

① draw streamlines around the airfoil

② plot  $C_p$

③ calculate  $C_L$  and compare with theory

④ Discuss the effect of panel design.

⑤ 3D POTENTIAL FLOWS (Ch. 5)

- physically, same as 2D flow, except the addition of  
one more dimension,

- methodologically, it is different!  $z = x + iy \rightarrow (x)$

↳ solve PDE (900. eq.)  
Laplace eq.

- Axisymmetric body (practical interest)  
in  $(r, \theta, \omega)$  spherical coord.

\* Velocity potential, and stream function.

$$\nabla^2 \phi = 0 \text{ and } \frac{\partial}{\partial \omega} = 0.$$

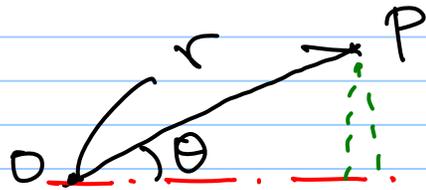
$$\hookrightarrow \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \cdot \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) = 0 \quad \text{--- } \otimes$$

$$\hookrightarrow \phi \rightarrow U_r \equiv \frac{\partial \phi}{\partial r}, \quad U_\theta \equiv \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad U_\omega \equiv 0.$$

In axisymmetric flow, a scalar function

satisfying continuity exists, even in a 3D flow.

$$\rightarrow u_r \equiv \frac{1}{r^2 \sin \theta} \cdot \frac{\partial \psi}{\partial \theta}, \quad u_\theta \equiv -\frac{1}{r \sin \theta} \cdot \frac{\partial \psi}{\partial r} \quad \text{: Stokes Stream fun.}$$



$$\psi = \psi(r, \theta)$$

ref. axis

$\Rightarrow 2\pi \cdot \psi =$  Volume of fluid crossing the surface of revolution formed by rotating the  $\vec{OP}$  around the ref. axis.

$$\psi = \psi_1$$

$$\rightarrow Q = \Delta \psi$$

$$\psi = \psi_2$$

\* Solution of Laplace eq. ( $\nabla^2 \phi = 0$ )

- separation of variables.

$$\phi(r, \theta) = R(r) \cdot T(\theta).$$

$$\therefore (*) \rightarrow \frac{T}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{R}{r^2 \sin \theta} \cdot \frac{d}{d\theta} \left( \sin \theta \cdot \frac{dT}{d\theta} \right) = 0$$

$$\therefore \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = - \frac{1}{T \cdot \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dT}{d\theta} \right) = \text{constant} \\ = l(l+1)$$

$$\cdot \text{For } R(r), \quad \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - l(l+1)R = 0$$

$$\uparrow R(r) = r^\alpha$$

$$\hookrightarrow \alpha = l, -(l+1)$$

$$\therefore R_l(r) = A r^l + \frac{B}{r^{l+1}}$$

$$\cdot \text{For } T(\theta), \quad \frac{1}{\sin \theta} \cdot \frac{d}{d\theta} \left( \sin \theta \frac{dT}{d\theta} \right) + l(l+1)T = 0$$

(Legendre eq.)

$$x = \cos \theta.$$

$$\frac{d}{dx} \left[ (1-x^2) \frac{dT}{dx} \right] + l(l+1)T = 0.$$

$$\rightarrow T_l(\theta) = C_l \cdot P_l(\cos \theta) + D_l \cdot Q_l(\cos \theta)$$

Legendre fn of 1<sup>st</sup> kind.  
 " " 2<sup>nd</sup> " "

practically,  
 we will use the  
 power series form.

diverges for  $\cos \theta = \pm 1$  and  
 for all values of  $l \rightarrow P_l \equiv 0$ .

$$\therefore T_l(\theta) = C_l \cdot P_l(\cos \theta)$$

also diverges for  $\cos \theta = \pm 1$ , when  $l \neq \text{integer}$ .

$$\therefore \phi_l(r, \theta) = \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta), \quad l : \text{integer.}$$

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta) : \text{general solution.}$$

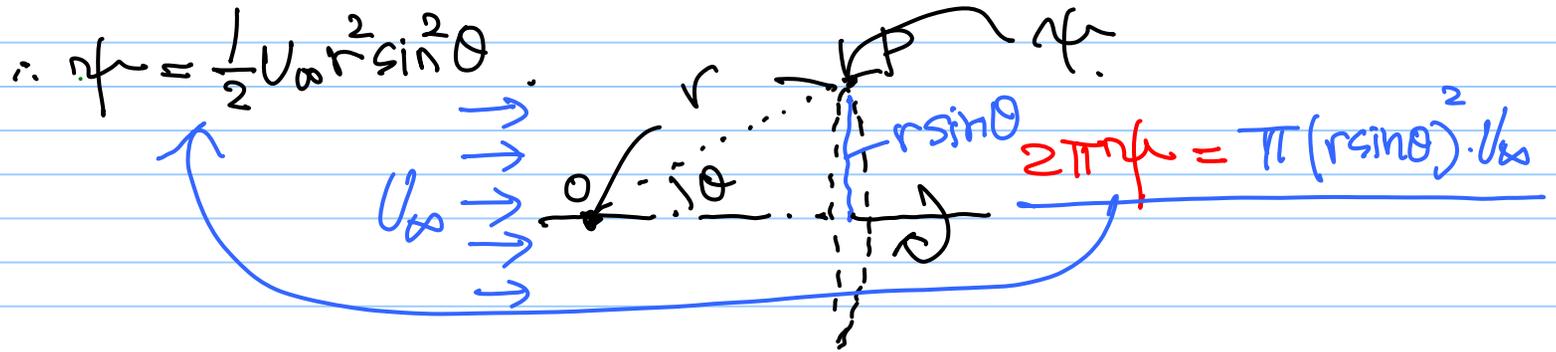
$$\rightarrow P_l(x) = \frac{1}{2^l \cdot l!} \cdot \frac{d^l}{dx^l} (x^2 - 1)^l \rightarrow \begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2} (3x^2 - 1) \\ &\vdots \end{aligned}$$

① Uniform flow.  $\phi(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$   
 $B_l = 0, A_l = \begin{cases} 0 & (l \neq 1) \\ U_{\infty} & (l = 1) \end{cases}$  ( $P_0(x) = 1, P_1(x) = x, \dots$ )

$\Rightarrow \phi(r, \theta) = U_{\infty} r \cos \theta$

$U_r \equiv \frac{\partial \phi}{\partial r} = U_{\infty} \cos \theta = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \rightarrow \psi = \frac{1}{2} U_{\infty}^2 r^2 \sin^2 \theta + f(r)$

$U_{\theta} \equiv \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -U_{\infty} \sin \theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \rightarrow \psi = \frac{1}{2} U_{\infty}^2 r^2 \sin^2 \theta + g(\theta)$



② source/sink

$$A_l = 0, \quad B_l = \begin{cases} 0 & (l \neq 0) \\ B_0 (\neq 0) & (l = 0) \end{cases}$$

$$\rightarrow \phi(r, \theta) = \frac{B_0}{r}$$

$$u_r = -\frac{B_0}{r^2}, \quad u_\theta = 0.$$

let  $Q$ , volume flow rate.

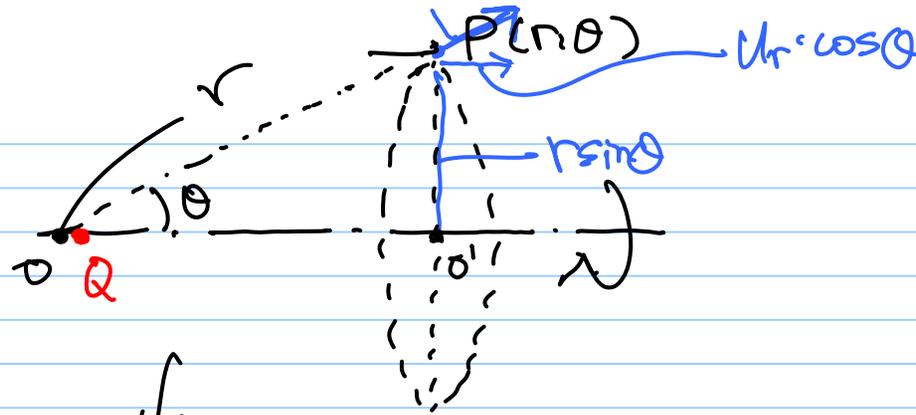
$$Q \equiv \int_S \vec{u} \cdot \vec{n} \, dS = \int_0^{2\pi} d\omega \int_0^\pi \left( \frac{-B_0}{r^2} \right) r^2 \sin\theta \, d\theta = -4\pi B_0$$

$$\therefore B_0 = -\frac{Q}{4\pi}$$

$$\therefore \phi(r, \theta) = -\frac{Q}{4\pi r} \text{ for source } \left( \frac{Q}{4\pi r} \text{ for sink} \right)$$

$\leftarrow u_r$

$\psi?$



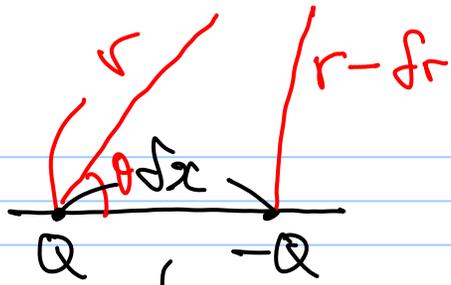
$$2\pi\psi + Q = \int_0^\theta (u_r \cos \theta) \cdot 2\pi (r \sin \theta) \cdot \frac{r d\theta}{\cos \theta}$$

$$\psi = -\frac{Q}{4\pi} (1 + \cos \theta) \quad \star$$

element area about  $OP$ .

⑤ Flow due to a doublet (source & sink)

$P(r, \theta)$   $\phi(r, \theta) = -\frac{Q}{4\pi r} + \frac{Q}{4\pi(r-f)}$



$$= -\frac{Q}{4\pi r} \left( 1 - \frac{1}{1 - dr/r} \right)$$

$$= \frac{Q}{4\pi r} \left( \frac{dr}{r} + \cancel{O\left(\frac{dr^2}{r}\right)} \right)$$

$$(r - dr)^2 = r^2 + dx^2 - 2r \cdot dx \cdot \cos\theta$$

$$\cos\theta = \frac{dr}{dx} \left( 1 + O\left(\frac{dr}{r}\right) \right)$$

$$\text{or, } dr = dx \cdot \cos\theta \left( 1 - O\left(\frac{dr}{r}\right) \right)$$

$$\rightarrow \phi(r, \theta) = \frac{Q}{4\pi r} \left\{ \frac{dx}{r} \cos\theta \left( 1 - \cancel{O\left(\frac{dr}{r}\right)} \right) \right\}$$

$$\text{let } \lim_{dx \rightarrow 0} (Q \cdot dx) = \mu \quad (\neq 0)$$

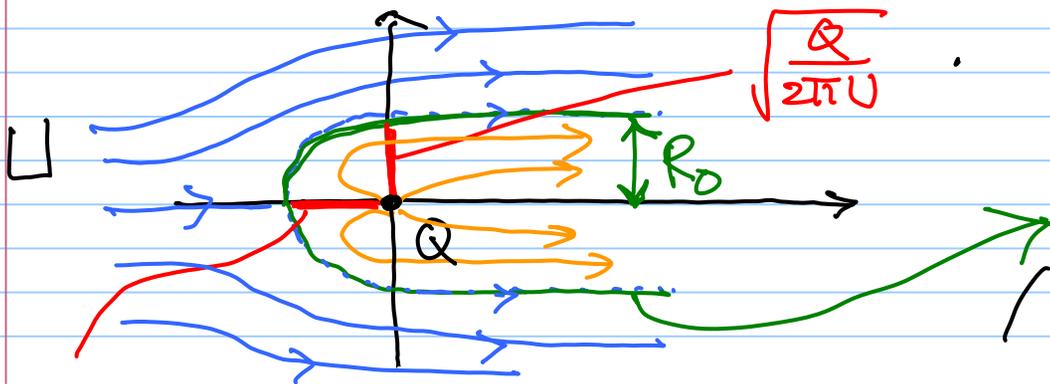
$$\therefore \phi(r, \theta) = \frac{\mu}{4\pi r^2} \cos \theta$$

$$\left( \begin{aligned} u_r &= \frac{\partial \phi}{\partial r} = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \\ u_\theta &= \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \end{aligned} \right)$$

$$\rightarrow \psi(r, \theta) = -\frac{\mu}{4\pi r} \sin^2 \theta$$

④ Flow near a blunt nose (aircraft fuselage, sub. hull ~)  
(uniform flow + source)

$$\psi(r, \theta) = \frac{1}{2} U r^2 \sin^2 \theta - \frac{Q}{4\pi} (1 + \cos \theta)$$



↓  
Surface of the body  
( $\psi = \text{constant}, = 0$ )

$$\sqrt{\frac{Q}{4\pi U}}$$

$$r = \left( \frac{2q}{U \sin^2 \theta} + \frac{Q}{4\pi U \sin^2(\theta/2)} \right)^{1/2}$$

w/  $\psi = 0$ ,  $r_0 = \sqrt{\frac{Q}{4\pi U} \cdot \frac{1}{\sin(\theta/2)}}$

$\theta = 0, r_0 \rightarrow \infty$

$\theta = \pi/2, r_0 = \sqrt{\frac{Q}{2\pi U}}$

$\theta = \pi, r_0 = \sqrt{\frac{Q}{4\pi U}}$

if we define  $R = r \sin \theta$

$$R_0 = r_0 \sin \theta = \sqrt{\frac{Q}{4\pi U} \cdot \frac{\sin \theta}{\sin(\theta/2)}}$$

as  $\theta \rightarrow 0$   $\frac{\sin \theta}{\sin(\theta/2)} \rightarrow 2 : R_0 \rightarrow \sqrt{\frac{Q}{\pi U}}$