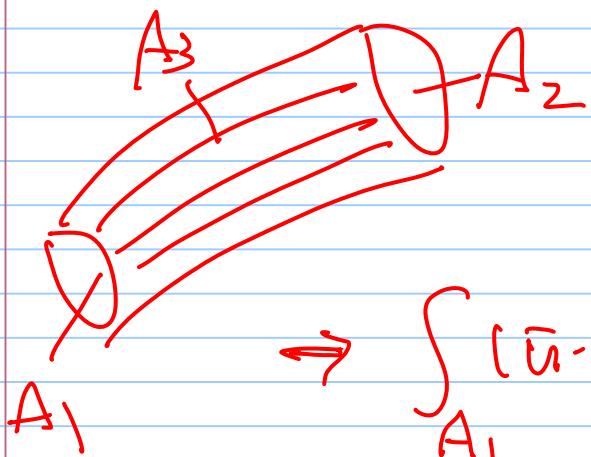


- Vortex line : whose tangents are everywhere parallel to the vorticity vector.  
→ constitute vortex tube.

•  $\nabla \cdot \bar{\omega} = \nabla \cdot (\nabla \times \bar{u}) = 0$  : vorticity is divergence free.  
↳ vortex line should form closed loop or terminate on the free (or solid) surf.  
(no source / sink of vorticity in the fluid itself).

$$\nabla \cdot \bar{u} = 0 \rightarrow \int (\nabla \cdot \bar{u}) dV = 0 \rightarrow \int (\bar{u} \cdot \bar{n}) dS = 0$$



$$\int_{A_1} (\bar{u} \cdot \bar{n}) dS + \int_{A_2} (\bar{u} \cdot \bar{n}) dS = 0 \Rightarrow -Q_1 + Q_2 = 0.$$

Gauss theorem

entire outer  
surf. of a  
streamtube.

in a similar way

$$\nabla \cdot \bar{\omega} = 0 \rightarrow \int_{A_1} (\bar{\omega} \cdot \bar{n}) dS + \int_{A_2} (\bar{\omega} \cdot \bar{n}) dS = 0 \Rightarrow \Gamma_1 = \Gamma_2.$$

## ⑤ SPECIAL FORMS OF EQU. EQUATIONS (Ch. 3)

\* Kelvin's Theorem (Circulation theorem)

- } -  $\mu = 0$  (inviscid)
- $\varphi = \text{constant}$  or  $P = P(\varphi)$
- conservative body force.  $\rightarrow f_j = \frac{\partial \varphi}{\partial x_j}$  ( $G$ : scalar function)

→ "the vorticity of each fluid particle is preserved!".

$$\underbrace{\rho \frac{du_j}{dt} + \rho u_{lc} \frac{\partial u_j}{\partial x_{lc}}}_{\int \frac{D u_j}{Dt}} = - \frac{\partial p}{\partial x_j} + \rho f_j : \text{Euler eq.}$$

$$\int \frac{D u_j}{Dt} \rightarrow \frac{D u_j}{Dt} = - \underbrace{\frac{\partial p}{\partial x_j}}_{\int \frac{\partial p}{\partial x_j}} + \underbrace{\frac{\partial G}{\partial x_j}}$$

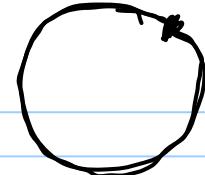
rate of change of velocity for a given fluid element.

$$\frac{Dp}{Dt} = \frac{D}{Dt} \int u_j dx_j = \oint \left[ \frac{\partial p}{\partial t} dx_j + u_j \frac{D(dx_j)}{Dt} \right]$$

$$d\left(\frac{Dx_j}{Dt}\right) = d\left(\frac{\partial x_j}{\partial t} + u_k \frac{\partial x_j}{\partial x_k}\right)$$

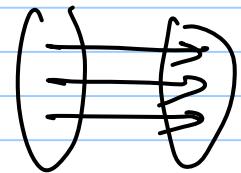
$$= du_j \quad (\text{Eulerian})$$

$$= \oint \left[ \frac{D u_i}{dt} dx_j + u_j du_i \right]$$

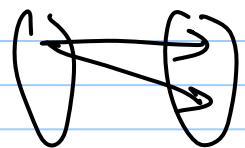


single-valued  
fn.

$$\oint dF = 0$$



vs



multi-valued

fn.

$$= \oint \left[ -\frac{dp}{q} + \frac{\partial f}{\partial x_j} dx_j + \frac{1}{2} d(u_i u_j) \right].$$

$f, u_i, u_j$ : single-valued.

$$\therefore \frac{dP}{dt} = - \oint \frac{dp}{q}$$

$$\textcircled{1} \quad \oint = \text{const.} \quad \frac{DP}{Dt} = - \frac{1}{\oint} \oint dp = 0. \quad \left. \begin{array}{l} \Rightarrow \frac{DP}{Dt} = 0 \\ (\text{conserved}) \end{array} \right\}$$

$$\textcircled{2} \quad P = g(\oint) \rightarrow dp = g'(\oint) d\oint$$

$$\frac{DP}{Dt} = - \oint \frac{g'(\oint)}{\oint} d\oint = 0.$$

→ total vorticity inside a given contour remains the same.

if one of above conditions is not valid,  $\frac{DP}{Dt} \neq 0$ .

## \* Bernoulli eq.

- inviscid, conservative body force.
- steady or irrotational.

• motion conservation

$$\frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} = - \frac{1}{\rho} \frac{\partial p}{\partial x_j} + \frac{\partial G}{\partial x_j} \quad (\text{Euler eq.})$$

$$\left. \begin{aligned} & \underbrace{+ u_k \frac{\partial u_j}{\partial x_k}}_{= (u \cdot \nabla) u} = \nabla \left( \frac{1}{2} u \cdot u \right) - u \times (\nabla \times u) \\ & = \nabla \left( \frac{1}{2} u \cdot u \right) - u \times \omega . \end{aligned} \right\}$$

$$\hookrightarrow \frac{\partial \mathbf{u}}{\partial t} + \nabla \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) - \mathbf{u} \times \boldsymbol{\omega} = - \frac{1}{\rho} \nabla p + \nabla G$$

• Consider a length element ( $dl$ ) in space.

$$\rightarrow dl \cdot \left( - \frac{1}{\rho} \nabla p \right) = \frac{1}{\rho} \underbrace{dl \cdot \nabla p}_{dp} = \frac{1}{\rho} dp = \underline{d} \int \frac{1}{\rho} dp = dl \cdot \nabla \int \frac{1}{\rho} dp$$

$$\underline{dl \cdot \nabla} = dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} = \underline{d}$$

$$\rightarrow \underline{\frac{1}{\rho} \nabla p} = \nabla \int \underline{\frac{1}{\rho} dp}.$$

∴ then, the Euler eq. becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left( \int \frac{dp}{\rho} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - G \right) = \mathbf{u} \times \boldsymbol{\omega}$$

for a steady flow,  $\underline{u} \cdot \nabla \left( \int \frac{dp}{\rho} + \frac{1}{2} u \cdot u - G \right) = u \times \omega$

$$\underline{u} \cdot \nabla \left( \int \frac{dp}{\rho} + \frac{1}{2} u \cdot u - G \right) = u \cdot (u \times \omega) = 0$$

$$\underline{\frac{D}{Dt}} = \cancel{\frac{\partial}{\partial t}} + \underline{u \cdot \nabla} \Rightarrow \frac{D}{Dt} \left[ \int \frac{dp}{\rho} + \frac{1}{2} u \cdot u - G \right] = 0.$$

$\therefore \int \frac{dp}{\rho} + \frac{1}{2} u \cdot u - G = \text{constant}$  along each streamline.

and if it is inviscid,  $\int \frac{dp}{\rho} + \frac{1}{2} u \cdot u - G = \text{constant}$  everywhere!

- unsteady, rotational flow,

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left( \int \frac{dp}{\rho} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - g \right) = \mathbf{u} \times \omega^0 = 0$$

scalar  
↓

$$\omega = \nabla \times \mathbf{u} = 0 \quad \mathbf{u} = \nabla \phi.$$

velocity  
potential.

$$\underline{\underline{dL \cdot \nabla}} \left( \frac{d\phi}{dt} + \int \frac{dp}{\rho} + \frac{1}{2} \nabla \phi \cdot \nabla \phi - g \right) = 0$$

$$\stackrel{\curvearrowleft}{d} \left( \frac{d\phi}{dt} + \int \frac{dp}{\rho} + \frac{1}{2} \nabla \phi \cdot \nabla \phi - g \right) = 0$$

$$\rightarrow \frac{d\phi}{dt} + \int \frac{dp}{\rho} + \frac{1}{2} \nabla \phi \cdot \nabla \phi - g = \underline{\underline{F(t)}}$$

unsteady Bernoulli const.

⇒ Bernoulli eq. usually helps to establish the irrotationality condition by relating the flow to a simpler flow. (e.g., free-stream)

\* Vorticity eq.

N-S eq,  $\rho, \mu = \text{const.}$

$$\nabla \times \left[ \frac{\partial \bar{u}}{\partial t} + \underbrace{(\bar{u} \cdot \nabla) \bar{u}}_{\hookrightarrow \nabla \left( \frac{1}{2} u \cdot u \right)} = -\nabla \left( \frac{P}{\rho} \right) + 2\nu \nabla^2 \bar{u} \right]$$

$$\frac{\partial \bar{\omega}}{\partial t} - \nabla \times (\bar{u} \times \bar{\omega}) = 2\nu \nabla^2 \bar{\omega}$$

$$\left\{ \begin{array}{l} \nabla \times (\bar{u} \times \bar{\omega}) = \bar{u}(\nabla \cdot \bar{\omega}) - \bar{\omega}(\nabla \cdot \bar{u}) - (\bar{u} \cdot \nabla) \bar{\omega} + (\bar{\omega} \cdot \nabla) \bar{u} \\ \cdot \nabla \cdot \bar{u} = 0 \\ \cdot \nabla \cdot \bar{\omega} = 0 \end{array} \right.$$

$$\therefore \frac{\partial \bar{\omega}}{\partial t} + (\bar{u} \cdot \nabla) \bar{\omega} = (\bar{\omega} \cdot \nabla) \bar{u} + 2\nu \nabla^2 \bar{\omega} ; \text{ Vorticity eq.}$$

• 2D flow, for example,  $(\bar{\omega} \cdot \nabla) \bar{u} = 0$ .

$$\frac{\partial \bar{\omega}}{\partial t} + (\bar{u} \cdot \nabla) \bar{\omega} = 2\nu \nabla^2 \bar{\omega} \quad (\text{diffusion eq.})$$

• Cf) Divergence of N-S eq?

$$\rightarrow \nabla \left( \frac{P}{\rho} \right) = \bar{\omega} \cdot \bar{\omega} + \bar{u} \cdot (\nabla^2 \bar{u}) - \frac{1}{2} \nabla^2 (\bar{u} \cdot \bar{u}) ; \text{ Poisson eq.}$$

• Interpretation of Vorticity eq.

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + 2\nu \frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

$$\omega_j (R_{ij} + S_{ij}) = \omega_j (-\frac{1}{2} \omega_k) + \omega_j S_{ij}$$

$$\therefore \frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j S_{ij} + V \frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

sink; diffusion of vorticity by viscosity.

Source : amplification and rotation of vorticity by  $S_{ij}$  (strain rate)

role of  $\omega_j S_{kj}$  ( $S_{kj} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right)$ )

$\Rightarrow$  e.g.)  $\omega_1 : \underbrace{\omega_1 S_{11}}_{\text{stretching or compressing}} + \underbrace{\omega_2 S_{12}}_{\text{tilting (rotation)}} + \underbrace{\omega_3 S_{13}}$