

# \* Conservation of Energy

- 1st Law of Thermodynamics

$dE = \delta W + \delta Q$  : valid for eq. state.  
 internal energy

for fluid flow?  
 (not eq. state)

$e_T = e + \frac{1}{2}u^2$   
 (total energy)

$$\therefore \frac{D}{Dt} \int_V \left[ \rho e + \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right] dV = \int_S \mathbf{u} \cdot \mathbf{P} dS + \int_V \mathbf{u} \cdot \rho \mathbf{f} dV - \int_S \mathbf{q} \cdot \hat{\mathbf{n}} dS$$

rate of change in total energy
rate of work done on the fluid
rate of heat leaving the fluid

RIT

$$\int_V \left\{ \frac{\partial}{\partial t} \left( \rho e + \frac{1}{2} \rho u_j u_j \right) + \frac{\partial}{\partial x_k} \left( \rho e + \frac{1}{2} \rho u_j u_j \right) u_k \right\} dV$$

$$\int_S \mathbf{u} \cdot \mathbf{P} dS = \int_S u_j \underline{\underline{\sigma_{ij} n_i}} dS = \int_V \frac{\partial}{\partial x_i} (u_j \sigma_{ij}) dV$$

$$\int_{\Sigma} \mathbf{f} \cdot \hat{\mathbf{n}} \, dS = \int_{\Sigma} f_j n_j \, dS = \int_V \frac{\partial f_j}{\partial x_j} \, dV.$$

$$u_j \frac{\partial \sigma_{ij}}{\partial x_i} + \sigma_{ij} \frac{\partial u_j}{\partial x_i}$$

$$\therefore \frac{\partial}{\partial t} \left( \rho e + \frac{1}{2} \rho u_j u_j \right) + \frac{\partial}{\partial x_k} \left[ \left( \rho e + \frac{1}{2} \rho u_j u_j \right) u_k \right] = \frac{\partial}{\partial x_i} (u_j \sigma_{ij}) + u_j \rho f_j - \frac{\partial f_j}{\partial x_j}$$

$$\text{LHS: } \frac{\partial}{\partial t} \left( \rho e + \frac{1}{2} \rho u_j u_j \right) = \rho \frac{\partial e}{\partial t} + e \frac{\partial \rho}{\partial t} + \rho \frac{\partial}{\partial t} \left( \frac{1}{2} u_j u_j \right) + \frac{1}{2} u_j u_j \frac{\partial \rho}{\partial t}$$

$$\frac{\partial}{\partial x_k} \left[ \left( \rho e + \frac{1}{2} \rho u_j u_j \right) u_k \right] = e \frac{\partial}{\partial x_k} (\rho u_k) + \rho u_k \frac{\partial e}{\partial x_k} + \frac{1}{2} u_j u_j \frac{\partial}{\partial x_k} (\rho u_k)$$

- from cont. eq.

$$\frac{\partial}{\partial x_k} (\rho u_k) = - \frac{\partial \rho}{\partial t}$$

$$+ \rho u_k \frac{\partial}{\partial x_k} \left( \frac{1}{2} u_j u_j \right)$$

$$\therefore \text{LHS} = \rho \left( \frac{\partial e}{\partial t} + u_k \frac{\partial e}{\partial x_k} + u_j \frac{\partial u_j}{\partial t} + u_j u_k \frac{\partial u_j}{\partial x_k} \right)$$

$$\Rightarrow \rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} + \rho u_j \frac{\partial u_j}{\partial t} + \rho u_j u_k \frac{\partial u_j}{\partial x_k} = \rho u_j \frac{\partial \sigma_{ij}}{\partial x_i} + \sigma_{ij} \frac{\partial u_j}{\partial x_i} + \rho u_j f_j - \rho \frac{\partial q_j}{\partial x_j}$$

Momentum conservation eq.  
(balance of mechanical energy)

$$= \rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = \sigma_{ij} \frac{\partial u_j}{\partial x_i} - \frac{\partial q_j}{\partial x_j} \rightarrow \text{rate of heat added due to convection.}$$

temporal convective  
rate of change of  
internal e.

↳ conversion of mech. e into  
thermal e due to surf. stress.

So we have,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k} (\rho u_k) = 0$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j$$

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = \sigma_{ij} \frac{\partial u_i}{\partial x_i} - \frac{\partial q_j}{\partial x_j} + \rho = \rho(P, T), e = e(P, T)$$

state eq.

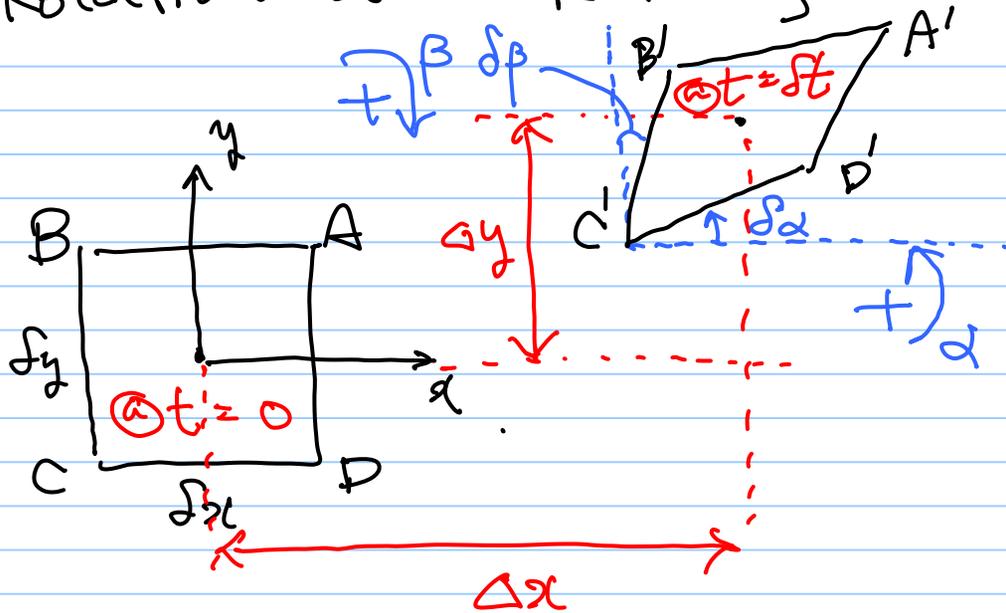
$\rho, e, u_j, f_j, \sigma_{ij} \rightarrow 17$  unknowns,  $\rightarrow$  "14"

$\sigma_{ij}$  symmetric

Constitutive

$\rightarrow \sigma_{ij}$  and  $q_j$  need to be specified!  $\rightarrow$  equations.

# \* Rotation and Rate of shear.



$$\Delta x = \int_0^{\Delta t} u(x(t), y(t)) dt \quad \left. \begin{array}{l} \text{Taylor} \\ \text{series} \\ \text{exp.} \end{array} \right\}$$

$$= \int_0^{\Delta t} \left[ u(0,0) + x(t) \frac{\partial u}{\partial x}(0,0) + y(t) \frac{\partial u}{\partial y}(0,0) + \text{HOT} \right] dt$$

$$= u(0,0) \Delta t + \int_0^{\Delta t} \left[ x(t) \frac{\partial u}{\partial x}(0,0) + y(t) \frac{\partial u}{\partial y}(0,0) + \text{HOT} \right] dt$$

$$\Delta y = \underline{v(0|0)} \Delta t + \int_0^{\Delta t} \left[ x(t) \frac{\partial v}{\partial x} (0|0) + y(t) \frac{\partial v}{\partial y} (0|0) + \text{HOT} \right] dt.$$

$$\Rightarrow \underline{\Delta \alpha} = \tan^{-1} (\text{slope of } \overline{CD'})$$

$$= \tan^{-1} \left\{ \frac{\left[ v\left(\frac{1}{2}\Delta x, -\frac{1}{2}\Delta y\right) \cdot \Delta t + \text{HOT} \right] - \left[ v\left(-\frac{1}{2}\Delta x, -\frac{1}{2}\Delta y\right) \cdot \Delta t + \text{HOT} \right]}{\Delta x + \text{HOT}} \right\}$$

↓ Taylor series expansion.

$$= \tan^{-1} \left\{ \frac{\left[ \Delta x \left( \frac{\partial v}{\partial x} (0|0) + \text{HOT} \right) \right] \Delta t}{\Delta x + \text{HOT}} \right\}$$

$$= \tan^4 \left\{ \frac{\left[ \frac{\partial v}{\partial x}(0,0) + t \dot{\theta} \right] \delta t}{1 + t \dot{\theta}} \right\} \sim \frac{\partial v}{\partial x}(0,0) \cdot \delta t$$

( $\sin d \sim d$ ,  $\tan d \sim d$  for  $d \ll 1$ )

$$\rightarrow \frac{\partial d}{\partial t} = \frac{\partial v}{\partial x}(0,0) = \dot{\alpha}, \quad \frac{\partial \beta}{\partial t} = \frac{\partial u}{\partial y}(0,0) = \dot{\beta}$$

( $\delta x, \delta y \rightarrow 0$ ), ( $\dot{\alpha}$ ), ( $\dot{\beta}$ )

• Rate of clockwise rotation about its centroid.

$$= \frac{1}{2}(\dot{\beta} - \dot{\alpha}) = \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right)$$

• Rate of shear.

$$= \frac{1}{2} (\dot{\beta} + \dot{\alpha}) = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

⇒ Rate of rotation:  $R_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$  : skew-sym.

“ shearing:  $S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$  : sym.

⇒  $R_{ij} + S_{ij} = \frac{\partial u_i}{\partial x_j} \equiv e_{ij}$  (deformation rate tensor)

# \* Constitutive equations

- in Newtonian fluid,

$$\underline{\sigma}_{ij} = -P \delta_{ij} + \tau_{ij} \sim \underline{e}_{kl} \equiv \frac{\partial u_k}{\partial x_l}$$

↑  
thermodynamic  
pressure
shear-stress

→ each none component in  $\tau_{ij}$  should be a linear combination of the nine elements of  $e_{kl}$ .

$$\tau_{ij} = \alpha_{ijkl} \frac{\partial u_k}{\partial x_l}$$

$$= \beta'_{ijkl} \cdot \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} - \frac{\partial u_l}{\partial x_k} \right) + \beta_{ijkl} \cdot \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$$

↑ rotation                      ↑ shearing

• Solid body rotation (no-shearing motion)

↳  $\tau_{ij} = 0$  (rotation itself is not

⇒  $\beta'_{ijkl} = 0$ ,  $\beta_{ijkl} \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) = 0$  (zero!)

$$\therefore \tau_{ij} = \beta_{ijkl} \cdot \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$$

↳ fluid property does not have a

preferred direction: isotropic:

$$\beta_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \gamma (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$$

(general form of isotropic tensor of 4<sup>th</sup> order)

→ from solid-body condition  $\Rightarrow \gamma = 0$ .  
/ rotation

$$\begin{aligned} \tau_{ij} &= \frac{1}{2} \left[ \lambda \delta_{ij} \delta_{kk} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right] \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) \\ &= \lambda \cdot \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \end{aligned}$$

$$\Rightarrow \sigma_{ij} = -P \delta_{ij} + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) : \text{Constitutive eq. for } \sigma_{ij}$$

should be determined empirically

$$q_j = -k \frac{\partial T}{\partial x_j} \quad (\text{Fourier's Law})$$

thermal conductivity

⊙  $\lambda$  and  $\mu$ ?

- for a simple shear flow,  $u = u(y)$ ,  $v = w = 0$

$$\sigma_{12} = \sigma_{21} = \mu \frac{du}{dy} \quad \text{dynamic viscosity } (\nu = \mu/\rho)$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -P$$

$$\sigma_{13} = \sigma_{31} = \sigma_{32} = \sigma_{23} = 0$$

→  $\eta$  : second viscosity coefficient.

· average of normal stress.

$$\frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) = -\bar{P} \quad \text{: mechanical press. (hydrostatic)}$$

$$= -P + \eta \frac{\partial u_k}{\partial x_k} + \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k}$$

$$= -P + \left( \eta + \frac{2}{3} \mu \right) \frac{\partial u_k}{\partial x_k}$$

↳ thermodynamic press.

$$\therefore P - \bar{P} = \left( \eta + \frac{2}{3} \mu \right) \frac{\partial u_k}{\partial x_k} = \underbrace{K}_{\text{bulk modulus}} \frac{\partial u_k}{\partial x_k}$$

## bulk viscosity

: measure of energy transfer from translational movement of molecule ( $\bar{p}$ ) to other (vib, rot.) movements ( $\bar{p}$ )

- for a monatomic gas (He, Ne, Ar)  
only translational mode is possible  $\Rightarrow K=0$   
 $\therefore \lambda = -\frac{2}{3}\mu$  (Stokes relation)
- even for polyatomic gas,  $K$  is quite small.

→ Not matters for incompressible flow!

\* Navier-Stokes equation.

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j, \quad \sigma_{ij} = -P \delta_{ij} + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)$$

$$\frac{\partial \sigma_{ij}}{\partial x_i} = -\frac{\partial P}{\partial x_j} + \frac{\partial}{\partial x_j} \left( \lambda \frac{\partial u_k}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \left[ \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \right]$$

$$\Rightarrow \rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = -\frac{\partial P}{\partial x_j} + \frac{\partial}{\partial x_j} \left( \lambda \frac{\partial u_k}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \left[ \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \right] + \rho f_j$$

• for incompressible flow, constant viscosity

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = - \frac{\partial p}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_i \partial x_i} + \rho f_j$$

• if  $\mu = 0$ ,

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = - \frac{\partial p}{\partial x_j} + \rho f_j \quad \text{: Euler equation.}$$

\* Revisit to energy eq.

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = \underbrace{\sigma_{ij} \frac{\partial u_j}{\partial x_i}} - \frac{\partial q_j}{\partial x_j}$$

$$\sigma_{ij} \frac{\partial u_j}{\partial x_i} = \left[ -p \delta_{ij} + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \right] \frac{\partial u_j}{\partial x_i}$$

$$= -p \frac{\partial u_k}{\partial x_k} + \lambda \left( \frac{\partial u_k}{\partial x_k} \right)^2 + \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \frac{\partial u_j}{\partial x_i}$$

reversible transfer  
of energy due to  
compression.

$\Phi$  : dissipation  
(irreversible)

(in incompressible flow  $\rightarrow$  zero)