Ch. 14. Collisions of Fast Ions in a Plasma

(*) Fast Ions in Fusion Plasmas.

1. $D + T \rightarrow \underline{He} + n$

   α particles with ~ 3.5 MeV

   born with an isotropic distribution in $\vec{v}$

2. Neutral Beam Injection for Plasma Heating.

   $E_b \sim 100$ keV presently

   1 MeV needed for reactors.

   " highly directed Beam " (anisotropic in $\vec{v}$)

3. Ion Cyclotron Heated Ions

   mostly in $\perp$ direction to $\vec{B}$.

14.2. Slowing-down of Beam Ions due to Collisions with electrons

⊛ Consider $v_{Ti} \ll \underline{V_b} \ll v_{Te}$.

of Beams with M_b and Z_b.

with Maxwellian background plasma electrons.

⇒ In the frame moving with the beam ions, the situation is similar to the case considered in Ch 11. i.e, electrons colliding with more massive (essentially stationary) ions.

In this case, the plasma electrons can transfer "momentum" to the beam ions, but not much energy.

The momentum gained by the beam will be exactly opposite to the velocity of beam ions. ⇒ "slow down".

⊛ Momentum Conservation $\Rightarrow -M_b \Delta\vec{V} = m_e \Delta\vec{v}$

⊛ Change in beam energy ; $\Delta W_b = \frac{M_b}{2}\left(|\vec{V}+\Delta\vec{V}| - V^2\right) \approx M_b \vec{V}\cdot\Delta\vec{V}$

Energy Conservation $\Rightarrow$ $\Delta W_e = \frac{1}{2} m_e |\Delta\vec{v}|^2 = \frac{m_e}{2}\left(\frac{M_b}{m_e}\right)^2 |\Delta\vec{V}|^2 = \frac{M_b^2}{2m}|\Delta\vec{V}|^2$.

$= -\Delta W_b$. (14.2),

$$\therefore \; -M_b V \Delta V_{\parallel} = \frac{M_b^2}{2m}|\Delta\vec{V}|^2 = \frac{M_b^2}{2m}\left[(\Delta V_{\parallel})^2 + (\Delta V_{\perp})^2\right] \qquad (14.3)$$

$$\Rightarrow \; \frac{M_b}{2}(\Delta V_\perp)^2 < mV|\Delta V_{\parallel}| \ll M_b V |\Delta V_{\parallel}|$$

$\therefore$ Energy in beam ion $\perp$ velocity components due to collisional deflection $\ll$ Energy decrease due to slowing-down without change of direction

In addition, from (14.3) $|\Delta V_{\parallel}| < \frac{2m}{M_b} V$, i.e.

beam ion momentum loss $\sim O\left(\frac{m}{M_b}\right)$ · original momentum

" energy loss $\sim O\left(\frac{m}{M_b}\right)$ " energy.

$\rightarrow \Delta V_\perp < \frac{2m}{M_b} V$ i.e., deflection through an angle $\sim \mathcal{O}(\frac{m}{M_b})$

$\therefore \Delta W_\perp = \mathcal{O}((\frac{m}{M_b})^2) \cdot W$.

In summary, the force of the background electrons on the beam ions is mostly in the nature of a "frictional drag."

(*) Momentum Conservation:

$$n_b M_b \frac{d}{dt} \underbrace{\langle \vec{V} \rangle}_{\text{mean velocity of beam ions.}} = -m \int \frac{d\langle \vec{v} \rangle}{dt} \underbrace{f_e(v)\, d^3v}_{\text{assume Maxwellian}} \quad (14.6)$$

(*) From Ch.11., we can deduce: $\frac{d}{dt}\langle \vec{v} \rangle = -\nu_{eb} (\vec{v} - \vec{V})$ (14.7)

(in the Laboratory frame).

where

$$\nu_{eb} = \frac{n_b Z_b^2 e^4 \ln\Lambda}{4\pi \epsilon_0^2 m^2 |\vec{v} - \vec{V}|^3}. \quad (14.8)$$

Using (14.6), $$\frac{d}{dt}\langle \vec{V} \rangle = \frac{Z_b^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m M_b} \int \frac{\vec{v}-\vec{V}}{|\vec{v}-\vec{V}|^3} f_e(v)\, d^3v \qquad (14.9).$$

Note that $$\frac{\vec{v}-\vec{V}}{|\vec{v}-\vec{V}|^3} = \frac{\partial}{\partial \vec{V}} \frac{1}{|\vec{v}-\vec{V}|}$$

(This identity ~~...~~ has appeared in the context of Coulomb force or gravity).

Then, $$\frac{d}{dt}\langle \vec{V} \rangle = -\frac{Z_b^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m M_b} \frac{\partial}{\partial \vec{V}} I(\vec{V}) \qquad (14.10)$$

where

$$I(\vec{V}) = -\int \frac{f_e(v)\, d^3v}{|\vec{v}-\vec{V}|} = -2\pi \int \frac{f_e(v)\, v^2 dv \sin\theta\, d\theta}{(v^2+V^2-2vV\cos\theta)^{1/2}}$$

in spherical coordinate

$$\cdots = -\frac{2\pi}{V} \int_0^\infty dv\, v f_e(v) \left\{ -|v-V| + v + V \right\}$$

See Fig 14.1.

$$\therefore \quad I = -4\pi \int_0^\infty v f_e(v)\, dv - \frac{4\pi}{V} \int_0^V v^2 f_e(v)\, dv$$

and

$$-\frac{\partial I}{\partial \vec{V}} = -\frac{4\pi \vec{V}}{V^3} \int_0^V v^2 f_e(v)\, dv.$$

For Maxwellian $f_e(v)$ and $V \ll v_{Te}$, we obtain

$$\frac{d}{dt}\langle \vec{V} \rangle = -\underbrace{\frac{2^{1/2} n_e Z_b^2 e^4 m^{1/2} \ln\Lambda}{12\pi^{3/2} \epsilon_0^2 M_b T_e^{3/2}}} \vec{V} \qquad (14.12)$$

(slowing down time)$^{-1}$: independent of $|\vec{V}|$!

depends on "T_e".

Taking $M_b \vec{V} \cdot$ of (14.12), we obtain

$$\frac{d}{dt} W_b = -\frac{2^{1/2} \underline{n_e} Z_b^2 e^4 m^{1/2} \ln\Lambda}{6\pi^{3/2} \epsilon_0^2 \underline{M_b}\, \underline{T_e^{3/2}}} W_b. \qquad (14.13)$$

14.3. Slowing-down of Beam Ions due to Collisions with backgnd Ions

(*) Consider $M_b \gg M$ case first.

The situation is similar to Beam slowing down due to collisions with e^-s up to some point. i.e.,

$$n_b M_b \frac{d}{dt}\langle \vec{V} \rangle = -M \int \frac{d\langle \vec{v} \rangle}{dt} f_i(v)\, d^3v \qquad (14.16)$$

where

$$\frac{d\langle \vec{v} \rangle}{dt} = -\nu_{ib}(\vec{v} - \vec{V}) \quad \text{and} \qquad (14.17)$$

$$\nu_{ib} = \frac{n_b Z^2 Z_b^2 e^4 \ln\Lambda}{4\pi \epsilon_0^2 M^2 |\vec{v} - \vec{V}|^3} \qquad (14.18).$$

For electron backgnd, we had to perform integration in $\vec{v}$, but Now we can assume $|\vec{V}| \gg |\vec{v}|$ and integration in $\vec{v}$ is unnecessary.

$$\Rightarrow \quad \frac{d}{dt}\langle \vec{V} \rangle = -\frac{n_i Z^2 Z_b^2 e^4 \ln\Lambda}{4\pi \epsilon_0^2 M M_b V^3} \vec{V} \qquad (14.19)$$

(*) $$\frac{d}{dt} W_b = - \frac{2^{1/2} n_i Z^2 Z_b^2 e^4 \ln\Lambda \, M_b^{1/2}}{8\pi \epsilon_0^2 M \quad W_b^{3/2}} \cdot W_b \qquad (14.20)$$

only depends on "W_b" not v_{Ti} or T_i of backgnd ion.

Recall this case assumed $M_b \gg M$.

(*) Now, we assume $M_b \ll M$. This case is somewhat similar to electrons being pitch-angle scattered by ~~t~~ heavier target ions.

While $(\Delta V_\perp)^2 \gg (\Delta V_\parallel)^2$ i.e., slowing-down is a subdominant process compared to the pitch-angle scattering,

A straighfwd analysis (~ the one for (11.11)) leads to

$$\frac{d}{dt} W_b = - \frac{M_b^2}{2M} \frac{d}{dt} (\Delta V_\perp)^2 = - \frac{2^{1/2} n_i Z^2 Z_b^2 e^4 M_b^{1/2} \ln\Lambda}{8\pi \epsilon_0^2 M \quad W_b^{3/2}} W_b \qquad (14.24)$$

i.e. the same with (14.20) for $M_b \gg M$ case.

→ We can deduce (14.20) and (14.24) apply to the case $M_b \sim M$ as well.

14.4. "Critical" Beam-Ion Energy.

Combining (14.13) and (14.20), we have

$$\frac{d}{dt} W_b = - \frac{2^{1/2} n_e Z_b^2 e^4 m^{1/2} \ln\Lambda}{6\pi^{3/2} \epsilon_0^2 M_b} \left(\frac{W_b}{T_e^{3/2}} + \frac{C}{W_b^{1/2}} \right) \qquad (14.25)$$

where $C = \dfrac{3\pi^{1/2} Z M_b^{3/2}}{4 m^{1/2} M} \simeq 57$, for proton M_b and M, Z.

(*) Two terms on RHS of (14.25) are equal for

$$\frac{W_{b,crit}}{T_e} = C^{2/3} \simeq 15.$$

(*) For $W_b > W_{b,crit}$, slowing-down on electrons (1st term) dominates,
and for $W_b < W_{b,crit}$ " ions (2nd term) dominates.

$\boxed{\text{HomeWork}}$ Problem 14.1 on page 239.

14.5. FOKKER-PLANCK EQN for Energetic Ions

(*) For isotropic source of energetic ions, we can ignore the effects of pitch angle scattering and consider only slowing down due to collisions with background electrons and ions as described by Eqs (14.12) and (14.19). Then F-P eqn without velocity space diffusion is:

$$\left(\frac{\partial f}{\partial t}\right)_{coll.} = -\frac{\partial}{\partial \vec{V}} \cdot \left(\frac{d\langle \Delta \vec{V}\rangle}{dt} f\right) \qquad (14.28)$$

From (14.12), (14.19) and (14.28),

$$\frac{\partial f_b}{\partial t} = \frac{n_e Z Z_b^2 e^4 \ln\Lambda}{4\pi \epsilon_0^2 M_b M} \frac{\partial}{\partial \vec{V}} \cdot \left[\frac{\vec{V}}{V^3}\left(1+\frac{V^3}{V_{crit}^3}\right) f_b\right] \qquad (14.29)$$

where

$$V_{crit} = \left(\frac{2 W_{b,crit}}{M_b}\right)^{1/2} = 3^{1/3} Z^{1/3} \left(\frac{\pi}{2}\right)^{1/6} \left(\frac{T_e}{m^{1/3} M^{2/3}}\right)^{1/2}.$$

(4) In spherical coordinates in velocity space, the divergence operator for this situation with isotropic $f_b(V)$ simplifies:

$$\frac{\partial}{\partial \vec{V}} \cdot \vec{A} = \frac{1}{V^2}\frac{\partial}{\partial V}\left(V^2 A_r\right) + \cancel{\frac{1}{V\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{V\sin\theta}\frac{\partial A_\phi}{\partial\phi}},$$

$\Rightarrow$ Eq (14.29) becomes,

$$\frac{\partial}{\partial t} f_b = \frac{n_e Z Z_b^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 M_b M} \frac{1}{V^2}\frac{\partial}{\partial V}\left[\left(1+\frac{V^3}{V_{crit}^3}\right) f_b\right] \qquad (14.30).$$

(*) Consider the energetic ions created (or injected) at $V = V_0$ continuously in time at a source rate "S".

$$\left(\frac{\partial f_b}{\partial t}\right)_{Source} = \frac{S\,\delta(V-V_0)}{4\pi V^2} \qquad (14.31)$$

By definition, $\left(\frac{\partial n}{\partial t}\right)_{source} = \int dV\, 4\pi V^2 \left(\frac{\partial f_b}{\partial t}\right)_{Source} = S.$

⊛ For $V \neq V_0$, $\frac{\partial f_b}{\partial t} = 0$ at steady state $\left((14.30) = 0 \right)$,

$$\Rightarrow \begin{cases} f_b = 0 & \text{for } V > V_0 \\ \left(1 + \frac{V^3}{V_{crit}^3}\right) f_b = C & \text{for } V < V_0 \end{cases}$$

* A constant "C" can be determined by applying

$$\lim_{\epsilon \to 0} \int_{V_0 - \epsilon}^{V_0 + \epsilon} dV \quad V^2 \cdot \left[\text{RHS of } (14.30) + \text{RHS of } (14.31) \right] = 0 ,$$

This leads to

$$-C \frac{n_e Z Z_b^2 e^4 \ln \Lambda}{4\pi \epsilon_0^2 M_b M} + \frac{S}{4\pi} = 0 \qquad (14.33).$$

Finally, the steady state beam distribution function is given by

$$\boxed{f_b(V) = \frac{S\epsilon_0^2 M M_b}{n_e Z Z_b^2 e^4 \ln\Lambda}\left(\frac{1}{1+V^3/V_{crit}^3}\right)}, \quad V \le V_0$$

$$= \quad 0 \qquad\qquad V > V_0$$

" Slowing Down Distribution Function "

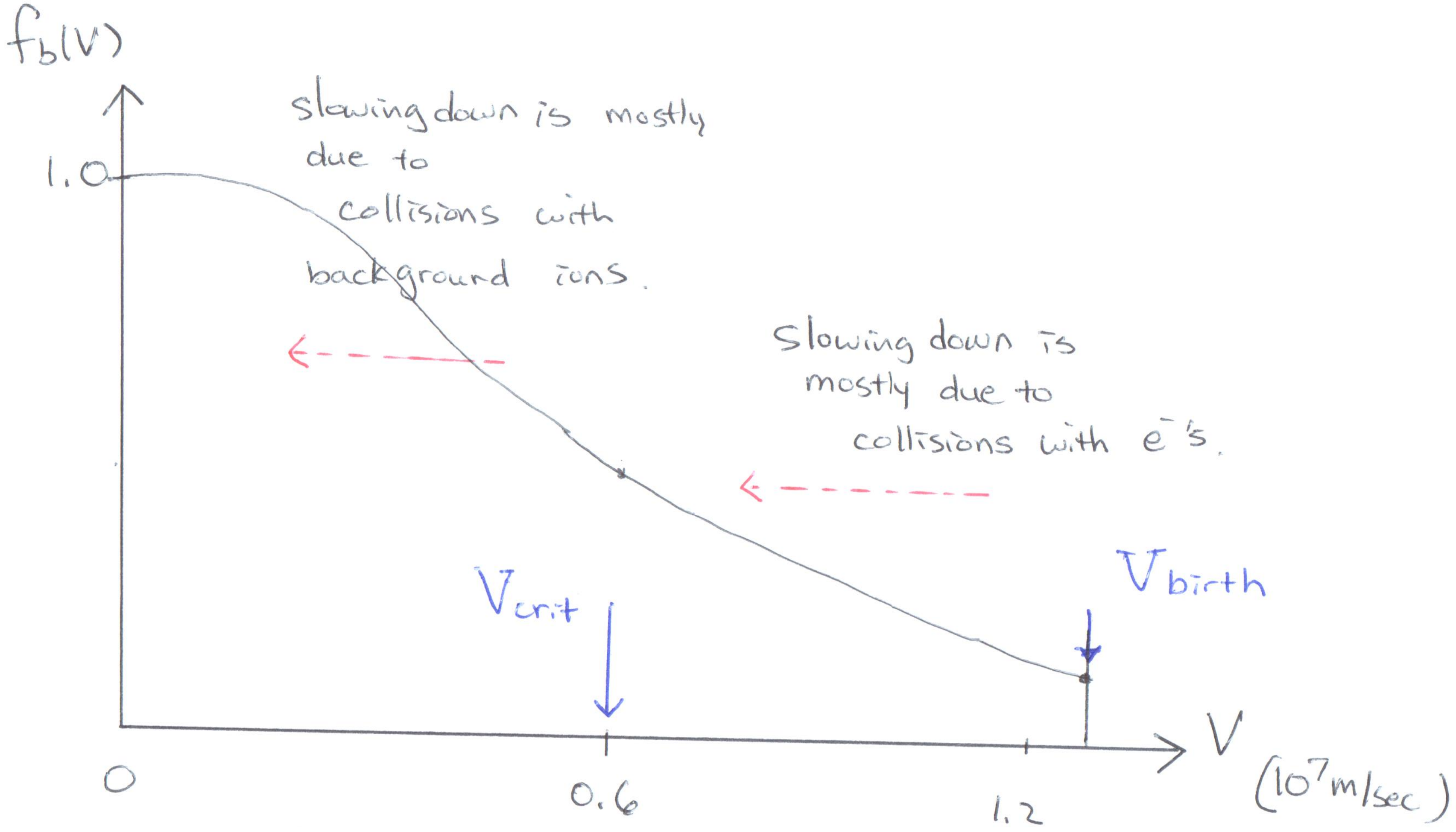
(*) Application to Fusion product Alpha particles from D-T fusion reaction :

$$S = n_D n_T \langle \sigma V \rangle_{DT},$$ strong ftn of temperature.

$$V_0 = 1.3 \times 10^7 \text{ m/sec} \quad \text{for} \quad E_\alpha = 3.5 \text{ MeV}$$

$$\langle \sigma V \rangle_{DT} = 4.2 \times 10^{-22} \text{ m}^3/\text{sec} \quad \text{at} \quad T_i = 20 \text{ keV}.$$

For this, $W_{b\text{-}crit} \simeq 30\, T_e$ (i.e., 600 keV for $T_e = 20$ keV),



For $T_e = 20$ keV, $E_b = 3.5$ MeV

Fig 14.2.