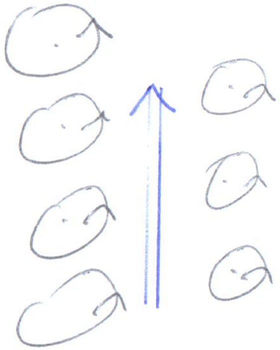
Bootstrap Current



Diamagnetic Current.

Bootstrap Current.

(*) "Non-uniform pressure" and "finite Larmor radius size" ⟹ Diamagnetic (drift) and Current.

(*) "Non-uniform pressure" and "finite banana width size" ⟹ "Bootstrap Current along $\vec{B}$."

⊛ Heuristic Estimation :

$$j_{b_e} = e\, v_{\parallel} \left(- \frac{d}{dr} n_{trapped} \right) \Delta_{banana} \simeq e \left(\sqrt{\epsilon}\, v_{Th,e} \right) (-) \left(\sqrt{\epsilon} \frac{\partial n}{\partial r} \right)$$

$$* \quad \frac{\sqrt{\epsilon}\, v_{Th,e}}{(eB_\theta / m_e)} \sim - \frac{\epsilon^{3/2}}{B_\theta} \frac{\partial P}{\partial r}.$$

This turns out to be an underestimation.

⊛ More systematic derivation (Ref: Miyamoto's book: pg 224 –)

* Electron Momentum Eqn :

$$m_e \frac{d}{dt} (n_e u_{\parallel}) = - n_e e B_\theta u_r + R_{ei}$$

where $R_{ei} \simeq - n_e m_e \nu_{ei} u_{\parallel}$

and

$$n_e u_r = - D_{Banana} \frac{\partial n_e}{\partial r}$$ in banana collisionality regime.

(*) Recall ; $D_{Banana} \simeq \underline{\sqrt{\epsilon}} \underline{\left(\frac{\nu_{ei}}{\epsilon}\right)} \underline{\left(\frac{q}{\sqrt{\epsilon}} \rho_e\right)^2} \sim \frac{q^2}{\epsilon^{3/2}} \nu_{ei} \rho_e^2$

fraction of trapped ptls

effective coll. freq.

banana orbit width.

$\therefore$ By ignoring electron inertia term and balancing ~~the~~ two terms on the RHS of electron momentum eqn,

we obtain,

$$\text{``}\ \underline{j_b} = n_e e u_\parallel \sim \underline{-\frac{\sqrt{\epsilon}}{B_\theta} \frac{\partial P_e}{\partial r}}\ \text{''}$$

* Note that $j_b \to 0$ as $r \to 0$.

(*) Bootstrap current does not rely on the induction electric field and is an essential part of the steady state (non-inductive) operation of tokamaks.

(*) Since $j_b \to 0$, its radial profile tends to be hollow and leads to non-monotonic (reversed) q profile $\Rightarrow$ It strongly affects stability of many possible instabilities.

Ch.13. The Fokker-Planck equation for Coulomb collisions

* The Fokker-Planck equation describes the evolution in time due to collisions of the function $f(\vec{v})$.

* Collisional effects depend only on the local properties $\rightarrow$ $f(\cancel{\vec{x}}, \vec{v})$.

ignored for present purposes.

* Define: $\phi(\vec{v}, \Delta\vec{v})$: the probability that a ptl with $\vec{v}$ acquires an increment of velocity $\Delta\vec{v}$ in a time interval Δt.

Then,

$$\underbrace{f(\vec{v}, t) = \int f(\vec{v}-\Delta\vec{v}, t-\Delta t)\,\phi(\vec{v}-\Delta\vec{v}, \Delta\vec{v})\,d^3\Delta v}_{\text{by definition}} \qquad (13.2)$$

and since sum of the probabilities of all possible $\Delta\vec{v}$ must be 1,

$$\int \phi(\vec{v}, \Delta\vec{v})\,d^3\Delta v = 1 \qquad (13.3).$$

⊛ Effects of Coulomb collisions can be described in terms of a sequence of <u>small-angle deflections</u>

⇒ We can expand $f\phi$ of equation (13.2) in powers of $\Delta\vec{v}$ (compared to $\vec{v}$).

- $$f(\vec{v}-\Delta\vec{v}, t-\Delta t) = f(\vec{v}, t-\Delta t) - \Delta\vec{v}\cdot\frac{\partial}{\partial\vec{v}} f(\vec{v}, t-\Delta t) + \frac{1}{2}\Delta\vec{v}\Delta\vec{v} : \frac{\partial^2}{\partial\vec{v}\partial\vec{v}} f(\vec{v}, t-\Delta t)$$

and

- $$\phi(\vec{v}-\Delta\vec{v}, \Delta\vec{v}) = \phi(\vec{v}, \Delta\vec{v}) - \Delta\vec{v}\cdot\frac{\partial}{\partial\vec{v}}\phi(\vec{v}, \Delta\vec{v}) + \frac{1}{2}\Delta\vec{v}\Delta\vec{v} : \frac{\partial^2}{\partial\vec{v}\partial\vec{v}}\phi(\vec{v}, \Delta\vec{v}).$$

— Note a dyadic form notation, for instance, eg.,

$$\Delta\vec{v}\Delta\vec{v} : \frac{\partial^2 f}{\partial\vec{v}\partial\vec{v}} = \left(\sum_{i,j}\right)\Delta v_i \Delta v_j \frac{\partial^2 f}{\partial v_i \partial v_j}$$; summation convention

— Then Eqs (13.2) and (13.3) for small Δt, up to the 2nd order in $\Delta\vec{v}$ yields →

$$\oplus \quad f(\vec{v},t) - f(\vec{v}, t-\Delta t) = - \int \Delta\vec{v} \cdot \left(\frac{\partial f}{\partial \vec{v}} \phi + \frac{\partial \phi}{\partial \vec{v}} f \right) d^3\Delta v$$

$$+ \frac{1}{2} \int \Delta\vec{v}\,\Delta\vec{v} : \left(\frac{\partial^2 f}{\partial\vec{v}\partial\vec{v}} \phi + 2 \frac{\partial f}{\partial \vec{v}} \frac{\partial \phi}{\partial \vec{v}} + \frac{\partial^2 \phi}{\partial\vec{v}\partial\vec{v}} f \right) d^3\Delta v$$

$$= - \frac{\partial}{\partial \vec{v}} \cdot \int f\phi\, \Delta\vec{v}\, d^3\Delta v + \frac{1}{2} \frac{\partial^2}{\partial\vec{v}\partial\vec{v}} : \int f\phi\, \Delta\vec{v}\,\Delta\vec{v}\, d^3\Delta v \qquad (13.4)$$

Then,

$$\circledast \quad \left(\frac{\partial f}{\partial t} \right)_{\text{coll.}} \equiv \frac{f(\vec{v},t) - f(\vec{v}, t-\Delta t)}{\Delta t}$$

$$= - \frac{\partial}{\partial \vec{v}} \cdot \left(\frac{d\langle\Delta\vec{v}\rangle}{dt} f \right) + \frac{1}{2} \frac{\partial^2}{\partial\vec{v}\partial\vec{v}} : \left(\frac{d\langle\Delta\vec{v}\,\Delta\vec{v}\rangle}{dt} f \right) \qquad (13.5)$$

since $f = f(\vec{v},t)$ is independent of $\Delta\vec{v}$, where

$$\frac{d\langle\Delta\vec{v}\rangle}{dt} \equiv \frac{1}{\Delta t} \int \phi\, \Delta\vec{v}\, d^3\Delta v \quad \text{and} \quad \frac{d}{dt}\langle\Delta\vec{v}\,\Delta\vec{v}\rangle \equiv \frac{1}{\Delta t} \int \phi\, \Delta\vec{v}\,\Delta\vec{v}\, d^3\Delta v \qquad (13.6)$$

; Dynamic friction term in the direction opposite to $\vec{v}$

; Velocity diffusion coefficient: describing spreading of ptl velocities over a wider region in $\vec{v}$-space.

Fokker-Planck Equation for Electron-Ion Collisions

* Consider electrons colliding with much heavier ions $\left(\frac{m}{M} \ll 1\right)$,

$$\boxed{\frac{d}{dt}\langle \Delta \vec{v} \rangle = -\frac{n_i Z^2 e^4 \ln \Lambda}{4\pi \epsilon_0^2 m^2 v^3} \vec{v}} \qquad (13.7) \quad \text{from } (11.14)$$

* For $\vec{v} = v\hat{z}$ initially for definiteness,

$$\frac{d}{dt}\langle \Delta v_x^2 \rangle = \frac{d}{dt}\langle \Delta v_y^2 \rangle = \frac{1}{2}\frac{d}{dt}\langle \Delta v_\perp^2 \rangle \qquad (13.8)$$

* Since there's no preferred direction for Δv_x and Δv_y

$$\frac{d}{dt}\langle \Delta v_x \Delta v_z \rangle = \frac{d}{dt}\langle \Delta v_y \Delta v_z \rangle = 0 \quad \text{and} \quad \frac{d}{dt}\langle \Delta v_x \Delta v_y \rangle = 0,$$

$\frac{d}{dt}\langle \Delta v_z^2 \rangle$ is negligible because $\Delta v_z \sim \frac{1}{2}\frac{(\Delta v_\perp)^2}{v}$.

⇒ Using Eq (11, 11)

$$\boxed{\frac{d}{dt}\langle \Delta \vec{v} \Delta \vec{v} \rangle = -\frac{n_i Z^2 e^4 \ln \Lambda}{4\pi \epsilon_0^2 m^2 v^3}\left(\overleftrightarrow{I} v^2 - \vec{v}\vec{v}\right)} \qquad (13.9)$$

(*)
$$\boxed{\left(\frac{\partial f_e}{\partial t}\right)_{coll.} = \frac{n_i Z^2 e^4 \ln\Lambda}{8\pi \epsilon_0^2 m^2} \frac{\partial}{\partial \vec{v}} \cdot \left(\frac{\overleftrightarrow{I} v^2 - \vec{v}\vec{v}}{v^3} \cdot \frac{\partial f_e}{\partial \vec{v}} \right)} \qquad (13.12)$$

Fokker-Planck Eqn describing the evolution of electron distn function due to collisions with fixed infinitely massive ions.

Here, an identity
$$\frac{\partial}{\partial \vec{v}} \cdot \left(\frac{\overleftrightarrow{I} v^2 - \vec{v}\vec{v}}{v^3} \right) = -\frac{2\vec{v}}{v^3} \qquad (13.11)$$
has been used. This can be verified using the index notation as shown in the text book.

(*) For electron-electron or ion-ion collisions, Eq (13.12) is no longer applicable. However, the general structure of FP-equation consisting of ✓dynamical friction and ✓velocity diffusion (as in Eq (13.5)) is preserved.

"Lorentz-Gas" Approximation

(*) Eq (13.12) is a good approximation for $Z \gg 1$ because $\nu_{ei} \gg \nu_{ee}$ in that case.

✓ In this approximation, the electron speed does not change

$\Rightarrow$ $\left(\frac{\partial f_e}{\partial t}\right)_{coll.} = 0$ for any f_e which is isotropic in $\vec{v}$

✓ In general $\left(\frac{\partial f_e}{\partial t}\right)_{coll} = 0$ for Maxwellian f_e for any valid collision operator, ∵ " implies

Thermodynamic equilibrium among the ptls.

(*) In spherical coordinates, $v_z = v\cos\theta$, $v_x = v\sin\theta\cos\phi$, $v_y = v\sin\theta\sin\phi$.

$$\Rightarrow \boxed{\left(\frac{\partial f_e}{\partial t}\right)_{coll.} = \frac{n_i Z^2 e^4 \ln\Lambda}{8\pi\epsilon_0^2 m^2 v^3}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial f_e}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} f_e\right]} \quad (13.14)$$

Plasma Resistivity In the Lorentz-Gas Approx.

A small electric field $\vec{E}$ will cause electrons to accelerate at a rate $-\frac{e}{m}\vec{E}$, so we have the following relation;

$$f_e(\vec{v}, t) = f_e(\vec{v} + e\vec{E}\Delta t/m, t - \Delta t) \qquad (13.16).$$

$\therefore$ The time rate of change in f_e due to $\vec{E}$ is given by

$$\lim_{\Delta t \to 0} \frac{f_e(\vec{v}, t) - f_e(\vec{v}, t - \Delta t)}{\Delta t} = \frac{e\vec{E}}{m} \cdot \frac{\partial f_{eo}}{\partial \vec{v}} \equiv \left(\frac{\partial f_e}{\partial t}\right)_{\vec{E}} \qquad (13.17).$$

In steady state, this change should be balanced by the collisional drag from the ions, i.e,

$$-\frac{e}{m}\vec{E} \cdot \frac{\partial f_{eo}}{\partial \vec{v}} = \left(\frac{\partial f_e}{\partial t}\right)_{coll.} = \left(\frac{\partial f_{e1}}{\partial t}\right)_{coll}, \qquad (13.19)$$

Here, we expanded $f_e(\vec{v})$ around $f_{eo}(\vec{v})$ = Maxwellian, i.e,

$f_e(\vec{v}) = f_{eoM}(\vec{v}) + f_{e1}(\vec{v})$ and used the fact that $\left(\frac{\partial f_{eoM}}{\partial t}\right)_{coll} = 0$.

* Taking $\vec{E} = E\hat{z}$, f_{e1} will be symmetric w.r.t. the azimuthal velocity angle about z-axis, (ϕ is an ignorable coordinate),

∴ Only the 1st term on the RHS of (13.14) contributes from $\left(\frac{\partial f_{e1}}{\partial t}\right)_{coll.}$ in Lorentz-gas approximation:

$$\frac{eEv f_{e0}}{T_e}\cos\theta = \frac{n_i Z^2 e^4 \ln\Lambda}{8\pi\epsilon_0^2 m^2 v^3}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial f_{e1}}{\partial\theta} \quad (13.20).$$

$$\Rightarrow f_{e1} = -\frac{4\pi\epsilon_0^2 m^2 E v^4 f_{e0}}{n_i Z^2 e^3 T_e \ln\Lambda}\cos\theta \quad (13.21).$$

Distortion of f_e from Maxwellian due to E, constrained by collisions.

* Then, it's straight forward to calculate the current density in z-direction:

$$j_z = -e\int f_{e1}(v\cos\theta)\, d^3v = (\text{const})\int_0^\infty v^7 f_{e0}\, dv \int_0^\pi \cos^2\theta \sin\theta\, d\theta.$$

$$= \frac{32\pi^{1/2}\epsilon_0^2}{m^{1/2} Z e^2 \ln\Lambda} E\,(2T_e)^{3/2} \quad (13.22)$$

$\therefore$ From $j_z = \eta E$ and Eq. (13.22), we obtain

$$\eta_{Lorentz} = \frac{m^{1/2} Z e^2 \ln \Lambda}{32\, \pi^{1/2} \epsilon_0^2 (2T_e)^{3/2}} \approx \frac{1}{3.4} \eta_{fluid} \quad \text{of Eq. (11.30).}$$

* The lower resistivity arises from the dominant contribution of high-velocity electrons in carrying current in the Lorentz-gas approx.

* If e-e collision is included in the analysis, it turns out

$$\eta_{SP} \simeq 1.7\, \eta_{Lorentz} \simeq \frac{1}{2} \eta_{fluid}$$

[L. Spitzer and R. Harm P. Rev. '53]

Homework

Problem 13.1. on page 224

Problem 13.2. on page 227