

$$\hookrightarrow U_R = \frac{c}{2R^{1/2}} \cos \frac{\theta}{2}, \quad U_\theta = -\frac{c}{2R^{1/2}} \sin \frac{\theta}{2}.$$

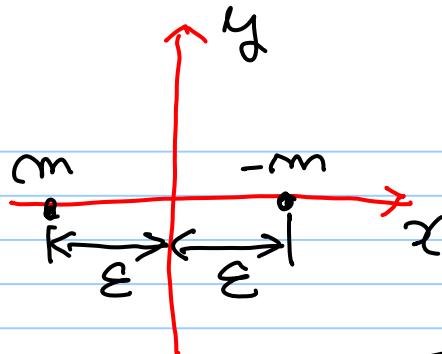
$$\begin{cases} 0 < \theta < \pi : U_R > 0, U_\theta < 0 \\ \pi < \theta < 2\pi : U_R < 0, U_\theta < 0 \end{cases}$$

\rightarrow singular point at the edge (velocity \rightarrow infinite)

\Rightarrow we need to treat the non-physical
sol. at the edge \rightarrow Kutta Condition.
(TE of airfoil).

⑤ Flow due to doublet. : $\frac{V}{z}$, (singularity at $z=0$).

\rightarrow coalescence of a source and a sink.



$$F(z) = \frac{m}{2\pi} \ln(z+\epsilon) - \frac{m}{2\pi} \ln(z-\epsilon).$$

$$= \frac{m}{2\pi} \ln \frac{1+\epsilon/z}{1-\epsilon/z}. \quad \underline{\epsilon \rightarrow 0. (\epsilon \ll 1)}.$$

• Using Taylor expansion.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{m}{2\pi} \ln \left[\left(1 + \frac{\epsilon}{z} \right) \left(1 + \frac{\epsilon}{z} + \left(\frac{\epsilon}{z} \right)^2 + \dots \right) \right]$$

$$= \frac{m}{2\pi} \ln \left[1 + 2 \cdot \frac{\epsilon}{z} + \cancel{+ \text{HOT}} \right] \underset{-1}{\sim} \frac{m}{2\pi} \ln \left(1 + 2 \cdot \frac{\epsilon}{z} \right).$$

for $x \ll 1$, $\ln(1+x) \approx x$. $\frac{m}{2\pi} \cdot 2 \cdot \frac{\epsilon}{z}$

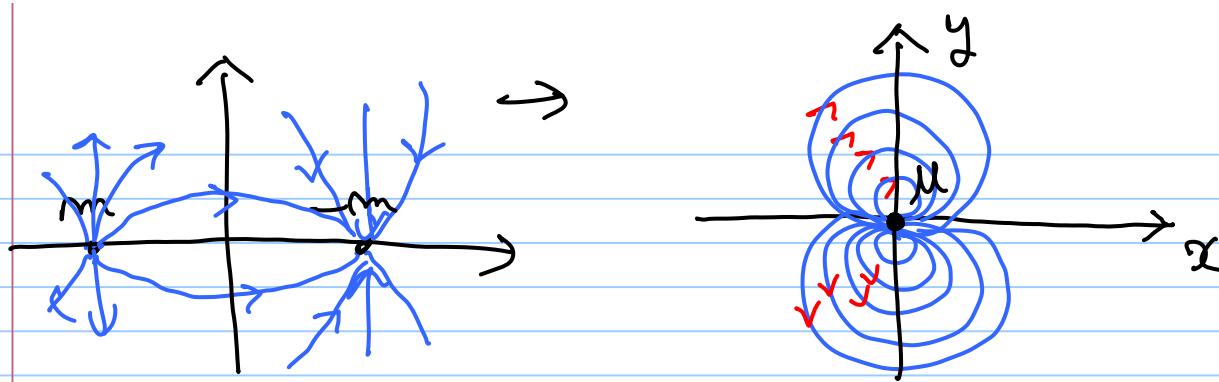
$$\therefore F(z) = \frac{m\epsilon}{\pi z}$$

To avoid trivial sol. define, m such that

$$\lim_{\epsilon \rightarrow 0} (m \cdot \epsilon) = \pi \mu. \quad (\mu: \text{constant, strength})$$

$$\Rightarrow F(z) = \frac{\mu}{z} = \frac{\mu}{x+iy} = \mu \frac{x-iy}{x^2+y^2} = \phi + i\psi$$

$$\Rightarrow x^2 + y^2 + \frac{\mu}{\psi} y = 0. \Rightarrow x^2 + \left(y + \frac{\mu}{2\psi}\right)^2 = \left(\frac{\mu}{2\psi}\right)^2.$$



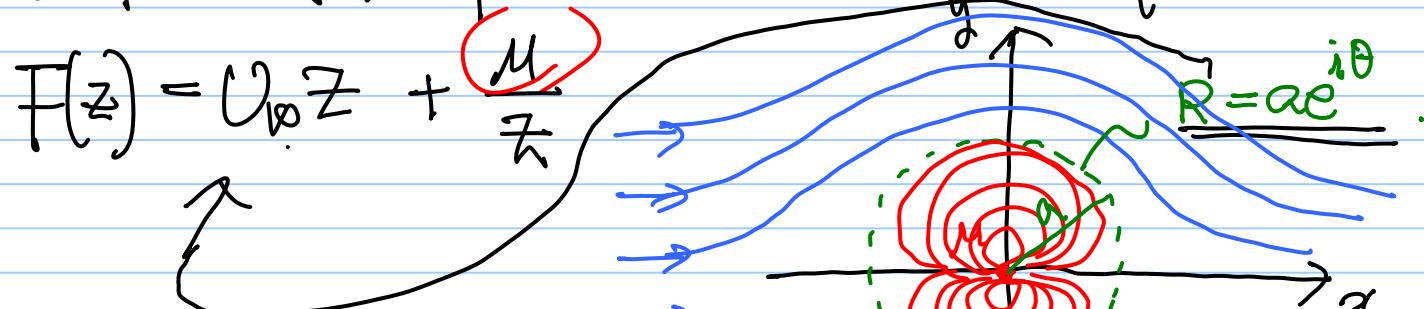
$$W(z) = \frac{dF}{dz} = -\frac{\mu}{z^2} = -\frac{\mu}{R^2} (\cos\theta - i\sin\theta) e^{-i\theta}$$

$$U_R = -\frac{\mu}{R^2} \cos\theta, \quad U_\theta = -\frac{\mu}{R^2} \sin\theta.$$

⑥ Flow around a circular cylinder

2019-04-10

→ uniform flow + doublet at the origin.



on the circle ($R=a$)

$$F(z) = U_\infty z + \frac{M}{z}$$

$$= \left(U_\infty a + \frac{M}{a} \right) e^{i\theta} + i \left(U_\infty a - \frac{M}{a} \right) e^{i\theta}$$

$\uparrow \phi$ $\uparrow \psi$

\cap Solid surface $\rightarrow \psi = 0$

$$\boxed{M = U_\infty a^2}$$

To find μ to make $R=a$ (the cylinder surface) a streamline ($r\phi=0$). \leftarrow

$$\therefore F(z) = U_p \left(z + \frac{a^2}{z} \right).$$

\rightarrow flow symmetry about x -axis : no lift force
 \therefore y -axis : no drag force.

\leftarrow D'Alembert Paradox.
effect of thin boundary-layer
where viscosity is dominant.

⑦ Flow around a circular cylinder w/ rotation.

: uniform flow + doublet + vortex

$$F(z) = U_\infty \left(z + \frac{a^2}{z} \right) + i \frac{\Gamma}{2\pi} \ln z + C \quad (\psi = 0 \text{ at } R=a)$$

(clockwise rotation)

$$\begin{matrix} \uparrow \\ z = a e^{i\theta} \end{matrix}$$

$$F(z) = 2U_\infty a \cos \theta - \frac{\Gamma}{2\pi} \theta + i \frac{\Gamma}{2\pi} \ln a + C$$

$$\therefore C = -i \frac{\Gamma}{2\pi} \ln a \text{ for } \psi=0 \text{ at } R=a.$$

$$\therefore F(z) = U_\infty \left(z + \frac{a^2}{z} \right) + i \frac{\Gamma}{2\pi} \ln \left(\frac{z}{a} \right)$$

$$\omega(z) = \frac{dF}{dz} = U_\infty \left(1 - \frac{a^2}{z^2} \right) + i \frac{P}{2\pi} \cdot \frac{1}{z} \quad (z = Re^{i\theta}).$$

$$= \left\{ U_\infty \left(1 - \frac{a^2}{R^2} \right) \cos\theta + i \left[U_\infty \left(1 + \frac{a^2}{R^2} \right) \sin\theta + \frac{P}{2\pi R} \right] \right\} e^{-i\theta}.$$

$$= (U_R - iU_\theta) e^{-i\theta}.$$

$$U_R = U_\infty \left(1 - \frac{a^2}{R^2} \right) \cos\theta, \quad U_\theta = -U_\infty \left(1 + \frac{a^2}{R^2} \right) \sin\theta - \frac{P}{2\pi R}.$$

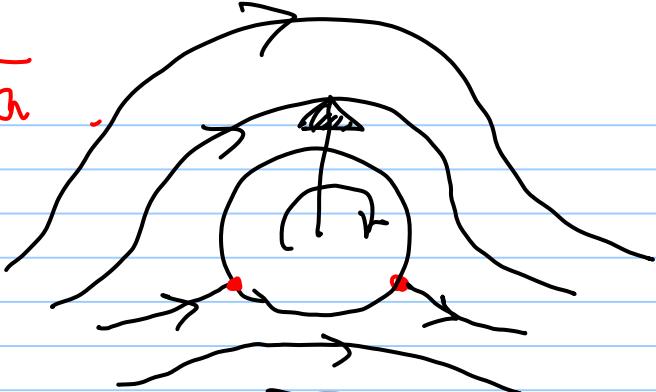
On $R=a$: $U_R = 0$, $U_\theta = -2U_\infty \sin\theta - \frac{P}{2\pi a}$.

→ find stagnation points ($U_R = U_\theta = 0$)

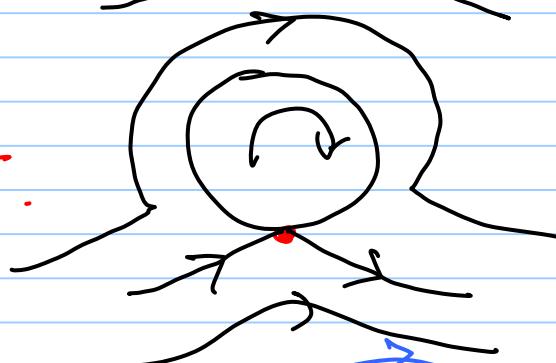
$$-2U_\infty \sin\theta - \frac{P}{2\pi a} = 0$$

$$\Rightarrow \sin\theta_s = -\frac{P}{4\pi U_\infty a}$$

$$0 < \frac{P}{4\pi U_\infty a} < 1 \rightarrow \text{two } \theta_s.$$

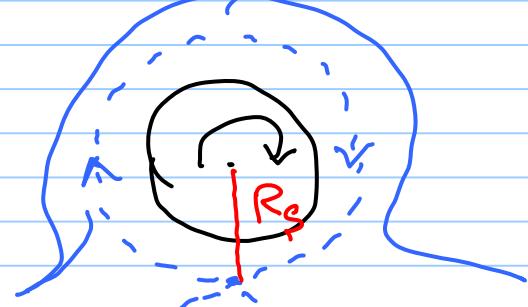


$$\frac{P}{4\pi U_\infty a} = 1 \rightarrow \text{one } \theta_s \\ = \frac{3}{2}\pi.$$



$$\frac{P}{4\pi U_\infty a} > 1 \rightarrow \text{stagn. pt is not on the surf.} \\ \rightarrow \text{in the flow.}$$

R_s, θ_s



$$\downarrow U_R = U_Q = 0.$$

$$\left\{ \begin{array}{l} U_\infty \left(1 - \frac{a^2}{R_s^2} \right) \cos \theta_S = 0 \\ U_\infty \left(1 + \frac{a^2}{R_s^2} \right) \sin \theta_S = - \frac{\Gamma}{2\pi R_s} \end{array} \right. \rightarrow R_s \neq a, \quad : \theta_S = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}.$$

$$\rightarrow \theta_S = \frac{\pi}{2}, \quad \frac{R_s}{a} = \frac{\Gamma}{4\pi U_\infty a} \left(1 - \sqrt{1 - \left(\frac{4\pi U_\infty a}{\Gamma} \right)^2} \right) \rightarrow \text{as } \Gamma \rightarrow \infty, \quad \frac{R_s}{a} < 1.$$

$$\boxed{\theta_S = \frac{3\pi}{2}, \quad \frac{R_s}{a} = \frac{\Gamma}{4\pi U_\infty a} \left(1 + \sqrt{1 - \left(\frac{4\pi U_\infty a}{\Gamma} \right)^2} \right)}$$

$R_s < a$.
NOT possible

X. Blasius Integral Laws.

- To calculate forces acting on the body
 - : velocity from $F(z)$ \rightarrow pressure from Bernoulli
 \rightarrow integration of pressure distribution .
- Blasius Law
 - : complex potential \rightarrow forces and moments by simple contour integrals
(w/o pressure information)

For steady, viscous fluid,
 force balance in $\rightarrow x$ direction.

$$-\int_{C_i} P dy - \int_{C_0} P dy = \int_{C_0} \rho u (udy - vdx)$$

(no mass transfer through C_i)
 (body = streamline)

$$\Rightarrow -X - \int_{C_0} P dy = \int_{C_0} \rho u (udy - vdx)$$

$$-Y + \int_{C_0} P dx = \int_{C_0} \rho v (udy - vdx)$$

→ contour integrals

$$\left. \begin{array}{l} X = \int_{C_0} (-P dy - \rho u^2 dy + \rho uv dx) \\ Y = \int_{C_0} (P dx - \rho uv dy + \rho u^2 dx) \end{array} \right\}$$

Bernoulli eq. $\rightarrow P + \frac{1}{2} \rho (u^2 + v^2) = B$ (constant).

$$P = B - \frac{1}{2} \rho (u^2 + v^2).$$

$$\Rightarrow X = \int_{C_0} \left[uv dx - \frac{1}{2} (u^2 - v^2) dy \right]$$

$$Y = \int_{C_0} \left[uv dy + \frac{1}{2} (u^2 - v^2) dx \right].$$

MIDTERM.
4/26 (Fri)
301 → 304
(If = 30 —)

$$X = \oint_{C_0} [u v dx - \frac{1}{2} (u^2 - v^2) dy]$$

$$Y = \oint_{C_0} [u v dy + \frac{1}{2} (u^2 - v^2) dx].$$

Evaluation of $i \frac{\Omega}{2} \int_{C_0} W(z) dz$, $W(z) = u - iv$, $z = x + iy$

$$= i \frac{\Omega}{2} \int_{C_0} (u - iv)^2 (dx + idy)$$

$$= i \frac{\Omega}{2} \int_{C_0} \left\{ [(u^2 - v^2) dx + z u v dy] + i [(u^2 - v^2) dy - z u v dx] \right\}$$

$$= X - i Y.$$

$$\therefore \left(x - \bar{z} \right) = i \frac{\rho}{2} \int_{C_0} w(z)^2 dz$$

drag lift

velocity, $w = dF/dz$.

: First Blasius' Integral Law.

how to do contour integration?
(Residue theorem).

- Moment balance.

, similarly, it can be shown that

$$(\text{+ } \text{M}) = - \frac{\rho}{2} \cdot \text{Re} \left[\int_{C_0} z \cdot \bar{w}^2 dz \right]$$

real part.

: Second Blasius' Integral Law.

- Laurent Series and Residue theorem.

if $F(z)$ is analytic for all points at $r_0 < r < r_1$
 whose center is at z_0 , then

$$F(z) = \dots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$\left\{ \begin{array}{l} b_n = \frac{1}{2\pi i} \int_C \frac{F(\xi)}{(\xi-z_0)^{n+1}} d\xi, \quad C: \text{Contour of } r=r_0 \\ \quad \quad \quad (n=0, 1, 2, \dots) \end{array} \right.$$

$$b_{-1} = \frac{1}{2\pi i} \int_{C_0} \frac{F(\xi)}{(\xi-z_0)^{-1}} d\xi, \quad C_0: \quad \quad \quad r=r_0$$

→ Residue : coefficient b_{-1} ($1/z$ -term)

$$\oint_C F(z) dz = 2\pi i \left(R_1 + R_2 + \dots + R_n \right)$$

(R_n is the residue at $\underline{z_n}$)

↑
singular pts of F(z).

④ Force and moment on a circular cylinder.

$$F(z) = U_\infty \left(z + \frac{a^2}{z} \right) + \frac{iP}{2\pi} \ln \frac{z}{a} \quad \begin{matrix} (a: \text{cylinder radius}) \\ (\text{w/ circulation}) \end{matrix}$$

↓

$$W(z) = \frac{dF}{dz} = U_\infty \left(1 - \frac{a^2}{z^2} \right) + \frac{iP}{2\pi z}$$

$$X - iY = i \frac{P}{2} \int_C \frac{W^2}{z} dz = i \frac{P}{2} \left[2\pi i \sum R_n \right]$$

$$W(z) = U_\infty^2 - \frac{2U_\infty^2 a^2}{z^2} + \frac{U_\infty^2 a^4}{z^4} + \frac{iU_\infty \Gamma}{\pi z} - \frac{iU_\infty \Gamma a^2}{\pi z^3} - \frac{\Gamma^2}{4\pi^2 z^2}$$

one singular pt, $z_0=0$, $R_0 = \frac{iU_\infty \Gamma}{\pi}$

$$X - iY = -i\rho U_\infty \Gamma.$$

$$\left. \begin{array}{l} X = 0 \\ Y = \rho U_\infty \Gamma \end{array} \right\} \text{(D'Alembert Paradox)}$$

(Kutta-Joukowski Law)

Moment,

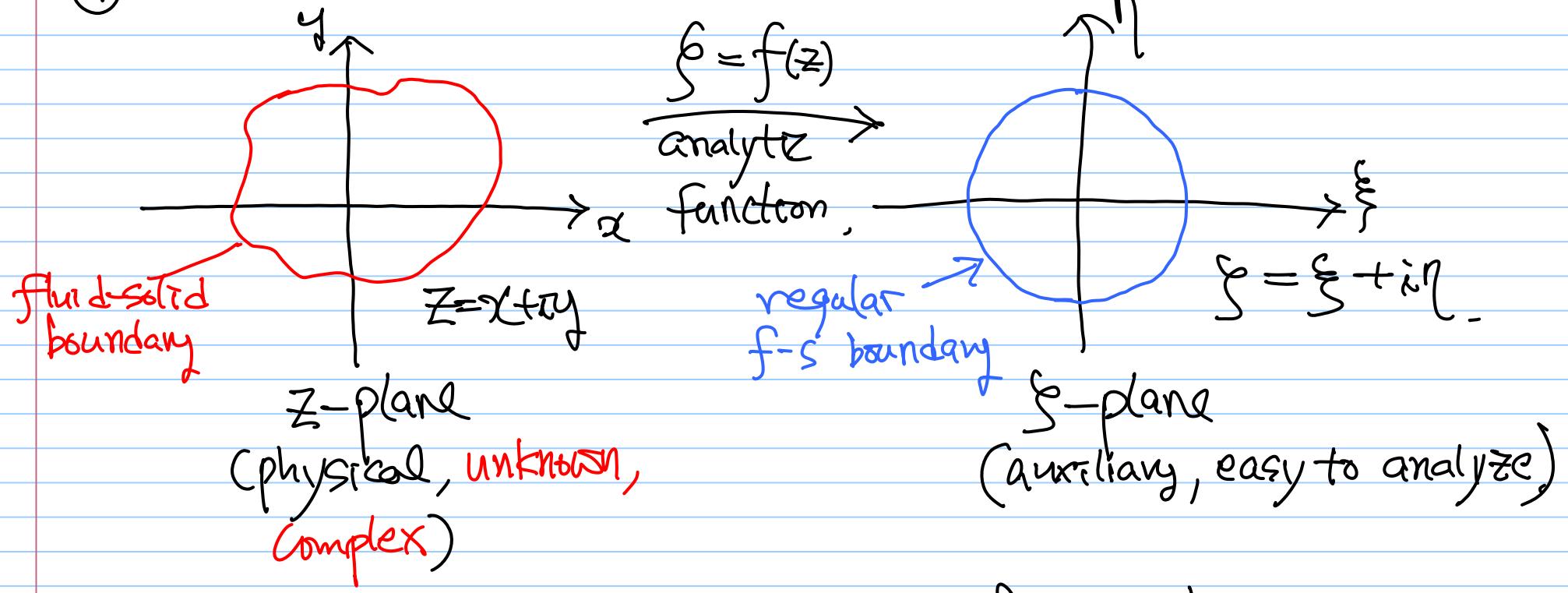
$$z \cdot W(z) = U_\infty z - \frac{2U_\infty^2 a^2}{z} + \frac{U_\infty^2 a^4}{z^3} + \frac{iU_\infty \Gamma}{\pi} - \frac{iU_\infty \Gamma a^2}{\pi z^2} - \frac{\Gamma^2}{4\pi^2 z}$$

$$M = -\frac{\rho}{2} \operatorname{Re} \left[\int_C z W^2 dz \right]$$

$$\downarrow R_0 = -2U_\infty^2 a^2 - \frac{\Gamma^2}{4\pi^2}$$

$$= -\frac{\rho}{2} \cdot \operatorname{Re} [z \pi i \cdot R_0] = 0$$

⑨ Conformal Transformation.



- Laplace equation in z -plane transforms into
the Laplace equation in s -plane.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \rightarrow \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0.$$

(refer to Curnie for verification)

- Complex potential in z -plane is also a valid complex potential in ξ -plane, and vice versa.

- Complex velocity. ($\xi = f(z)$; $F(z) \rightarrow F(\xi)$)

$$W(z) = \frac{dF(z)}{dz} = \frac{dF(\xi)}{d\xi} \cdot \frac{d\xi}{dz} = \frac{d\xi}{dz} \cdot W(\xi).$$

- strength of a singular point β maintained after the transformation. (source, sink, vortex, ...)