

⑩ Joukowski transformation : Solution of the flow around ellipse and airfoil.
; mapping function.

$$z = \zeta + \frac{c^2}{\zeta} \quad (c^2: \text{real } \#)$$

• For $\zeta \rightarrow \pm\infty$, $z \approx \zeta$. $\rightarrow \frac{dz}{d\zeta} = 1 \rightarrow w(z) = w(\zeta)$.

more away from the body. \downarrow same free-stream condition.

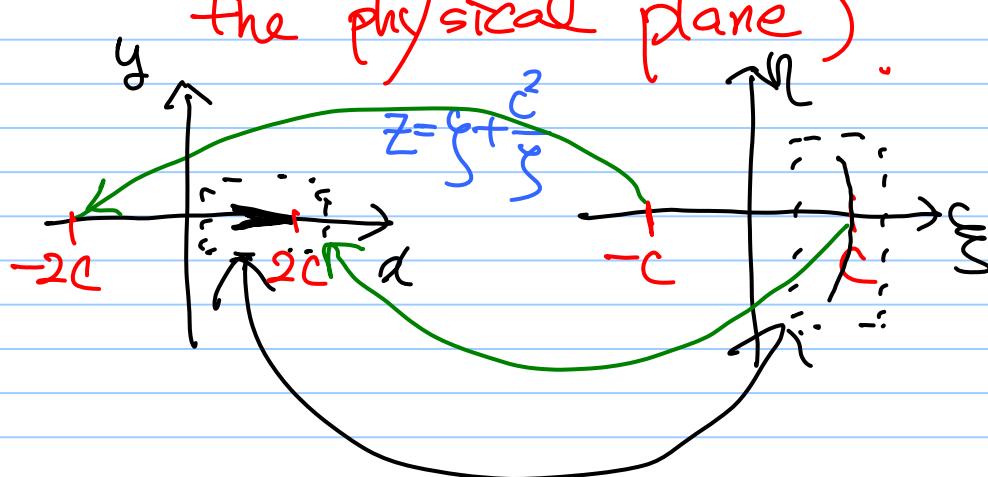
• singular point of transformation.

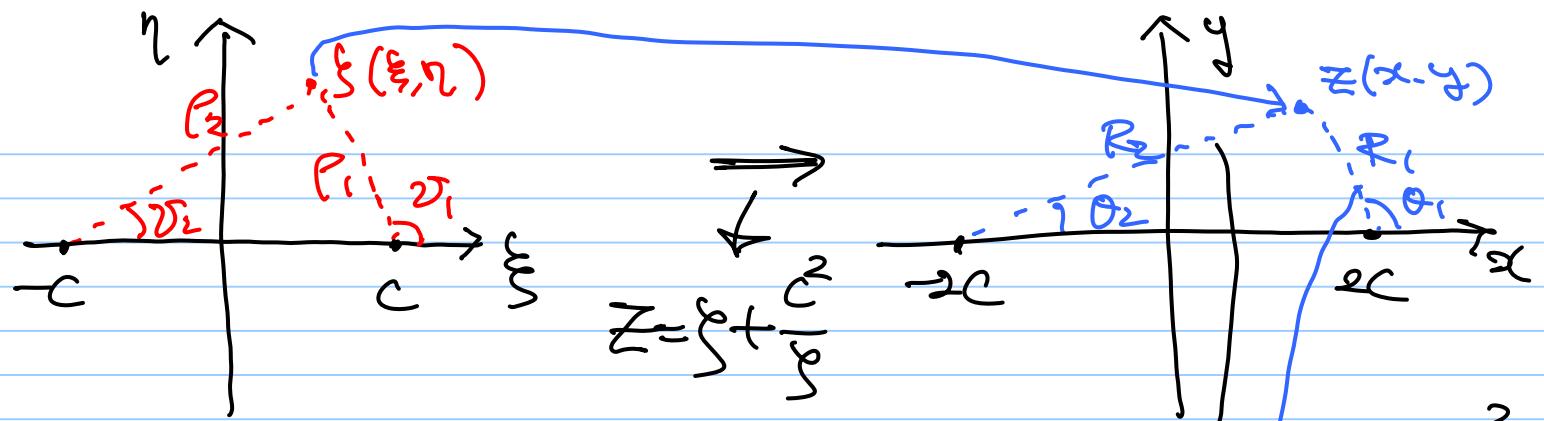
• @ $\zeta = 0$. (normally inside the body, not in the fluid, no significance in general)

- critical point of transformation ($\frac{dz}{d\xi} = 0$)

$$\rightarrow @ \xi = \pm c.$$

(Smooth curves passing through critical points may become sharp corners in the physical plane).



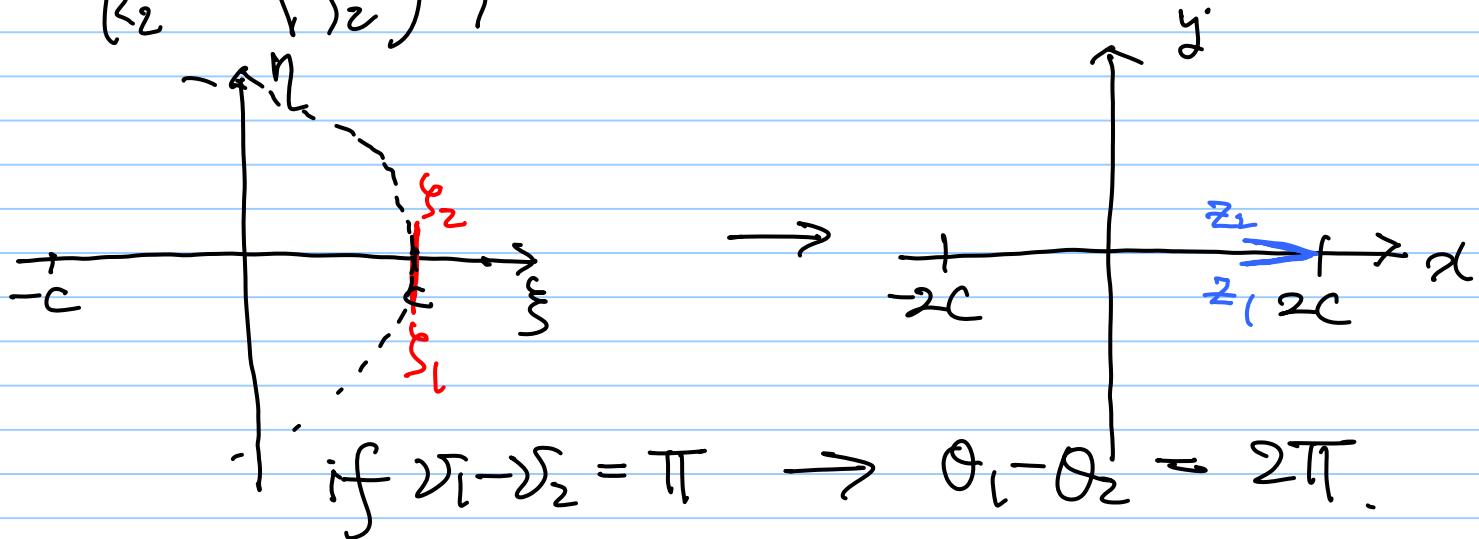


$$z = \xi + \frac{c^2}{\xi}$$

$$\begin{aligned} z+2c &= \frac{(\xi+c)^2}{\xi^2} \\ z-2c &= \frac{(\xi-c)^2}{\xi^2} \end{aligned}$$

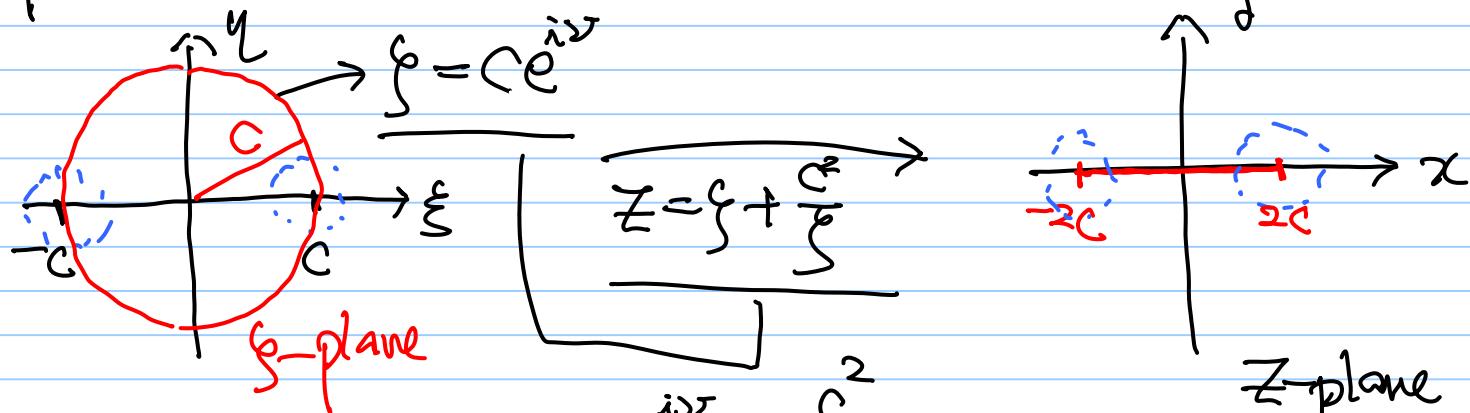
$$\frac{z-2c}{z+2c} = \left(\frac{\xi-c}{\xi+c} \right)^2 \rightarrow \frac{R_1 e^{i \theta_1}}{R_2 e^{i \theta_2}} = \left(\frac{R_1 e^{i \omega_1}}{R_2 e^{i \omega_2}} \right)^2$$

$$\Rightarrow \frac{R_1}{R_2} = \left(\frac{\rho_1}{\rho_2} \right)^2, \quad \theta_1 - \theta_2 = 2(\varphi_1 - \varphi_2)$$



$$z = \xi + \frac{c^2}{\xi}$$

e.g) circle centered at the origin, radius c , in ξ -plane



$$z = ce^{i\varphi} + \frac{c^2}{ce^{i\varphi}}$$

$$= ce^{i\varphi} + c e^{-i\varphi} = 2c \cdot \cos \varphi = x + iy$$

$$x = 2c \cdot \cos \varphi, \quad y = 0.$$

① Flow around ellipse. (in z -plane).

- circle in ξ -plane, radius 'a' ($> c$) $\xrightarrow{\xi = \xi + \frac{c^2}{\xi}}$? (in z -plane)

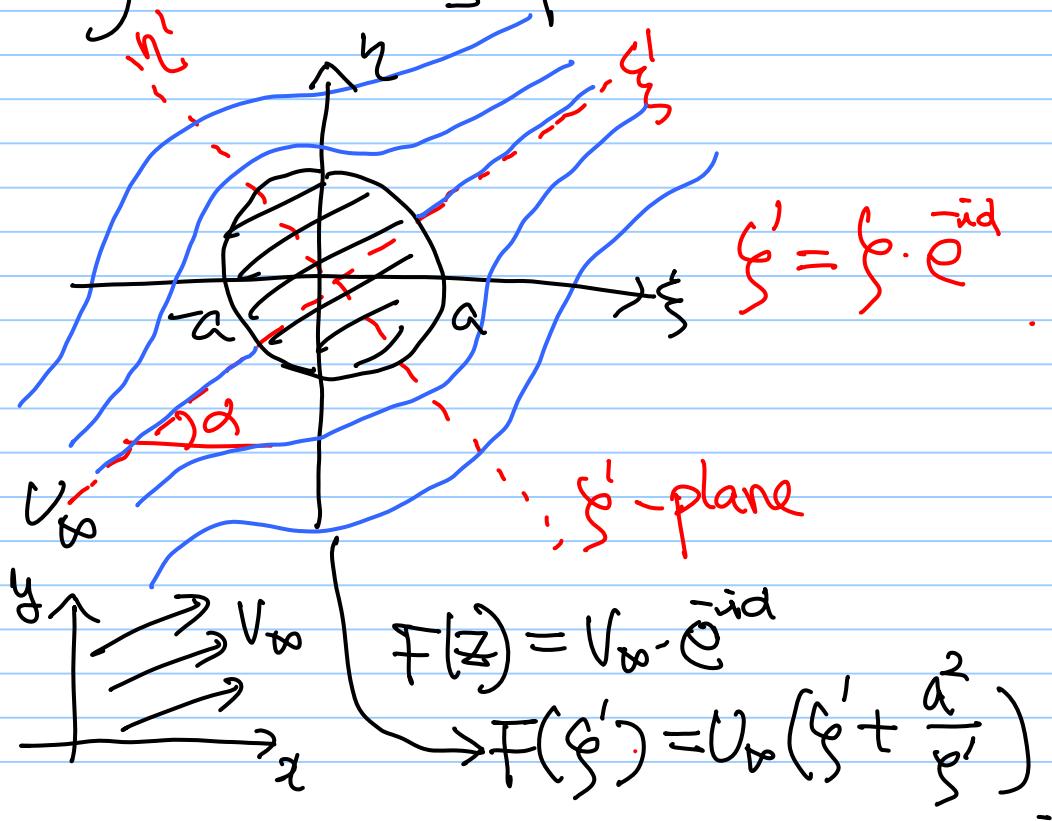
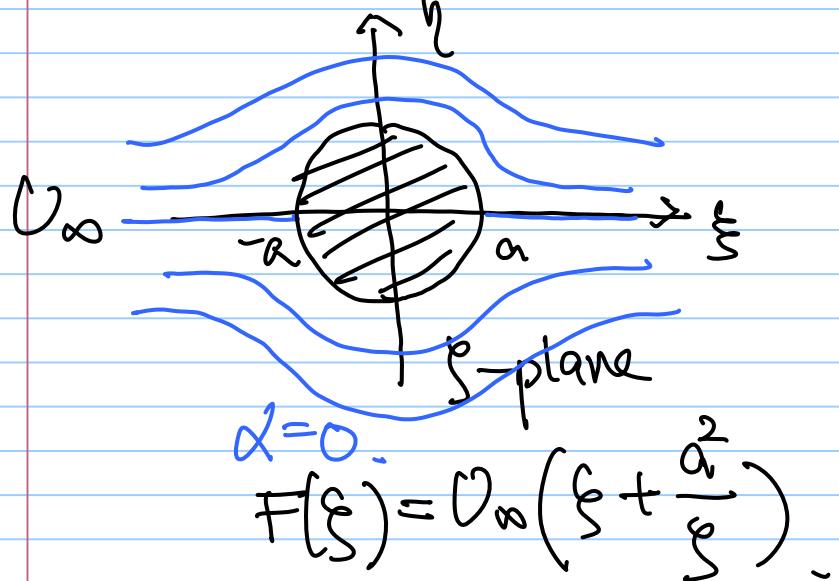
$$\xi = ae^{i\omega t}$$

$$\rightarrow z = ae^{i\omega t} + \frac{c^2}{a} e^{-i\omega t} = \underbrace{\left(a + \frac{c^2}{a}\right) \cos \omega t}_{= x} + i \underbrace{\left(a - \frac{c^2}{a}\right) \sin \omega t}_{= y}.$$

$$\rightarrow \left(\frac{x}{a + c^2/a}\right)^2 + \left(\frac{y}{a - c^2/a}\right)^2 = 1. \text{ : ellipse in } z\text{-plane.}$$

- Joukowski Tr : as $\xi \rightarrow \infty$, $\omega(z) = \omega(\xi)$.

- Uniform flow (U_∞) approaching this ellipse (in ζ -plane)
at the angle of attack (α) \rightarrow same flow to approach the circular cylinder in ξ -plane.



$$\rightarrow F(\xi) = U_\infty \left(\xi \vec{e}^{id} + \frac{\alpha^2}{\xi} \vec{e}^{-id} \right)$$

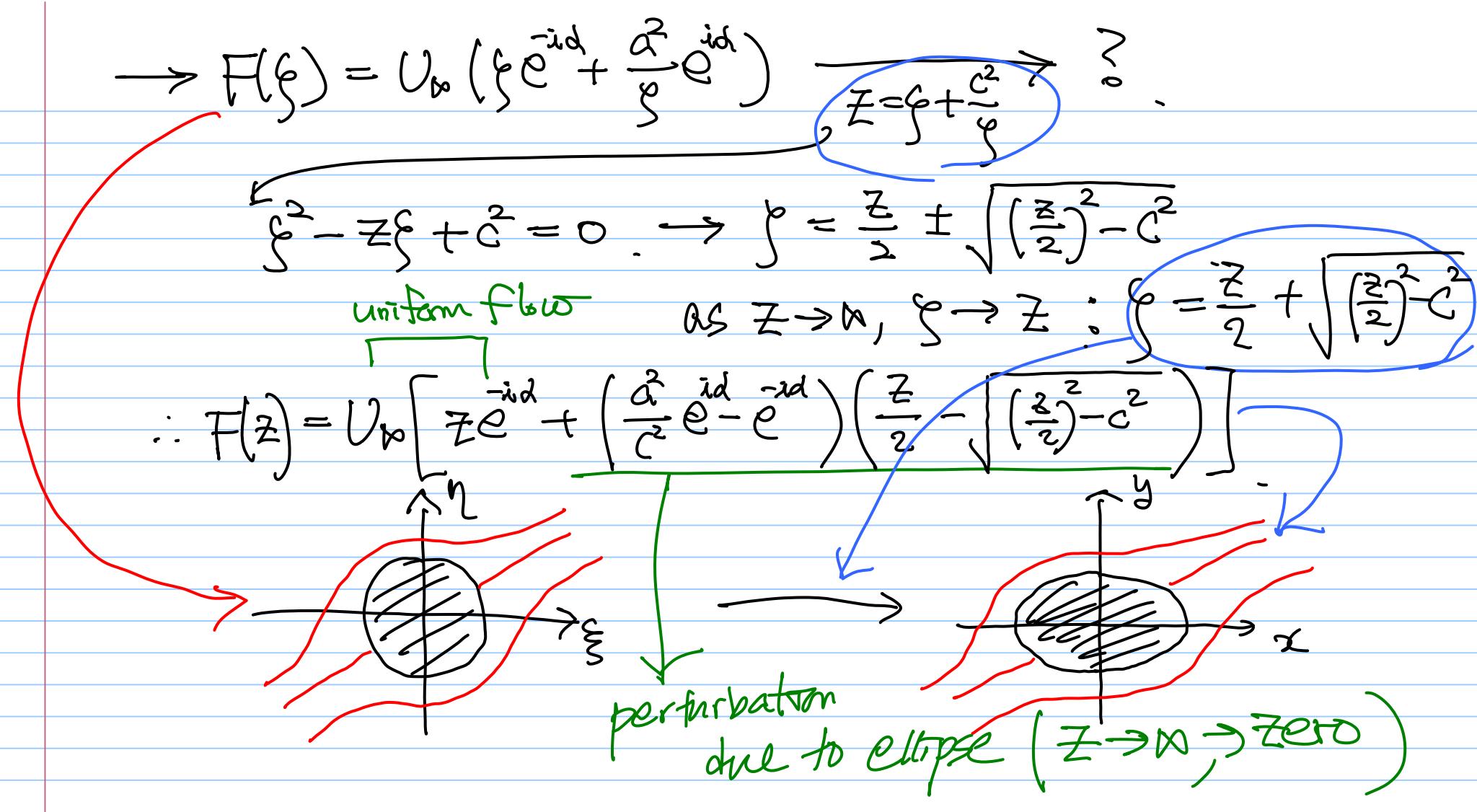
$$\xi = \xi_0 + \frac{c^2}{\xi_0}$$

$$\xi^2 - z\xi + c^2 = 0 \rightarrow \xi = \frac{z}{2} \pm \sqrt{\left(\frac{z}{2}\right)^2 - c^2}$$

uniform flow

$$\therefore F(z) = U_\infty \left[z \vec{e}^{-id} + \left(\frac{\alpha^2}{c^2} \vec{e}^{id} - \vec{e}^{-id} \right) \left(\frac{z}{2} - \sqrt{\left(\frac{z}{2}\right)^2 - c^2} \right) \right]$$

$$\text{as } z \rightarrow \infty, \xi \rightarrow z \therefore \xi = \frac{z}{2} + \sqrt{\left(\frac{z}{2}\right)^2 - c^2}$$



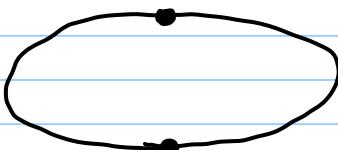
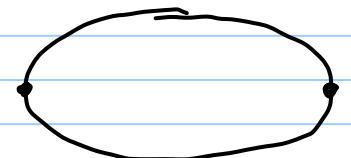
- stagn. point.

In ζ -plane, $\zeta = \pm ae^{id}$ $\xrightarrow{\text{corresponding pts}}$
 in z -plane.

$$z = \zeta + \frac{c^2}{\zeta}$$

$$z = \pm ae^{id} \pm \frac{c^2}{a} e^{-id} = \pm \left(a + \frac{c^2}{a} \right) \cos d \pm i \left(a - \frac{c^2}{a} \right) \sin d$$

$$\delta = 0$$



$$\uparrow \delta = \frac{\pi}{2}$$

How about



\rightarrow Modified Joukowski Transformation.

$$z = \zeta + \frac{c^2}{\zeta}, \quad c = ib. \quad (b: \text{real} \#)$$

$$\zeta = ae^{iz} \rightarrow \left(\frac{x}{a-b^2/a} \right)^2 + \left(\frac{y}{a+b^2/a} \right)^2 = 1 \quad \text{in } z\text{-plane.}$$

then, $F(z) = V_\infty \left[z - \left(1 + \frac{a^2}{b^2} \right) \left(\frac{z}{2} - \sqrt{\left(\frac{z}{2} \right)^2 + b^2} \right) \right]$

② kutta Condition and flat-plate airfoil.

- potential flow solution around a sharp edge.
 → singularity ($U \rightarrow \infty$) at the edge itself.
 (physically X).
 ↓ Correction!

① place a separation point at the edge
 ($V = \text{finite at the edge}$)

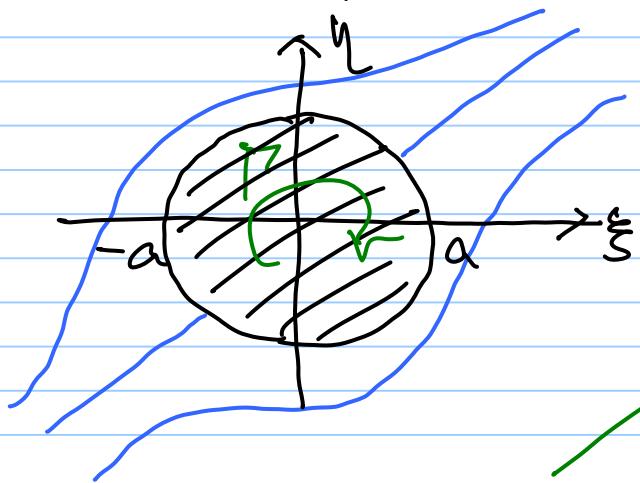
↳ free-streamline (be back later)

② place a stag. point at the edge.
($V = 0$ at the edge)

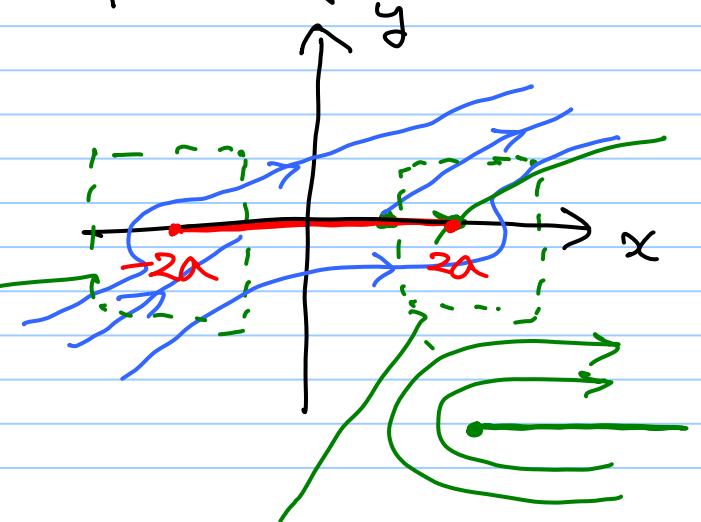
↳ Kutta Condition

- Let $c \rightarrow a$ in the flow around an ellipse.

⇒ ellipse shrinks to a flat plate ($-2a \leq x \leq 2a$)



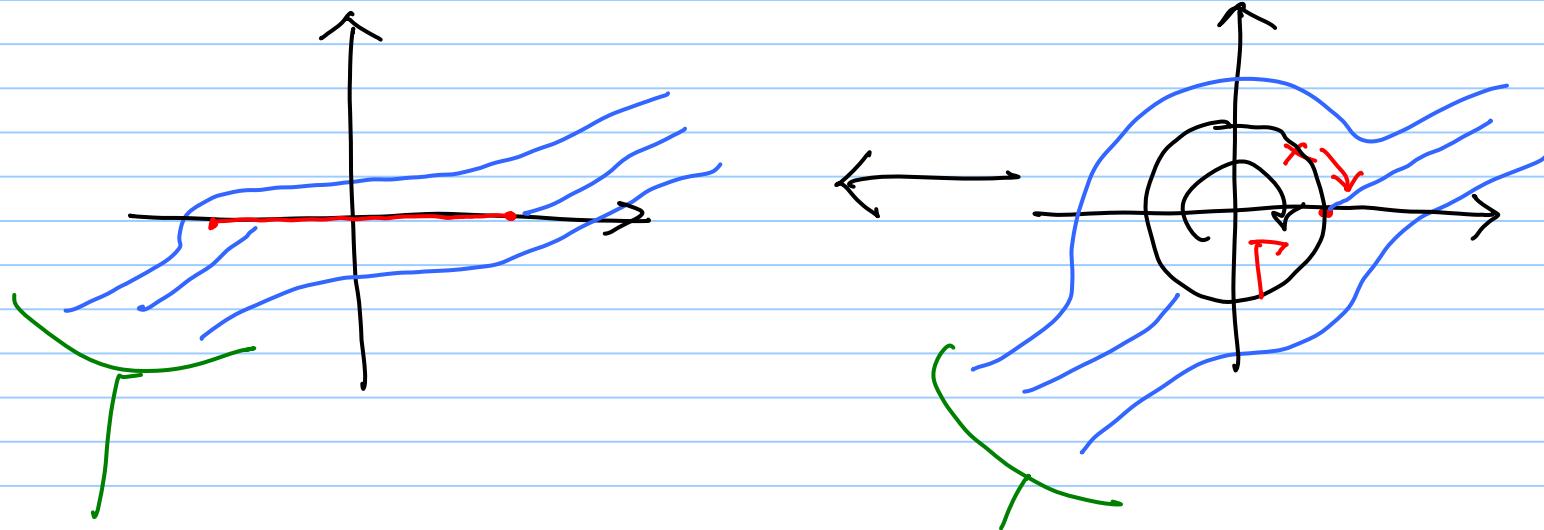
$$z = \xi + \frac{a^2}{\xi}$$



Leading edge : real airfoil has a thick (finite curvature)
→ OK.

Trailing edge : normally sharp.

⇒ place Γ' to move rear stag. pt
to the trailing edge → kutta Condition



Recall, $\sin \theta_s = -\frac{\Gamma}{4\pi U_\infty a}$. $\Rightarrow \Gamma = 4\pi U_\infty a \sin \alpha$
 (clockwise direction)

in ξ -plane, $F(\xi) = U_\infty \left[\xi e^{i\alpha} + \frac{a^2}{\xi} e^{-i\alpha} \right] + \lambda \left(2U_\infty a \sin \alpha \cdot \ln \frac{\xi}{a} \right)$.

$$z = \xi + \frac{a^2}{\xi}$$

$$F(z)$$

Kutta Condition

$$\Gamma = 4\pi U_\infty a \cdot \sin \alpha$$

\Rightarrow Kutta-Joukowski Law.

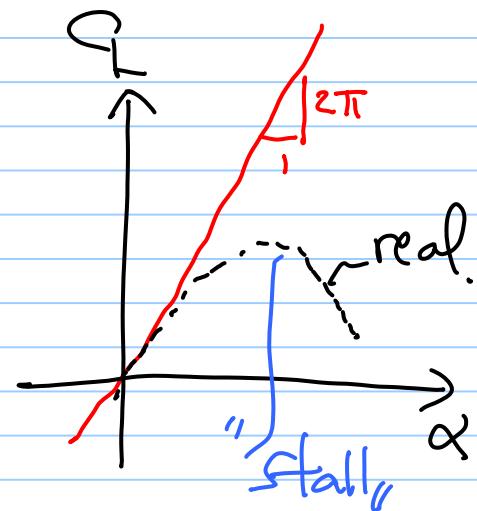
$$L = \rho U_\infty \Gamma = \rho U_\infty \cdot 4\pi C_{l0} a \cdot \sin \alpha$$

$$= 4\pi \rho U_\infty^2 a \cdot \sin \alpha$$

$$C_L = \frac{L}{\frac{1}{2} \rho U_\infty^2 L} = 2\pi \cdot \sin \alpha \sim 2\pi \alpha$$

chord length ($= 4a$)

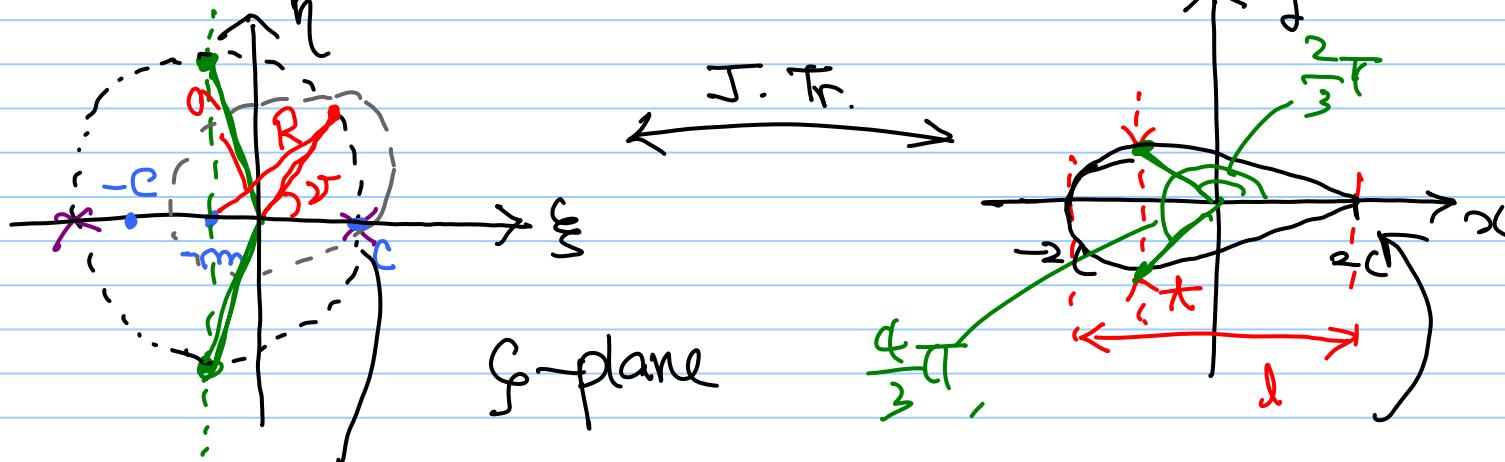
$$C_L \sim 2\pi \alpha \quad (\alpha \ll 1)$$



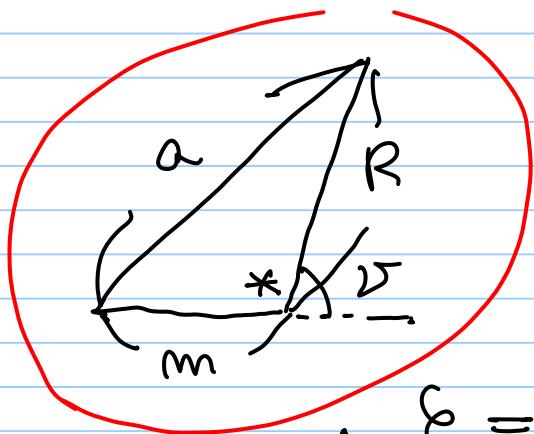
(B) Symmetrical Joukowsky Airfoil.

- family of airfoils obtained by Joukowsky
 Tr. from a series of circles in ξ -plane,
 whose centers are slightly displaced
 from the origin.

- Displacement of the circle in real axis



- C^k : sharp T.E.
- $-C$: inside the circle \rightarrow to have a finite curvature (L.E.)



$$\alpha = C + m = C \left(1 + \frac{m}{C}\right) \text{ if } \varepsilon \leq \frac{m}{C} \quad (\ll 1)$$

$$= C(1 + \varepsilon)$$

$$\begin{cases} \xi = C \\ \xi = -C - 2m \end{cases}$$

$$z = \xi + \frac{c^2}{\xi}$$

$$z = C + \frac{c^2}{C} = 2C : \text{IE}$$

$$z = -(C + 2m) - \frac{c^2}{C + 2m} \quad (m < 1)$$

$$\approx -2C(\text{LE})$$

\rightarrow Chord length,
 $l = 4C$.

(not affected by the
shift of circle)



$$a^2 = R^2 + m^2 - 2Rm \cos(\pi - 2\varphi)$$

($R \geq c$)

$$= R^2 + m^2 + 2Rm \cdot \cos \varphi.$$

$$(c+m)^2 = R^2 \left(1 + \frac{m^2}{R^2} + 2 \cdot \frac{m}{R} \cos \varphi \right), \quad \left(\varepsilon = \frac{m}{c} \geq \frac{m}{R} \right)$$

$$\Rightarrow c+m = R \left(1 + 2 \cdot \frac{m}{R} \cos \varphi \right)^{1/2} = R \left(1 + \frac{m}{R} \cos \varphi + O(\varepsilon^2) \right)$$

Taylor series exp.

$$\therefore \underline{R = c [1 + \varepsilon (1 - \cos \varphi)]}$$

$$f = Re^{i\omega t} \longrightarrow z = f + \frac{\alpha^2}{f}$$

$$= C [1 + \varepsilon(1 - \cos \omega t)] e^{i\omega t} + \boxed{\frac{C e^{-i\omega t}}{1 + \varepsilon(1 - \cos \omega t)}}$$

$$= C [1 + \varepsilon(1 - \cos \omega t)] e^{i\omega t} + C [1 - \varepsilon(1 - \cos \omega t) + \theta(\varepsilon)] e^{-i\omega t}$$

$$= C \left[\underbrace{2 \cos \omega t}_x + i \underbrace{2\varepsilon(1 - \cos \omega t) \sin \omega t}_y \right]$$

$$\therefore y = \pm 2C\varepsilon \left(1 - \frac{x}{2C}\right) \sqrt{1 - \left(\frac{x}{2C}\right)^2} \quad \because \text{profile of the symmetric airfoil.}$$

• max. thickness \rightarrow @ $\frac{dy}{d\gamma} = 0 \rightarrow \gamma = \frac{2}{3}\pi, \frac{4}{3}\pi$.

$$y = \pm \frac{3\sqrt{3}}{2} \epsilon c, t_{\max} = 3\sqrt{3} \epsilon c$$

$$\cdot \frac{t_{\max}}{l} = \frac{3\sqrt{3} \epsilon c}{4c} = \frac{3\sqrt{3}}{4} \epsilon$$

$$\text{or } \epsilon = 0.07 \frac{t_{\max}}{l}$$

\rightarrow then, eq. for airfoil becomes

$$\frac{y}{t_{\max}} = \pm 0.385 \left(1 - 2 \frac{x}{l} \right) \sqrt{1 - \left(2 \frac{x}{l} \right)^2}$$

• circulation for kutta condition.

$$\Gamma = 4\pi U_\infty a \sin \alpha = \pi U_\infty l \left(1 + 0.07 \frac{t_{\max}}{l} \right) \sin \alpha$$

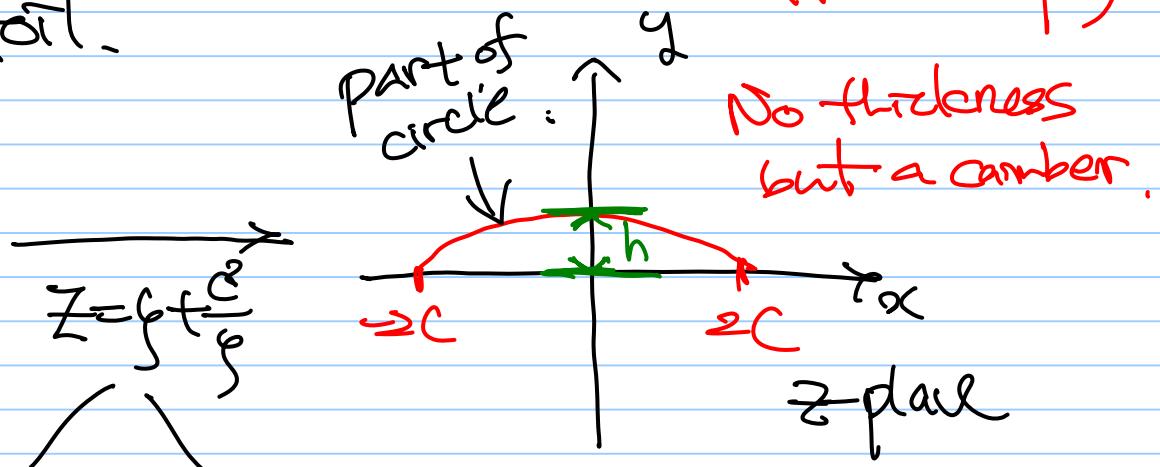
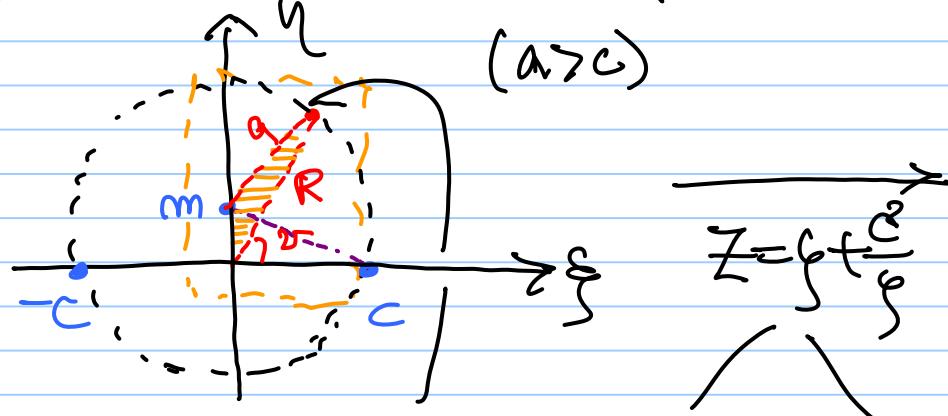
$$\rightarrow \left\{ \begin{array}{l} L = \rho U_\infty^2 = \pi \rho U_\infty^2 l \left(1 + 0.77 \frac{t_{max}}{l} \right) \sin \delta \\ C_L = 2\pi \left(1 + 0.77 \frac{t_{max}}{l} \right) \sin \delta \end{array} \right.$$

i) as $t \rightarrow 0$, $C_L \approx 2\pi \sin \delta$.

ii) as $t_{max} \uparrow$, $\rightarrow C_L \uparrow$. practically good? No!

(flow sep.)

⑭ circular arc airfoil.



$$f = Re^{i\omega t}$$

$$z = Re^{i\omega t} + \frac{c^2}{R} e^{-i\omega t}$$

$$= \underbrace{\left(R + \frac{c^2}{R} \right) \cos \omega t}_{x} + i \underbrace{\left(R - \frac{c^2}{R} \right) \sin \omega t}_{y}$$

→ eliminate R .

$$: x^2 \sin^2 \omega t - y^2 \cos^2 \omega t = 4c^2 \sin^2 \omega t \cdot \cos^2 \omega t.$$

$$m^2 + c^2 = a^2 = R^2 + m^2 - 2 \cdot Rm \cos\left(\frac{\pi}{2} - \omega t\right)$$

$$\Rightarrow x^2 + y^2 + 2 \left(\frac{c^2}{m} - m \right) y = 4c^2$$

$$\rightarrow x^2 + \left[y + c \left(\frac{c}{m} - \frac{m}{c} \right) \right]^2 = c^2 \left[4 + \left(\frac{c}{m} - \frac{m}{c} \right)^2 \right]$$

using $\epsilon = \alpha y/c \ll 1$.

$$x^2 + \left(y + \frac{c^2}{m}\right)^2 = c^2 \left(1 + \frac{c^2}{m^2}\right)$$

- chord length, $l = 4c$.
- camber height, $h = 2m$, ($x=0, \rightarrow y=2m$)
- Kutta Condition, $T = 4\pi U_\infty C \sin(\alpha + \frac{m}{c})$

$$\begin{aligned} L &= 4\pi \rho U_\infty^2 C \cdot \sin(\alpha + \frac{m}{c}) \\ C_L &= 2\pi \sin(\alpha + \frac{2h}{l}) \end{aligned}$$

- increase the effective angle of attack.
- i) @ $\alpha = 0$, $C_L \neq 0$.
 - ii) as $h \uparrow$, $C_L \uparrow \rightarrow$ flow separation.