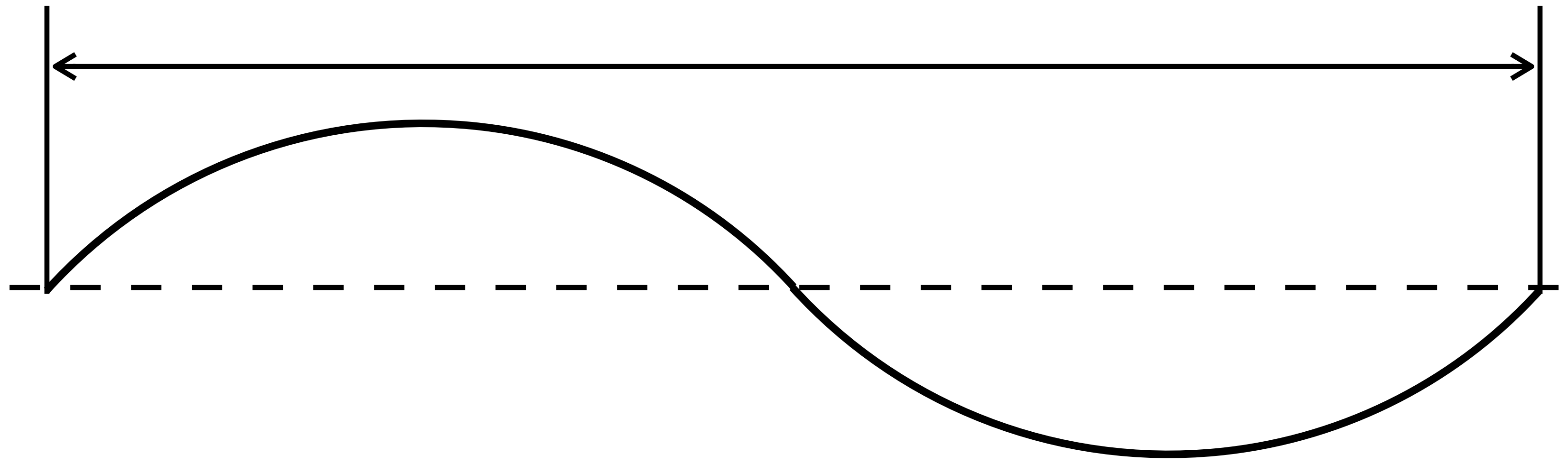
→ The equation of motion : $\dfrac{\boldsymbol{p}}{dt} = -\dfrac{\boldsymbol{p}}{\tau} - e\left(\boldsymbol{E} + \overline{\dfrac{\boldsymbol{p}}{m} \times \boldsymbol{B}}\right)$

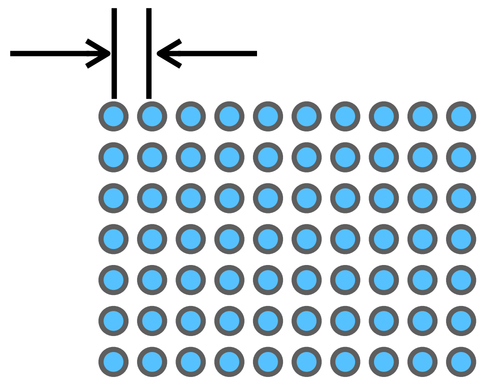
Negligible compared to **E**

(2) **E**-field of the EM wave depends on both space and time (i.e., $\boldsymbol{E}(\boldsymbol{r}, t)$)!

→ But, if a wavelength of the EM wave >> a mean free path of electrons

(e.g. 380 ~ 780 nm for visible light) (1 ~ 10 Å by the Drude's model)





→ Then, $\boldsymbol{J}(\boldsymbol{r}, t)$ is entirely determined by $\boldsymbol{E}(\boldsymbol{r}, t)$ at $\boldsymbol{r}$!

Complex permittivity

- Maxwell's equations (in phasor notation)

$$\begin{cases} \nabla \cdot \boldsymbol{E} = 0 \\ \nabla \cdot \boldsymbol{H} = 0 \\ \nabla \times \boldsymbol{E} = -\mu \dfrac{\partial H}{\partial t} \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} + \epsilon \dfrac{\partial E}{\partial t} \end{cases} \longrightarrow \begin{cases} \nabla \cdot \tilde{\boldsymbol{E}} = 0 \\ \nabla \cdot \tilde{\boldsymbol{H}} = 0 \\ \nabla \times \tilde{\boldsymbol{E}} = -\mu \left(-j\omega \tilde{\boldsymbol{H}}\right) = j\omega\mu \tilde{\boldsymbol{H}} \\ \nabla \times \tilde{\boldsymbol{H}} = \tilde{\boldsymbol{J}} - j\omega\epsilon \tilde{\boldsymbol{E}} \end{cases} \qquad \left(\because \boldsymbol{E}(\boldsymbol{r}, t) = \Re\left(\tilde{\boldsymbol{E}}(\boldsymbol{r})\, e^{-j\omega t}\right) \right)$$

- Complex permittivity

$$\nabla \times \tilde{\boldsymbol{H}} = \tilde{\boldsymbol{J}} - j\omega\epsilon \tilde{\boldsymbol{E}} = \left(\sigma - j\omega\epsilon\right) \tilde{\boldsymbol{E}} = -j\omega \left(\epsilon - \frac{\sigma}{j\omega}\right) \tilde{\boldsymbol{E}} \triangleq -j\omega\epsilon_c \tilde{\boldsymbol{E}}$$

$$\therefore \epsilon_c = \epsilon - \frac{\sigma}{j\omega}, \quad \text{where} \quad \sigma = \frac{\sigma_0}{1 - j\omega\tau}$$

Plasma oscillation (1/3)

Formula from previous slides

$$\sigma = \frac{\sigma_0}{1 - j\omega\tau}, \epsilon_c = \epsilon - \frac{\sigma}{j\omega}, \sigma_0 = \frac{ne^2\tau}{m}$$

- Plasma frequency
  - High frequency approximation $(\omega\tau \gg 1)$

$$\lim_{\omega\tau \gg 1} \sigma = \frac{\sigma_0}{-j\omega\tau} \longrightarrow \epsilon_c = \epsilon - \frac{\sigma_0/(-j\omega\tau)}{j\omega} = \epsilon\left(1 - \frac{\sigma_0/\epsilon\tau}{\omega^2}\right) \triangleq \epsilon\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

$$\epsilon_c = \epsilon\left(1 - \frac{\omega_p^2}{\omega^2}\right),$$ where $$\omega_p^2 = \frac{\sigma_0}{\epsilon\tau} = \frac{ne^2\tau/m}{\epsilon\tau} = \frac{ne^2}{m\epsilon}$$

  - Wave equation :

$$\nabla \times \tilde{\boldsymbol{H}} = -j\omega\epsilon_c\tilde{\boldsymbol{E}}$$

$$\nabla \times \tilde{\boldsymbol{E}} = j\omega\mu\tilde{\boldsymbol{H}} \longrightarrow \left[\nabla \times \left(\nabla \times \tilde{\boldsymbol{E}}\right) = \nabla\left(\nabla \cdot \tilde{\boldsymbol{E}}\right) - \nabla^2\tilde{\boldsymbol{E}}\right] = \left[j\omega\mu\left(\nabla \times \tilde{\boldsymbol{H}}\right)\right]$$

$$\rightarrow -\nabla^2\tilde{\boldsymbol{E}} = \omega^2\mu\epsilon_c\tilde{\boldsymbol{E}} \triangleq k_c^2\tilde{\boldsymbol{E}} \longrightarrow \nabla^2\tilde{\boldsymbol{E}} + k_c^2\nabla^2\tilde{\boldsymbol{E}} = 0 \quad \begin{cases} E_x \sim e^{-jk_cx} \\ E_y \sim e^{-jk_cy} \\ E_z \sim e^{-jk_cz} \end{cases}$$, where $k_c = \omega\sqrt{\mu\epsilon_c}$

Plasma oscillation (2/3)

Formula from previous slides

$$\epsilon_c = \epsilon\left(1 - \frac{\omega_p^2}{\omega^2}\right), \quad \begin{cases} E_x \sim e^{-jk_c x} \\ E_y \sim e^{-jk_c y} \\ E_z \sim e^{-jk_c z} \end{cases} \quad \text{where } k_c = \omega\sqrt{\mu\epsilon_c}$$

- Plasma frequency (contd.)
  - (Case 1) $\omega < \omega_p \rightarrow \epsilon_c$: real and negative

    $\tilde{\boldsymbol{E}}$ decays exponentially into a metal, and gets reflected off

    (i.e. no radiation can propagate "through" the metal)
  - (Case 2) $\omega > \omega_p \rightarrow \epsilon_c$: real and positive

    $\tilde{\boldsymbol{E}}$ oscillates with a frequency ω without attenuation

    (i.e. radiation can propagate through the metal = the metal becomes transparent!)
- In reality?
  - Plasma frequency in other form:

$$\nu_p = \frac{\omega_p}{2\pi} \quad \text{and} \quad \lambda_p = \frac{c}{\nu_p}$$

  - Alkali metals satisfy the Drude's prediction!
  - But other metals do not!

Table 1.5
OBSERVED AND THEORETICAL WAVELENGTHS BELOW WHICH THE ALKALI METALS BECOME TRANSPARENT

ELEMENT	THEORETICAL[a] λ (10^3 Å) = 100nm	OBSERVED λ (10^3 Å)
Li	1.5	2.0
Na	2.0	2.1
K	2.8	3.1
Rb	3.1	3.6
Cs	3.5	4.4

[a] From Eq. (1.41).
Source: M. Born and E. Wolf, *Principles of Optics*, Pergamon, New York, 1964.

Plasma oscillation (3/3)

- What is really happening in a metal?
  - In an electron gas, a charge density oscillates in response to the external ***E***-field.

$$\begin{cases} \text{Continuity equation : } \nabla \cdot \boldsymbol{J} = -\dfrac{\partial \rho}{\partial t} \quad \longrightarrow \quad \nabla \cdot \tilde{\boldsymbol{J}} = j\omega\tilde{\rho} \\ \text{Gauss's law : } \nabla \cdot \tilde{\boldsymbol{E}} = \dfrac{\tilde{\rho}}{\epsilon} \end{cases}$$

Since $\tilde{\boldsymbol{J}} = \sigma\tilde{\boldsymbol{E}}$, $\left[\nabla \cdot \tilde{\boldsymbol{J}} = j\omega\tilde{\rho}\right] = \sigma\nabla \cdot \tilde{\boldsymbol{E}} \;\rightarrow\; \nabla \cdot \tilde{\boldsymbol{E}} = \dfrac{j\omega\tilde{\rho}}{\sigma} \;\rightarrow\; \dfrac{j\omega\tilde{\rho}}{\sigma} = \dfrac{\tilde{\rho}}{\epsilon} \;\rightarrow\; j\omega\epsilon = \sigma$

  - High frequency approx. $(\omega\tau \gg 1)$

$$j\omega\epsilon = \left[\sigma = \frac{\sigma_0}{1 - j\omega\tau} \simeq \frac{\sigma_0}{-j\omega\tau}\right] \longrightarrow \boxed{\omega^2 = \frac{\sigma_0}{\epsilon\tau}}$$ (Same as a plasma frequency!)

  - Such a charge density wave = Plasma oscillation

Intro. to Electro-physics

Drude's model (3rd)

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The most notable success of Drude's model

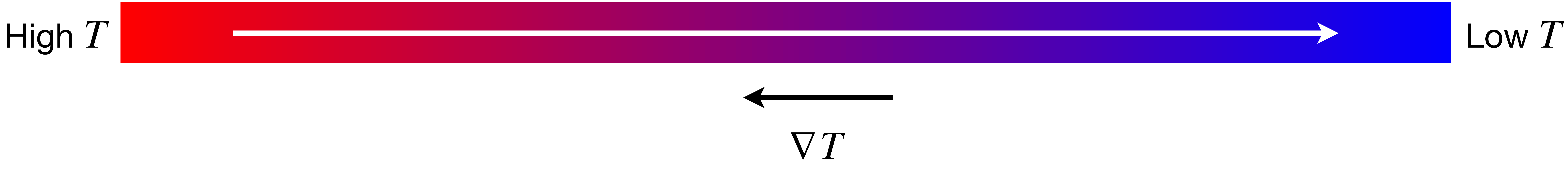
G. H. Wiedemann
(Germany)
1826-1899

- Explanation of the empirical law of Wiedemann and Franz (1853)

$$\frac{\kappa}{\sigma} = CT,$$ where κ : thermal conductivity, σ : electrical conductivity

  - C : a proportionality constant that is, to a fair accuracy, the same for all metals!

- Key assumptions to derive thermal conductivity (κ)
  - Thermal conduction in a metal is done by the conduction electrons
  - Thermal conduction by the ions is negligible
  - Thermal energy flows opposite to temperature gradient (∇T)

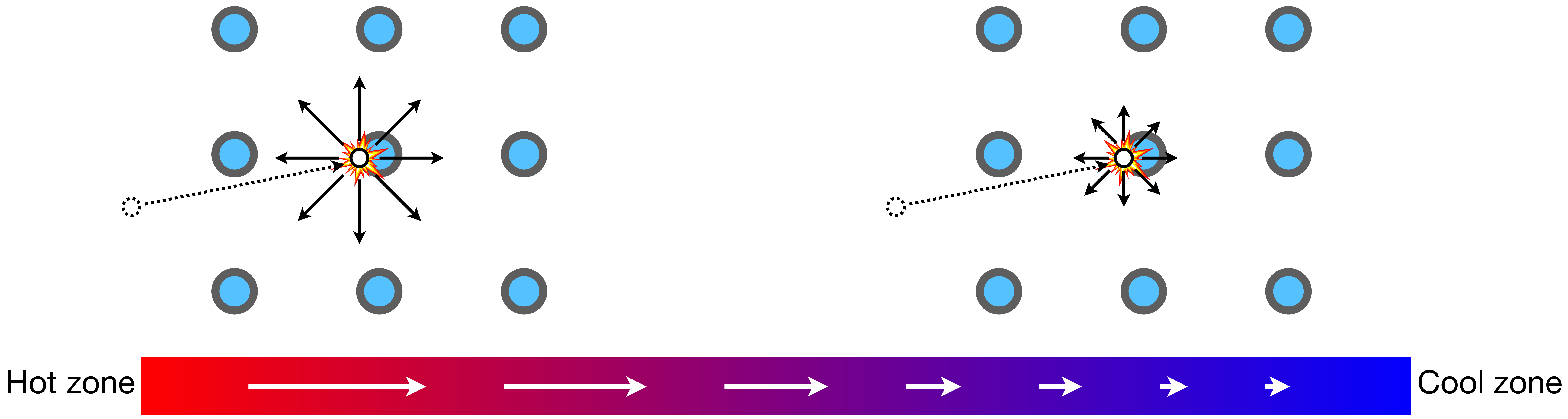


Thermal conductivity

- Thermal flux (J_T)
  - Direction = Parallel to the direction of heat flow
  - Magnitude = thermal energy per unit time crossing a unit area perpendicular to the flow (W/m^2)

$$J_T = -\kappa \nabla T,$$ where κ : thermal conductivity (W/m-K), ∇T : temperature gradient (K/m)

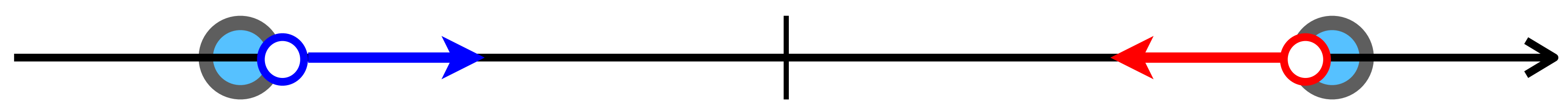
- Revisited assumption of Drude's model
  - After each collision, an electron emerges with a speed appropriate to the local temperature



Derivation of thermal conductivity (1/2)

- Simple 1D model
  - A half of electrons that had their last collisions at $x - v\tau$ move towards x with a velocity v
  - A half of electrons that had their last collisions at $x + v\tau$ move towards x with a velocity $-v$

* $x - v\tau = l$: mean free path



Position	$x - v\tau$	x	$x + v\tau$
Temp	$T(x - v\tau)$		$T(x + v\tau)$
Energy	$\mathscr{E}\left(T(x - v\tau)\right)$		$\mathscr{E}\left(T(x + v\tau)\right)$

- Thermal flux at x

$$J_T = \frac{n}{2} \cdot v \cdot \mathscr{E}\left(T(x - v\tau)\right) + \frac{n}{2} \cdot (-v) \cdot \mathscr{E}\left(T(x + v\tau)\right),$$ where n : Electron density per unit volume

$$= \frac{nv}{2}\left[\mathscr{E}\left(T(x - v\tau)\right) - \mathscr{E}\left(T(x + v\tau)\right)\right]$$

$$= \frac{nv}{2} \cdot 2v\tau \cdot \left[\frac{\mathscr{E}\left(T(x - v\tau)\right) - \mathscr{E}\left(T(x + v\tau)\right)}{2v\tau}\right] \xrightarrow[v\tau \to 0]{} nv^2\tau \frac{d\mathscr{E}}{dT}\left(-\frac{dT}{dx}\right)$$

Derivation of thermal conductivity (2/2)

Thermal flux in 1D

$$J_T = nv^2\tau\frac{d\mathcal{E}}{dT}\left(-\frac{dT}{dx}\right)$$

- Thermal conduction in 3D
  - Electronic velocity

$$\boldsymbol{v} = \boldsymbol{a}_x v_x + \boldsymbol{a}_y v_y + \boldsymbol{a}_z v_z, \text{ where } \langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle \triangleq \frac{1}{3}v^2$$

mean-square electronic speed

$$\left(v^2 = \langle v_x^2\rangle + \langle v_y^2\rangle + \langle v_z^2\rangle\right)$$

* Note that $\langle v_x\rangle = \langle v_y\rangle = \langle v_z\rangle = 0$ in thermal equilibrium. Corrections to this due to ∇T is exceedingly small.

  - Electronic specific heat

$$n\frac{d\mathcal{E}}{dT} = \frac{N}{V}\frac{d\mathcal{E}}{dT} = \boxed{\frac{1}{V}\frac{dE}{dT} \triangleq c_V}$$

: Required energy to raise unit temperature of an electron gas in the unit volume

- Thermal flux in 3D

$$\boldsymbol{J}_T = \boldsymbol{a}_x n\langle v_x^2\rangle\tau\frac{d\mathcal{E}}{dT}\left(-\frac{dT}{dx}\right) \longrightarrow \boldsymbol{J}_T = \frac{1}{3}v^2\tau c_V(-\nabla T) = \kappa(-\nabla T)$$

Thermal conductivity

$$\boxed{\therefore \kappa = \frac{1}{3}v^2\tau c_V}$$

(Revisited) Wiedemann and Franz law [1/2]

- Wiedemann and Franz law interpreted by Drude's model

$$\frac{\kappa}{\sigma} = \frac{\frac{1}{3}v^2\tau c_V}{ne^2\tau/m} = \frac{1}{3}\frac{c_V m v^2}{ne^2}$$

According to the classical kinetic theory of gases: $\begin{cases} c_V = \frac{3}{2}nk_B \\ \frac{1}{2}mv^2 = \frac{3}{2}k_B T \end{cases}$

$$\therefore \frac{\kappa}{\sigma} = \frac{3}{2}\left(\frac{k_B}{e}\right)^2 T$$

$$\rightarrow \quad \frac{\kappa}{\sigma T} = \frac{3}{2}\left(\frac{k_B}{e}\right)^2 \simeq 1.11\ \left[10^{-8}\ W\cdot\Omega/K^2\right]$$

Measured $\frac{\kappa}{\sigma T}\ \left[10^{-8}\ W\cdot\Omega/K^2\right]$

Element	273 K	373 K
Li	2.22	2.43
Na	2.12	
K	2.23	
Rb	2.42	
Cu	2.20	2.29
Ag	2.31	2.38
Au	2.32	2.36
Be	2.36	2.42
Mg	2.14	2.25
Nb	2.90	2.78
Fe	2.61	2.88
Zn	2.28	2.30
Cd	2.49	
Al	2.14	2.19
In	2.58	2.60
Tl	2.75	2.75
Sn	2.48	2.54
Pb	2.64	2.53
Bi	3.53	3.35
Sb	2.57	2.69

Kaye and Laby, Table of Physical and Chemical Constants, Longmans Green, Longdon, 1966.

(Revisited) Wiedemann and Franz law [2/2]

- In reality
  - It was fortunate that the Drude's model roughly predicts Wiedemann-Franz Law!

	Drude's model	In reality
$\frac{\kappa}{\sigma T}\ \left[10^{-8}\ W \cdot \Omega / K^2\right]$	~1.1	~2.2
Electronic heat capacity	c_V	$\sim \frac{1}{100} c_V$
Mean square electronic velocity	$\frac{1}{2} m v^2$	$\sim 100 \cdot \frac{1}{2} m v^2$

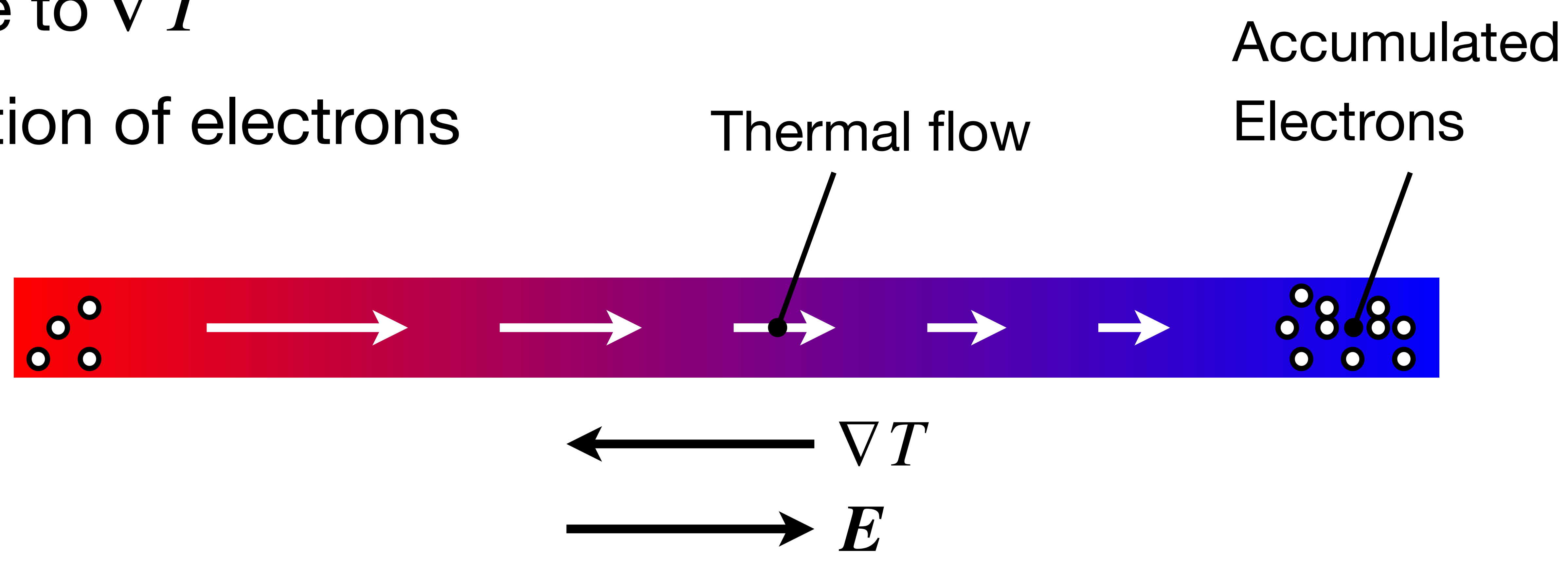
$$\frac{\kappa}{\sigma} = \frac{1}{6} \frac{c_V \left(\frac{1}{2} m v^2\right)}{n e^2}$$

Errors in c_V and v^2 cancel each other out, which leads to "nearly correct" results!

- Sommerfeld model (Drude's model + quantum theory) can correct the discrepancy in each parameter and hence more accurately predicts the W-F law!

Thermoelectric field (1/2)

- Condition
  - A temperature gradient (∇T) in an open-circuited, long and thin bar
  - ∇T forces electrons to move in a direction opposite to ∇T
  - An electric field is induced to oppose the accumulation of electrons



- Thermoelectric field

$$\boldsymbol{E} = Q\nabla T,$$ where Q : thermopower (V/K)

  - Mean electronic velocity at x due to ∇T

$$v_Q = \frac{1}{2}\left[v(x - v\tau) - v(x + v\tau)\right] = \frac{1}{2}(2v\tau)\left[\frac{v(x - v\tau) - v(x + v\tau)}{2v\tau}\right] = v\tau\left(-\frac{dv}{dx}\right)$$

$$= -\tau\frac{d}{dx}\left(\frac{v^2}{2}\right) = -\tau\frac{d}{dT}\left(\frac{v^2}{2}\right)\frac{dT}{dx} = -\tau\frac{d}{dT}\left(\frac{v^2}{2}\right)\nabla T$$

Thermoelectric field (2/2)

1D velocity due to ∇T

$$v_Q = -\tau \frac{d}{dT}\left(\frac{v^2}{2}\right)\nabla T$$

- Thermoelectric field in **3D** (contd.)
  - Mean square electronic velocity :

$$\langle v_x^2 \rangle = \left\langle v_y^2 \right\rangle = \langle v_z^2 \rangle = \frac{1}{3}v^2 \longrightarrow \boldsymbol{v}_Q = -\frac{\tau}{6}\left(\frac{dv^2}{dT}\right)\nabla T$$

  - Mean electronic velocity due to thermoelectric field : $\boldsymbol{v}_E = -\dfrac{e\boldsymbol{E}\tau}{m}$

  - In thermal equilibrium : $\boldsymbol{v}_Q + \boldsymbol{v}_E = 0 \longrightarrow -\dfrac{e\boldsymbol{E}\tau}{m} = \dfrac{-\tau}{6}\left(\dfrac{dv^2}{dT}\right)\nabla T$

$$\boldsymbol{E} = -\frac{1}{3e}\frac{d}{dT}\left(\frac{mv^2}{2}\right)\nabla T \triangleq Q\nabla T, \quad \text{where} \quad \boxed{Q = -\frac{1}{3e}\frac{d}{dT}\left(\frac{mv^2}{2}\right)}$$

Thermopower : the ideal and the real

- Thermopower by Drude's model

$$Q = -\frac{1}{3e}\frac{d}{dT}\left(\frac{mv^2}{2}\right) = -\frac{1}{3ne}n\frac{d\mathscr{E}}{dT} = -\frac{c_V}{3ne} = \frac{-k_B}{2e} = -0.43 \times 10^{-4}\ [V/K]$$

$$c_V = n\frac{d\mathscr{E}}{dT} = \frac{3}{2}nk_B$$

- Measured thermopower

Element	Q [V/K]
Na	-5×10^{-6}
K	-12.5×10^{-6}
Cu	1.8×10^{-6}
Be	1.5×10^{-6}
Al	-1.8×10^{-6}

- A factor of 100 smaller!
- A sign of thermopower is positive for some metals!

Drude vs. Sommerfeld models

- Drude model shows…
  - Good agreement with Wiedemann-Franz law ($\kappa/\sigma T = C$)
  - discrepancy in specific heat $\left(c_V\right)$, mean-square electronic velocity $\left(v^2\right)$, thermopower $\left(Q\right)$ and etc.
- What is wrong about Drude's model?
  - Assumption that electronic velocity distribution follows the Maxwell-Boltzmann distribution **(X)**

    ($\because$ Electrons are Fermion that must obey Fermi-Dirac Distribution **(O)**!)

	Drude model	Sommerfeld model
Electronic velocity distribution	Maxwell-Boltzmann Distribution	Fermi-Dirac Distribution
Formula	$f_{MB}(\boldsymbol{v}) = n\left(\frac{m}{2k_BT}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2k_BT}}$	$f_{FD}(\boldsymbol{v}) = \frac{1}{4}\left(\frac{m}{\pi\hbar}\right)^3 \frac{1}{\exp\left(\frac{\frac{1}{2}mv^2 - E_0}{k_BT}\right)+1}$

The number of electrons per unit volume with velocities in the range of $d\boldsymbol{v}$ about $\boldsymbol{v}$: $f(\boldsymbol{v})\,d\boldsymbol{v}$

- **Sommerfeld model = Drude's free electron model + Fermi-Dirac Distribution!**