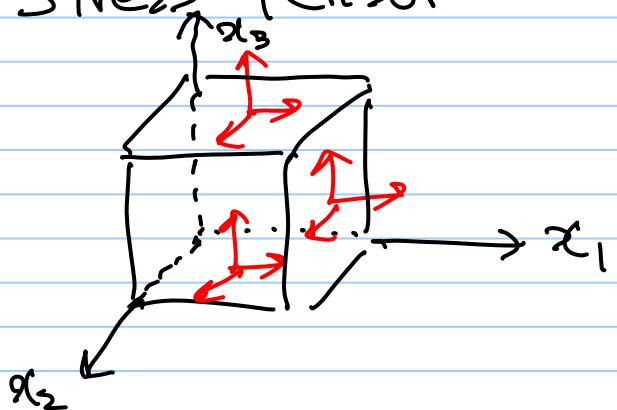


* Conservation of Momentum (Newton's 2nd Law)

$$\frac{D}{Dt} \int_V g u \cdot dV = \int_S \underline{P} \cdot dS + \int_V \underline{f} \cdot f \cdot dV . \leftarrow$$

rate of
mtm change
Surf.
force
body
force .

• Stress tensor



$$\rightarrow \sigma_{ij} \quad (i, j = 1, 2, 3)$$

↑ acting on x_i -plane, in
 x_j -direction .

- Surface pressure, P

$$P_1 = \sigma_{11} \cdot \hat{n}_1, P_2 = \sigma_{12} \cdot \hat{n}_1, P_3 = \sigma_{13} \cdot \hat{n}_1$$

\vdots \rightarrow unit normal (directional) vector
of x_1 -plane.

$$\Rightarrow P_j = \sigma_{ij} \hat{n}_i$$

$$\frac{D}{Dt} \int_V \rho u_j dV = \int_S \sigma_{ij} \hat{n}_i dS + \int_V \rho f_j dV.$$

RTT

$$= \int_V \left[\frac{\partial}{\partial t} (\rho u_j) + \frac{\partial}{\partial x_k} (\rho u_j u_k) \right] dV$$

$$\downarrow \int_V \frac{\partial \sigma_{ij}}{\partial x_i} dV \quad (\text{Gauss Theorem}), \int_S (F \cdot \hat{n}) dS = \int_V (\nabla \cdot F) dV$$

$$\Rightarrow \frac{\partial}{\partial t} (\rho u_j) + \frac{\partial}{\partial x_k} (\rho u_j u_{kj}) = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j$$

$$\cancel{\rho \frac{\partial u_j}{\partial t} + u_j \frac{\partial \rho}{\partial t} + u_j \frac{\partial}{\partial x_k} (\rho u_{kj}) + \rho u_k \frac{\partial u_j}{\partial x_k}} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j$$

continuity eq. = 0.

$$\therefore \cancel{\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k}} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j \quad \text{body force.}$$

temporal convective acceleration \rightarrow gradient of surface stress.

rate of change
in metm.

symmetry of stress tensor. ($\sigma_{ij} = \sigma_{ji}$)

- angular mom conservation

$$\frac{D}{Dt} \int_V [r \times (\rho u)] dV = \int_S (r \times P) dS + \int_V [r \times (\rho f)] dV.$$

revisit CCT.

$$\left[\frac{D}{Dt} \int_V \rho \beta dV \right] = \int_V \left[\frac{\partial (\rho \beta)}{\partial t} + \frac{\partial (\rho \beta u_k)}{\partial x_k} \right] dV$$

$$= \int_V \left[\cancel{\beta \frac{\partial \rho}{\partial t}} + \cancel{\rho \frac{\partial \beta}{\partial t}} + \cancel{\beta u_k \frac{\partial \rho}{\partial x_k}} + \cancel{\rho u_k \frac{\partial \beta}{\partial x_k}} + \cancel{\rho \beta \frac{\partial u_k}{\partial x_k}} \right] dV$$

continuity.

$$= \int_V \rho \left(\frac{\partial \beta}{\partial t} + u_k \frac{\partial \beta}{\partial x_k} \right) dV = \int_V \rho \frac{D\beta}{Dt} dV.$$

]

• LHS of $\textcircled{*}$

$$\begin{aligned} \frac{D}{Dt} \int_V [r \times (\rho u)] dV &= \int_V \rho \frac{D}{Dt} (r \times u) dV \\ &= \int_V \rho \left[\frac{\partial (r \times u)}{\partial t} + u_k \frac{\partial (r \times u)}{\partial x_k} \right] dV \end{aligned}$$

$$= \int_V \rho \left[\frac{\partial r}{\partial t} \times u + r \times \frac{\partial u}{\partial t} + u_k \frac{\partial r}{\partial x_k} \times u + u_k r \times \frac{\partial u}{\partial x_k} \right] dV.$$

~~u~~

$\delta_{ik} \hat{e}_i$

u

$$= \int_V \left[r \times \rho \left(\frac{\partial u}{\partial t} + u_k \frac{\partial u}{\partial x_k} \right) \right] dV = \int_V \left(r \times \rho \frac{Du}{dt} \right) dV.$$

• RHS of \otimes

$$\int_V (r \times P) dV = \int_V \epsilon_{ijk} r_j P_k \hat{e}_i dV = \int_V \epsilon_{ijk} r_j \sigma_{pk} \hat{n}_p \cdot \hat{e}_i dS$$

$$= \int_V \frac{\partial}{\partial x_p} (\epsilon_{ijk} r_j \delta_{pk} \hat{e}_i) dV \leftarrow \begin{matrix} \text{Gauss} \\ \text{Theorem} \end{matrix}$$

$$= \int_V \epsilon_{ijk} \left[\frac{\partial r_i}{\partial x_p} \delta_{pk} + r_j \frac{\partial \delta_{pk}}{\partial x_p} \right] \hat{e}_i dV.$$

$$= \int_V \epsilon_{ijk} \left(\delta_{jp} \delta_{pk} + r_j \frac{\partial \delta_{pk}}{\partial x_p} \right) \hat{e}_i dV.$$

→ into \otimes .

$$\int_V \left(r_x \rho f + \underbrace{\epsilon_{ijk} r_j \frac{\partial \rho f_k}{\partial x_p} \hat{e}_i}_{\epsilon_{ijk} r_j \rho f_k \hat{e}_i} - r_x \rho \frac{Du}{Dt} \right) dV + \int_V \epsilon_{ijk} \delta_{jk} \hat{e}_i dV = 0.$$

$\int_V = 0$ (momentum conserv. eq.)

$$\Rightarrow \int_V \epsilon_{ijk} \delta_{jk} \hat{e}_i dV = 0 \rightarrow \underbrace{\epsilon_{ijk} \delta_{jk}}_{\text{Skew-symmetric}} \hat{e}_i = 0.$$

Skew-symmetric

$\Rightarrow \delta$ should be symm!

HW box 313-222.