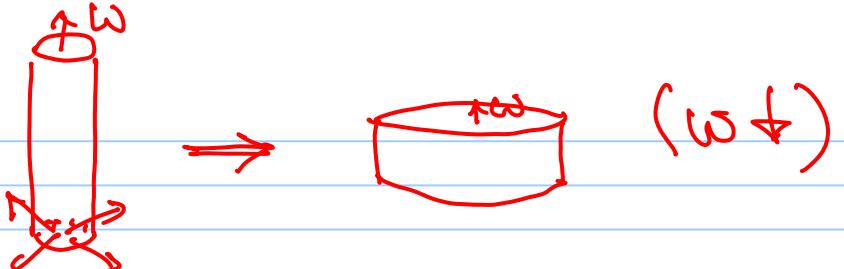


HW #3. $\frac{2-t}{3} \cdot b, \delta$ (due 4/15)
 $\frac{3}{3} - z_1, z_2$.



⑥ 2D POTENTIAL FLOWS (Ch. 4)

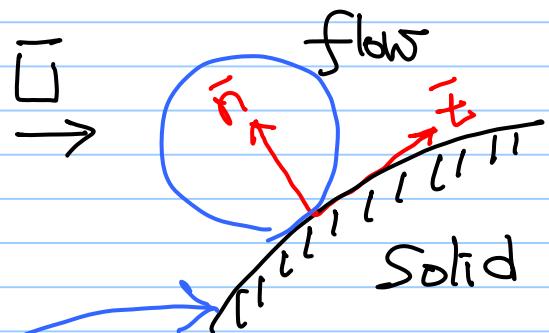
- Ideal fluid flow
 - incompressible
 - inviscid. (inertia dominant)

$$\rightarrow \begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla) \psi = -\frac{1}{\rho} \nabla P + \mathbf{f} \end{cases}$$

• BC's for Euler eq?

$$\left. \begin{aligned} \frac{\partial u_i}{\partial x_i} &= 0 \\ \frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} &= -\frac{1}{\rho} \frac{\partial p}{\partial x_j} + f_j \end{aligned} \right\}$$

or



no-slip BC

$$: \bar{u} \cdot \bar{n} = \bar{U} \cdot \bar{n}$$

$\bar{u} \cdot \bar{t}$: unspecified ($\neq 0$)

tangential velocity is obtained
from the solution!
Instead, the surface of a body becomes a streamline

- potential flow

: if an ideal fluid flow is irrotational, the flow

will remain irrotational even near the solid body

(Kelvin's theorem)

$$\rightarrow \bar{\omega} = \nabla \times \bar{u} = 0 \text{ everywhere.}$$

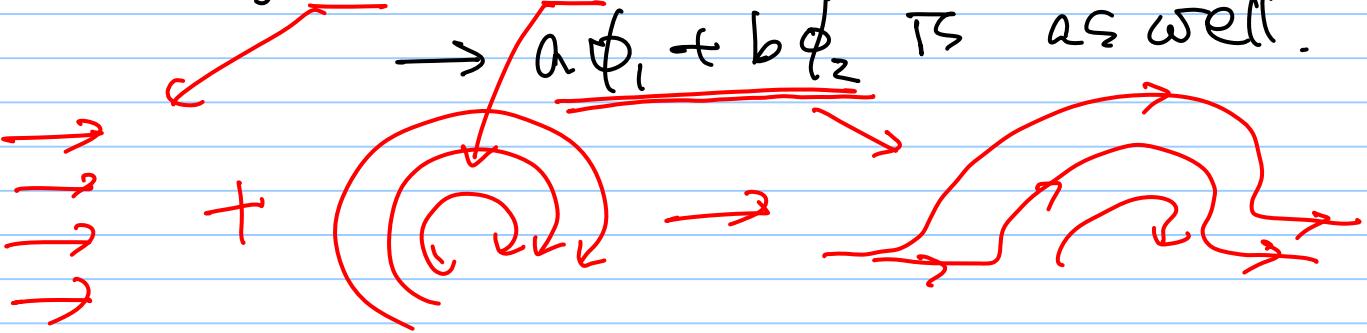
$$\rightarrow \bar{u} = \nabla \phi \quad (\phi: \text{velocity potential, scalar})$$

$$\rightarrow \nabla \cdot \bar{u} = \nabla \cdot \nabla \phi = \boxed{\nabla^2 \phi = 0} : \text{Laplace eq.}$$

\hookrightarrow sol. of ϕ .

Can get velocity field w/o

Solving eq. of motion (Euler eq.)

- how about pressure? Bernoulli eq!
- $\nabla^2 \phi = 0$: linear eq. (superposition)
if ϕ_1 and ϕ_2 are sols of $\nabla^2 \phi = 0$.
 $a\phi_1 + b\phi_2$ is as well.

- Obviously, rotational flow differs from real flows in certain aspects.
- role of viscosity in internal vs. external flows.

* Stream function (in a complementary way for velocity potential, ϕ), a second function (ψ) may be defined.

i) automatically satisfies the continuity.

ii) irrotationality -

$$\rightarrow U \equiv \frac{\partial \psi}{\partial y}, \quad V \equiv -\frac{\partial \psi}{\partial x}, \quad W = 0. \quad (\text{in 2D})$$

$$\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = \frac{\partial}{\partial x} \left(-\frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y} \right) = 0 \rightarrow \boxed{\nabla^2 \psi = 0}$$

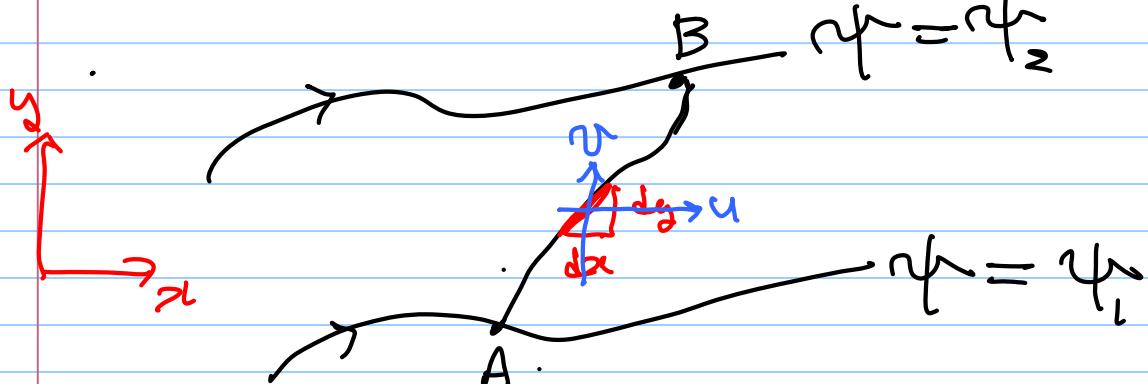
MIDTERM 4/26(Fri) 18:30 (2hrs) Rooms to be announced.

2019-04-03

$$\text{1. } \psi = \psi(x, y)$$

$$\boxed{\frac{d\psi}{dx}} = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = \boxed{-v dx + u dy} = 0 \quad (\psi: \text{const.})$$

$$\rightarrow \frac{dy}{dx} \Big|_{\psi=\text{const.}} = \frac{u}{v}$$



$$\psi_2 - \psi_1 = ?$$

control surface \overline{AB} , arbitrary but w/ positive slope.

→ Volume flow rate (per unit depth) through \overline{AB}

$$\equiv Q = \int_A^B u \cdot dy - \int_A^B v \cdot dx$$

$$\rightarrow \eta_2 - \eta_1 = Q$$

$$d\eta = -v dx + u dy$$

Streamlines ($\eta = \text{const.}$) and (equi)potential lines ($\phi = \text{const.}$)
are orthogonal. ($\phi = \phi(x, y)$, $\eta = \eta(x, y)$)

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u dx + v dy = 0$$

$$\therefore \left(\frac{dy}{dx} \right)_{\phi=\text{const}} = -\frac{u}{v} \quad \rightarrow \quad \left(\frac{dy}{dx} \right)_{\eta=\text{const}} \cdot \left(\frac{dy}{dx} \right)_{\phi=\text{const}} = -1.$$

* Complex Potential / Velocity.

- Complex variable theory.

. Analytic function, $F(z)$, $z = x + iy$.

: dF/dz is existing at $z = z_0$ and around it.

, and dF/dz is indep. of direction.

- Singular point : any points at which $F(z)$ is
not analytic.

. Derivatives of analytic function, $F(z)$

$$\Rightarrow \frac{dF}{dz} = \frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y}$$

\uparrow $\uparrow =$

$$- u = \boxed{\frac{\partial \phi}{\partial x}} = \boxed{\frac{\partial \psi}{\partial y}}$$

$$v = \boxed{\frac{\partial \phi}{\partial y}} = - \boxed{\frac{\partial \psi}{\partial x}}$$

→ Cauchy-Riemann equations.

→ ϕ and ψ are solutions of C-R equations.

(necessary condition for analytic fn.)

∴ Let's define complex potential $F(z) = \phi(x, y) + i\psi(x, y)$.

if $F(z)$ is analytic, ϕ and ψ

$$(z = x + iy)$$

automatically satisfy the C-R equation.

→ For every analytic $f(z)$, real part of $F \rightarrow \phi$
imaginary $\approx \rightarrow \psi$)

Solve for u & v .
Solve P from Bernoulli eq.

- | | |
|----------------------------------|---|
| PROS | CONS |
| - powerful. | : - inverse (get sol's \rightarrow physical
meaning) |
| - simpler than solving
PDE's. | : - can't be generalized
to 3D, |

- Complex velocity, $\underline{\omega}(z)$.

$$\underline{\omega(z)} = \frac{dF}{dz} = \frac{\partial F}{\partial z} = \frac{\partial}{\partial x} (\phi + i\psi) = \underline{\underline{u - iv}}.$$

or $\rightarrow i \frac{\partial F}{\partial y} = -i \frac{\partial}{\partial y} (\phi + i\psi) = \boxed{\underline{\underline{u - iv}}}.$

Complex conjugate

$$\rightarrow \underline{\omega} \overline{\omega} = (u - iv)(u + iv) = \underline{\underline{u^2 + v^2}}$$

in polar coordinates (R, θ)

$$u_R = \frac{1}{R} \frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial R}, \quad u_\theta = -\frac{\partial \psi}{\partial R} = \frac{1}{R} \frac{\partial \phi}{\partial \theta}$$

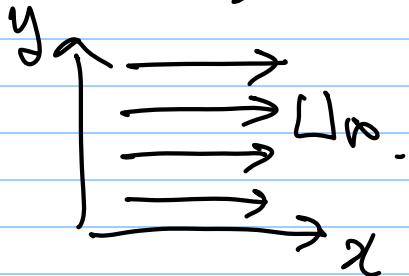
$$F(z) = \phi(z) + i\psi(z), \quad z = R(\cos\theta + i\sin\theta) = Re^{i\theta}.$$

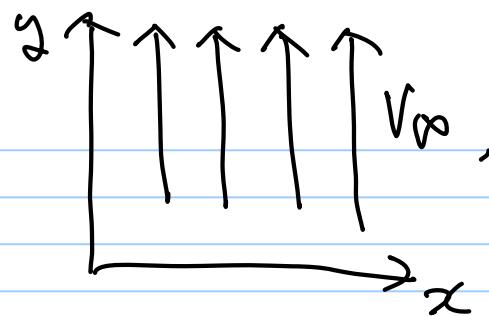
$$\left. \begin{array}{l} u = U_R \cos \theta - U_\theta \sin \theta \\ v = U_R \sin \theta + U_\theta \cos \theta \end{array} \right\} \Rightarrow w(z) = \frac{(U_R - iU_\theta)e^{-i\theta}}{z}$$

① Uniform flow, $F(z) \sim z$.

$$F(z) = U_\infty z, \quad (U_\infty: \text{real #}).$$

$$\rightarrow w(z) = \frac{dF}{dz} = U_\infty = u - iv, \quad \rightarrow \begin{cases} u = U_\infty \\ v = 0 \end{cases}$$

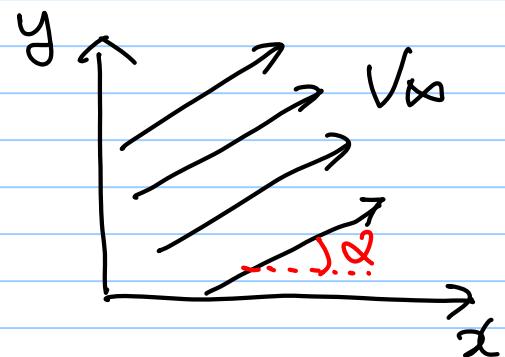




$$F(z) = -iV_\infty z.$$

$$\omega(z) = \frac{dF}{dz} = -iV_\infty = u - i\nu.$$

$$u=0, \nu=V_\infty,$$



$$\Rightarrow F(z) = V_\infty z \cdot e^{-i\alpha}$$

$$\omega(z) = V_\infty \cos\alpha - iV_\infty \sin\alpha = u - i\nu.$$

$$u = V_\infty \cos\alpha, \nu = V_\infty \sin\alpha,$$

② Source, sink and vortex flows : $F(z) \sim \ln z$.

- Source / sink,

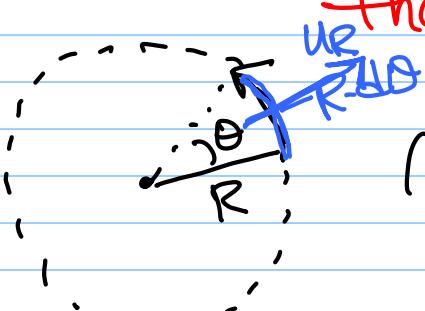
$$F(z) = C \cdot \ln z . = C \cdot \ln (R e^{i\theta}) .$$

$$= C \cdot \ln R + iC\theta . = \underline{\phi} + i\underline{\psi} .$$

$$\omega(z) = \frac{dF}{dz} = \frac{C}{z} = \frac{C}{R e^{i\theta}} = \frac{C}{R} e^{-i\theta} \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \underline{u_R = \frac{C}{R}}, \underline{C_\theta = 0}.$$

$$= (u_R - iC\theta) e^{-i\theta} \left. \begin{array}{l} \\ \end{array} \right\}$$

Strength of a source (cm^3/s) as a volume of fluid
that passes per unit time & unit depth.



$$m = \int_0^{2\pi} u_R \cdot R d\theta . = \int_0^{2\pi} C \cdot d\theta = 2\pi C .$$

$$\Rightarrow C = m / 2\pi .$$

$$\therefore F(z) = \frac{m}{2\pi} \ln z \text{ or } F(z) = \frac{m}{2\pi} \ln(z - z_0) \quad \left. \begin{array}{l} m > 0 : \text{source} \\ m < 0 : \text{sink} \end{array} \right\}$$

• Vortex.

$$\downarrow z = R \cdot e^{i\theta}$$

$$F(z) = -iC \cdot \ln z, = -iC \cdot \ln(R \cdot e^{i\theta}) = C\theta - iC \cdot \ln R = \phi + i\psi$$

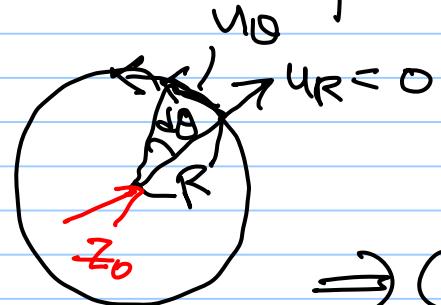
$$\phi = C\theta, \psi = -C \cdot \ln R$$

$$W(z) = \frac{dF}{dz} = -i \frac{C}{z} = -i \frac{C}{R} e^{-i\theta} = (u_R - i\alpha_0) e^{-i\theta}$$

$$\hookrightarrow u_R = 0, \alpha_0 = \frac{C}{R} \quad \left. \begin{array}{l} C > 0 : \text{CCW} \\ C < 0 : \text{clockwise} \end{array} \right\}$$

- vortex strength, circulation.

$$\Gamma \equiv \oint_C \frac{dz}{z} = \oint_C \frac{U_0 \cdot R \cdot d\theta}{z} \\ = \int_0^{2\pi} C \cdot d\theta = 2\pi C$$

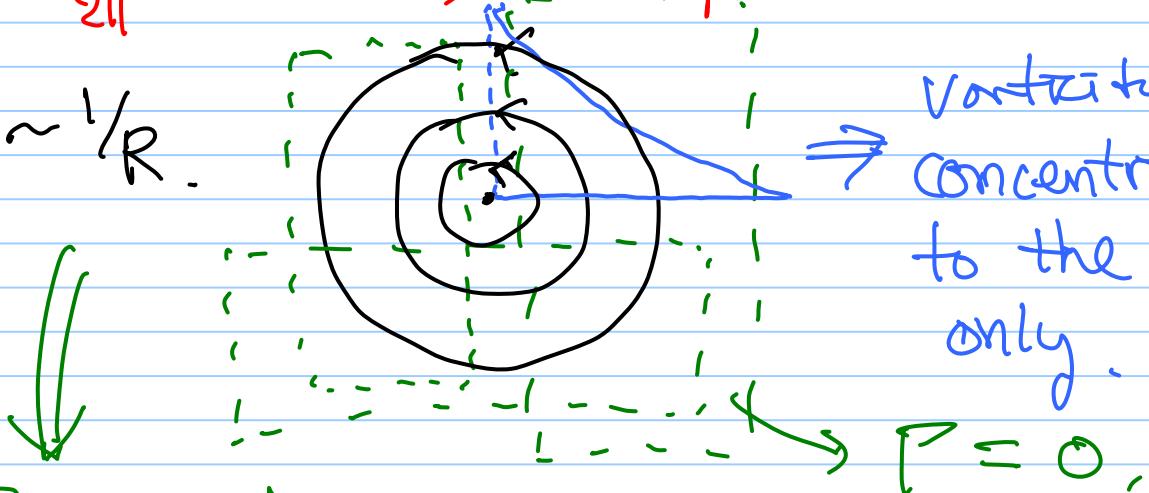


$$\Rightarrow C = \Gamma / 2\pi$$

$$\therefore F(z) = -i \frac{\Gamma}{2\pi} \ln(z - z_0), \quad z_0: \text{position of center.}$$

$$z = z_0$$

$$U_0 = \frac{C}{R}, \sim 1/R.$$



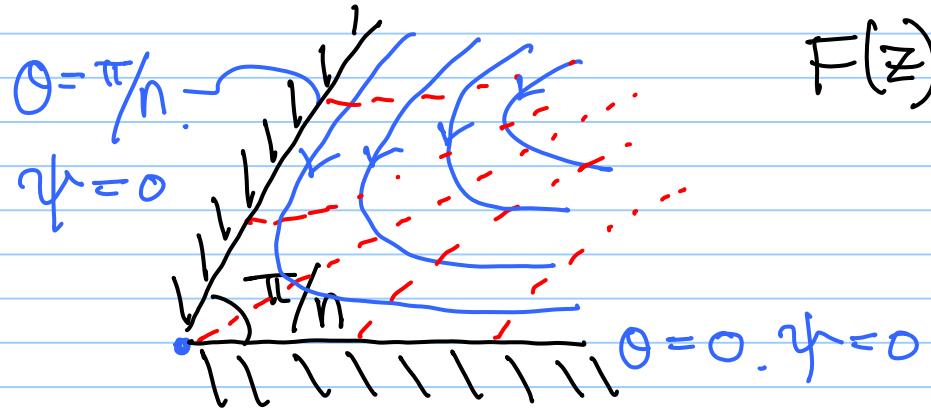
(vorticity) free vortex.

: vorticity is concentrated at $z = z_0$ (singular pt.).

cf) forced vortex. (solid-body rotation)

$$U_R = 0, \quad U_0 \sim R. \Rightarrow \omega = 2\Omega. \quad (\neq 0)$$

③ Flow in a corner., $F(z) \sim z^n$ ($n \geq 1$)



$$F(z) = U_\infty \cdot z^n$$

$$= U_\infty (R e^{i\theta})^n$$

$$= U_\infty R^n e^{in\theta}$$

$$= U_\infty R^n (\cos(n\theta) + i \sin(n\theta))$$

$$= \phi + i\psi$$

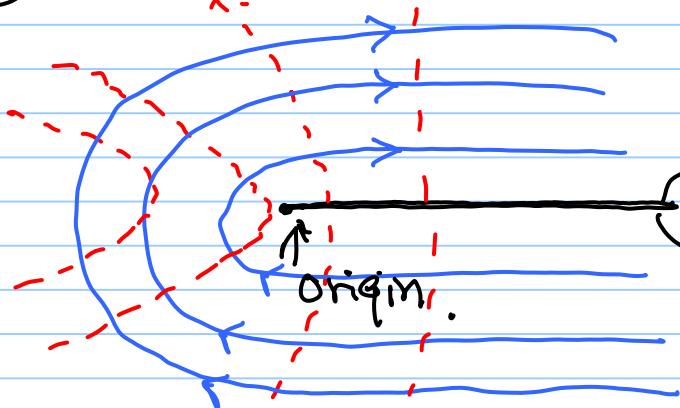
$$\omega(z) = \frac{dF}{dz} = n \cdot U_p \cdot z^{n-1} = (U_R - iU_\theta) e^{-i\theta}$$

$$\therefore U_R = n \cdot U_p R^{n-1} \cos(n\theta), \quad U_\theta = -n \cdot U_p R^{n-1} \sin(n\theta)$$

|

$$\begin{cases} 0 < \theta < \frac{\pi}{2n}, \\ \frac{\pi}{2n} < \theta < \frac{\pi}{n} \end{cases} \quad : \quad U_R > 0, \quad U_\theta < 0 \rightarrow \text{flow direction.}$$

④ Flow around a sharp edge, $F(z) \sim z^{1/2}$



$$w(z) = \frac{dF}{dz} = \frac{C}{z z^{1/2}}$$

$$\begin{aligned} \theta = 2\pi, \varphi = 0 \\ \theta = 0, \varphi = 0 \end{aligned}$$

$$F(z) = C \cdot z^{1/2}$$

$$= C \cdot R^{1/2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)$$

$$\begin{aligned} \rightarrow \phi &= CR^{1/2} \cos \frac{\theta}{2} \\ (\psi &= CR^{1/2} \sin \frac{\theta}{2}) \end{aligned}$$

$$\hookrightarrow U_R = \frac{c}{2R^{1/2}} \cos \frac{\theta}{2}, \quad U_\theta = -\frac{c}{2R^{1/2}} \sin \frac{\theta}{2}.$$

$$\begin{cases} 0 < \theta < \pi : U_R > 0, U_\theta < 0 \\ \pi < \theta < 2\pi : U_R < 0, U_\theta < 0 \end{cases}$$

\rightarrow singular point at the edge (velocity \rightarrow infinite)

\Rightarrow we need to treat the non-physical
sol. at the edge \rightarrow Kutta Condition.
(TE of airfoil).

⑤ Flow due to doublet. : $\frac{V}{z}$, (singularity at $z=0$).

\rightarrow coalescence of a source and a sink.