

⑤ Flow around a sphere (uniform flow + doublet)

$$\eta_f(r, \theta) = \frac{1}{2} U r^2 \sin^2 \theta - \frac{\mu}{4\pi r} \sin \theta$$

radius of a  $\rightarrow \eta_f(a, \theta) = 0$ .

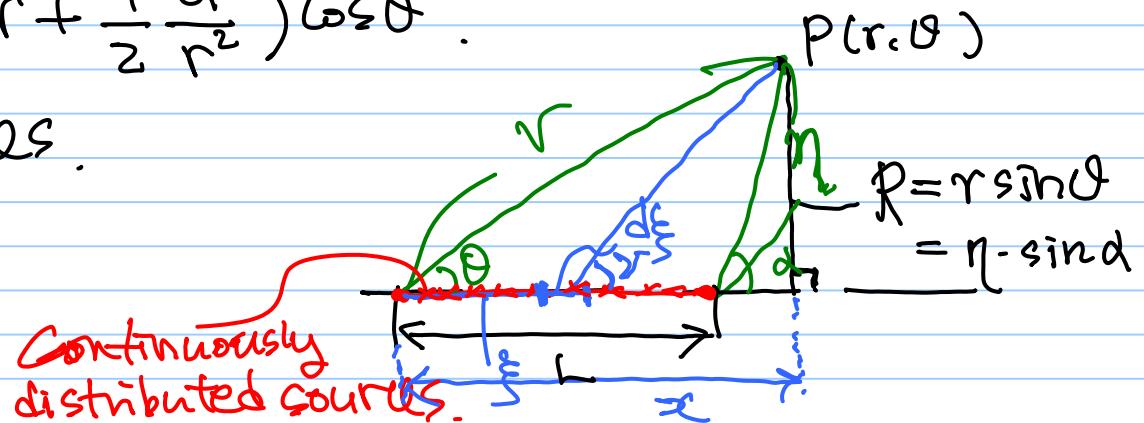
$$\downarrow$$

$$\mu = 2\pi U a^3$$

$$\therefore \eta_f(r, \theta) = \frac{1}{2} U \left( r^2 - \frac{a^3}{r} \right) \sin^2 \theta,$$

$$\phi(r, \theta) = U \left( r + \frac{1}{2} \frac{a^3}{r^2} \right) \cos \theta.$$

⑥ Line-distributed sources.



•  $\eta$ : sound strength per unit length.

$$\eta \psi = -\frac{Q}{4\pi} (1 + \cos \theta), \text{ for a single source } (Q)$$

$$\Rightarrow \eta \psi = \int_0^L -\frac{\eta \cdot d\xi}{4\pi} (1 + \cos \xi) \cdot d\xi, \text{ for a source array.}$$

$$\cdot R = r \sin \theta = \eta \sin \alpha = \text{constant.}$$

$$\cdot \alpha - \xi = R \cdot \cot \alpha \rightarrow -d\xi = -R \cdot \operatorname{cosec}^2 \alpha \cdot d\alpha.$$

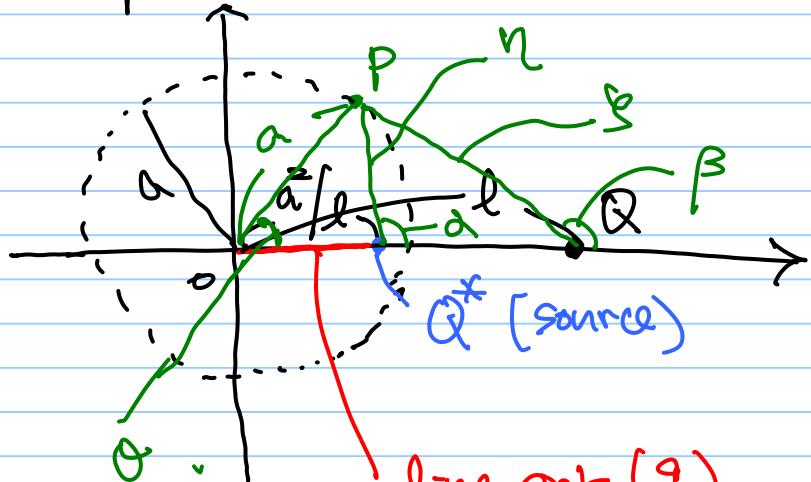
$$\therefore \eta \psi = -\frac{qR}{4\pi} \int_0^\alpha \operatorname{cosec}^2 \alpha (1 + \cos \alpha) d\alpha$$

$$= -\frac{qR}{4\pi} \left( \cot \alpha - \cot 0 + \frac{1}{\sin \alpha} - \frac{1}{\sin 0} \right)$$

$$= -\frac{q}{f\pi} (L + r - \eta)$$

· No class next week.

① Sphere in the flow field of a source.



no flow through  $R=a$ .

$$\rightarrow q^* = \frac{q \cdot a^2}{l}$$

$$\psi(r, \theta)$$

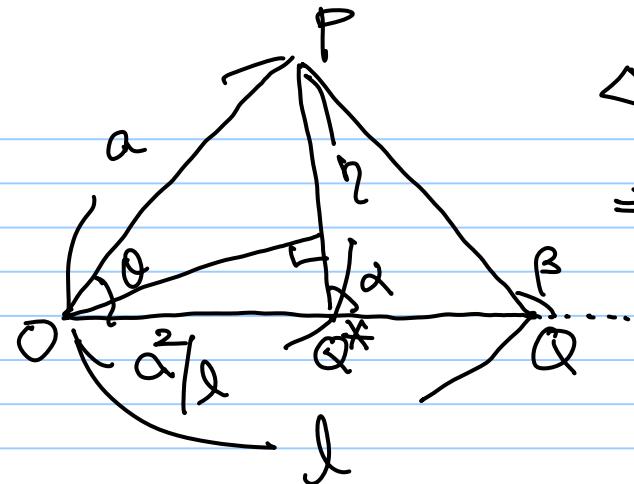
$$= -\frac{Q}{4\pi} (1 + \cos \beta) - \frac{Q^*}{4\pi} (1 + \cos \alpha)$$

$$+ \frac{q}{4\pi} \left( \frac{a^2}{l} + r - l \right)$$

then, on the sphere surface, ( $r=a$ )

$$\psi(a, \theta) = -\frac{Q}{4\pi} (1 + \cos \beta) - \frac{Q^*}{4\pi} (1 + \cos \alpha) + \frac{q}{4\pi} \left( 1 + \frac{r}{a} - \frac{lq}{a^2} \right)$$

$$\left( q = Q^* \cdot \frac{l}{a^2} \right)$$



$$\triangle OPQ^* \sim \triangle OQP.$$

$$\Rightarrow \gamma = \frac{a^2}{l} \cos(\pi - \alpha) + a \cdot \cos(\pi - \beta).$$

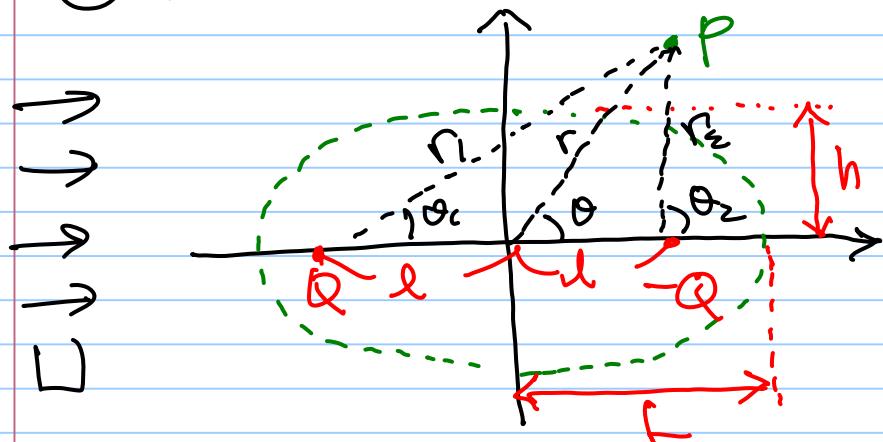
$$= -\frac{a^2}{l} \cos \alpha - a \cdot \cos \beta$$

$$\therefore \psi(a, \theta) = (1 + \cos \beta) \left( -\frac{Q}{4\pi} + \frac{Q^*}{4\pi} \cdot \frac{a}{l} \right) = 0.$$

$$\rightarrow Q^* = Q \cdot \frac{a}{l} \text{ to make } \psi(a, \theta) = 0.$$

$$\Rightarrow \psi(r, \theta) = -\frac{Q}{4\pi} (1 + \cos \beta) - \frac{Q}{4\pi} \cdot \frac{l}{a} (1 + \cos \alpha) + \frac{Q}{4\pi} \left( \frac{l}{a} + \frac{r}{a} - \frac{\gamma}{a} \right)$$

⊕ Rankine Solids = uniform flow + source + sink.



$$R = r \sin \theta,$$

$$\psi(r\theta)$$

$$= \frac{1}{2} U r^2 \sin^2 \theta - \frac{Q}{4\pi} (\cos \theta_1 - \cos \theta_2)$$

at  $r=r_0$  (surface of the body)

$$\rightarrow \phi = \frac{1}{2} U r_0^2 \sin^2 \theta - \frac{Q}{4\pi} (\cos \theta_1 - \cos \theta_2),$$

if  $R_0 = r_0 \sin \theta$ .

$$\rightarrow R_0^2 = \frac{Q}{2\pi U} (\cos \theta_1 - \cos \theta_2).$$

$$\left. \begin{array}{l} \theta_1 = \theta_2 = 0 \\ \theta_1 = \theta_2 = \pi \end{array} \right\} R_0 = 0.$$

• max.  $R_0$ ,  $\omega s \theta_1 = -\cos \theta_2$   
 $(\theta = \pi/2 \text{ or } 3\pi/2)$

- $\frac{L}{h}$  and  $h$ .

→ downstream stag. pt.

- sum of induced velocity should be zero!

$$\hookrightarrow U + \frac{Q}{4\pi(L+l)^2} - \frac{Q}{4\pi(L-l)^2} = 0.$$

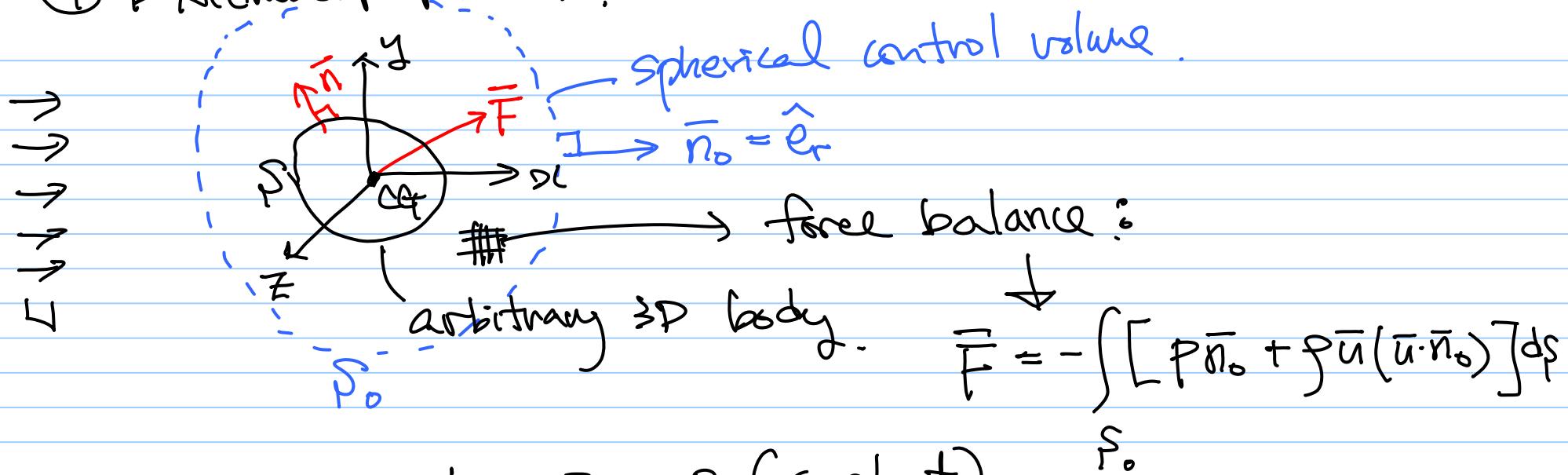
$$\therefore (L^2 - l^2)^2 - \frac{Ql}{\pi L} L = 0. \rightarrow \text{find } L.$$

- $h^2$ ?  $R_o Q \cos\theta_1 = -\cos\theta_2$ .

$$h^2 = \frac{Q}{2\pi U} \left[ \frac{l}{\sqrt{h^2 + l^2}} + \frac{l}{\sqrt{h^2 + l^2}} \right] \Rightarrow h^2 \sqrt{h^2 + l^2} = \frac{Ql}{\pi U}.$$

→ find  $h$ .

### ④ D'Alembert Paradox.



then,  $P + \frac{1}{2} \rho \bar{u} \cdot \bar{u} = B$  (constant)

$$\left( \int_{S_0} B \bar{n}_o dS = 0 \right)$$

$$\rightarrow \bar{F} = \rho \int_{S_0} \left[ \frac{1}{2} (\bar{u} \cdot \bar{u}) \bar{n}_o - \bar{u} (\bar{u} \cdot \bar{n}_o) \right] dS.$$

vector.

Let's define,  $\bar{U} = \underline{U} + \bar{U}'$

$\uparrow$   
Uniform flow

$\uparrow$  perturbation due to the solid body

( $\bar{U}'$ : large near the body,  
zero as  $r \rightarrow \infty$ )

$$= \rho \int_{S_0} \left[ -\underline{U} \times (\bar{U}' \times \bar{n}_0) \bar{n}_0 + \frac{1}{2} (\bar{U}' \cdot \bar{U}') \bar{n}_0 - \bar{U}' (\bar{U}' \cdot \bar{n}_0) \right] dS.$$

$$\bar{U}' = \nabla \phi'. \quad \left( \phi = \sum_{l=0}^{\infty} A_l \frac{P_l(\cos \theta)}{r^{l+1}} \right) \rightarrow \phi' \sim O\left(\frac{1}{r}\right)$$

$$\rightarrow \bar{U}' \sim O\left(\frac{1}{r^2}\right)$$

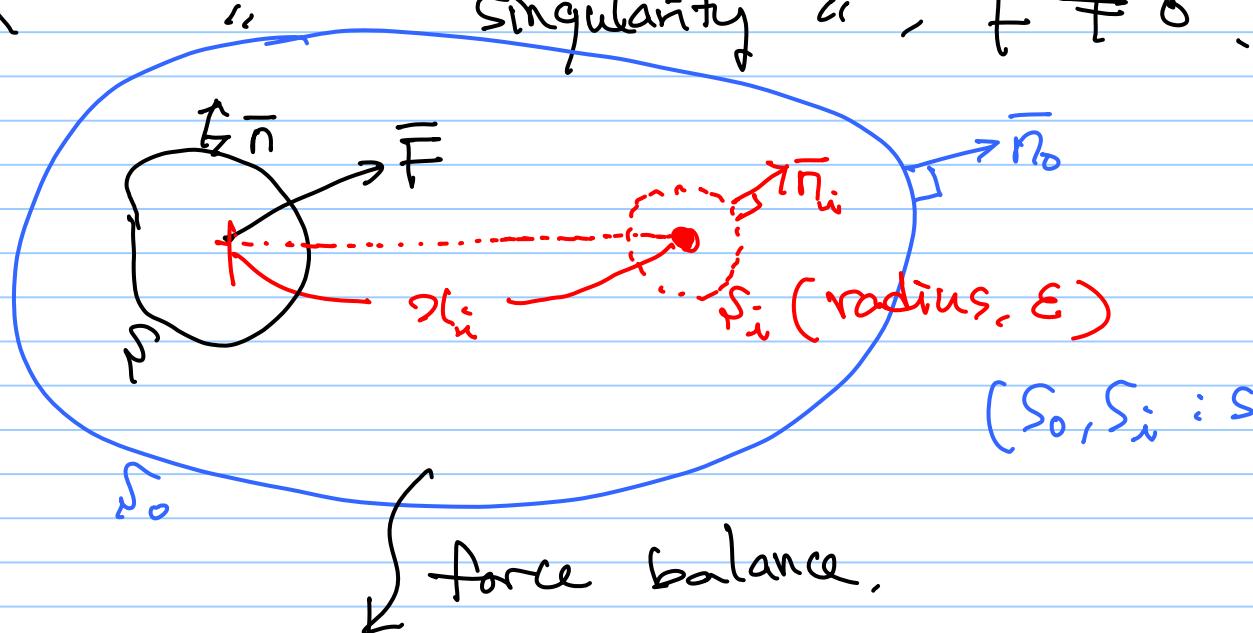
If radius of sphere ( $S_0$ ) is taken to be large enough,  
(i.e.,  $r \rightarrow \infty$ )

$\rightarrow \bar{F} \neq 0$ . (D'Alembert Paradox).

⑩ Forces induced by singularities.

{ 3D body in uniform flow,  $\bar{F} = 0$ .

" singularity " ,  $\bar{F} \neq 0$ .



( $s_0, s_i$  : spherical)

$$\frac{D}{Dt} \int_V \rho \bar{u} dV = \int_{\Sigma_i} \frac{\partial}{\partial t} (\rho \bar{u}) dS + \int_{\Sigma_i} \rho \bar{u} (\bar{u} \cdot \bar{n}) dS = \int_{\Sigma_i} \bar{P} dS + \int_{\Sigma_i} \rho \bar{f} dS.$$

$$\rightarrow \int_{\Sigma_i} \bar{P} dS + \int_{\Sigma_0} \bar{P} dS + \int_{\Sigma_0} \bar{P} dS = \int_{\Sigma_i} \rho \bar{u} (\bar{u} \cdot \bar{n}) dS + \int_{\Sigma_i} \rho \bar{u} (\bar{u} \cdot \bar{n}) dS + \int_{\Sigma_0} \rho \bar{u} (\bar{u} \cdot \bar{n}) dS$$

---  $\bar{F}$

$$\therefore -\bar{F} - \int_{\Sigma_i} P(-\bar{n}_i) dS - \int_{\Sigma_0} P \bar{n}_0 dS = \int_{\Sigma_i} \rho \bar{u} (\bar{u} \cdot (-\bar{n}_i)) dS + \int_{\Sigma_0} \rho \bar{u} (\bar{u} \cdot \bar{n}_0) dS.$$

$\boxed{\quad}$

D'Alembert Paradox.

$$\therefore \bar{F} = \int_{S_i} [P \bar{n}_i + P \bar{u} (\bar{u} \cdot \bar{n}_i)] dS . \leftarrow P = B - \frac{1}{2} \rho (\bar{u} \cdot \bar{u}) .$$

$$= \rho \int_{S_i} \left[ -\frac{1}{2} (\bar{u} \cdot \bar{u}) \bar{n}_i + \bar{u} (\bar{u} \cdot \bar{n}_i) \right] dS .$$

Final. 6/12 (Wed). (11:00 →).

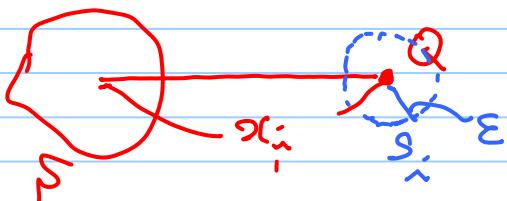
• force by singularity

$$\rightarrow \bar{F} = \rho \int_{S_i} \left[ -\frac{1}{2} (\bar{u} \cdot \bar{u}) \bar{n}_i + \bar{u} (\bar{u} \cdot \bar{n}_i) \right] dS.$$

$\bar{u}$  velocity on  $S_i$ .  $\leftarrow$  depend on the singularity.

Ⓐ In case of a source ( $Q$ )

$$\text{velocity on } S_i : \bar{u} = \frac{Q}{4\pi\epsilon^2} \hat{e}_\epsilon + \bar{u}_i$$



contributions from  
others (than  
the  $Q$  itself)

$$\bar{u} \cdot \bar{u} = \frac{Q^2}{16\pi^2 \epsilon^4} + \frac{Q}{2\pi \epsilon^2} \hat{e}_\epsilon \cdot \bar{u}_i + \bar{u}_i \cdot \bar{u}_i$$

$$\bar{u} \cdot \bar{n}_i = \frac{Q}{4\pi \epsilon^2} + \bar{u}_i \cdot \hat{e}_\epsilon$$

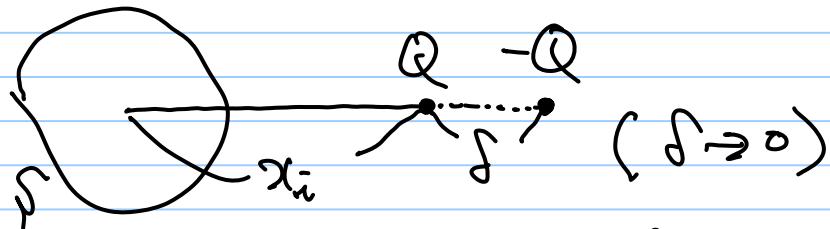
$$\rightarrow \bar{F} = \oint_{S_i} \left[ \frac{Q^2}{32\pi^2 \epsilon^4} \hat{e}_\epsilon - \frac{1}{2} (\bar{u} \cdot \bar{u}_i) \hat{e}_\epsilon + \frac{Q}{4\pi \epsilon^2} \bar{u}_i + (\bar{u}_i \cdot \hat{e}_\epsilon) \bar{u}_i \right] dS$$

let  $\epsilon \rightarrow 0$ .  $\Rightarrow \bar{u}_i \rightarrow \text{constant}$ .

$$= \oint_{S_i} \frac{Q}{4\pi \epsilon^2} \bar{u}_i dS = \cancel{\frac{Q \bar{u}_i}{4\pi \epsilon^2} \int_{S_i} dS} = Q \bar{u}_i$$

(  $\cancel{\frac{Q \bar{u}_i}{4\pi \epsilon^2} \int_{S_i} dS}$  for a sink )

B) in case of a doublet.



$$\begin{aligned} \text{velocity at } x = x_i : & \frac{Q}{4\pi f^2} \hat{e}_x + \bar{u}_i & \text{contributed by singularities except } Q \text{ and } -Q, \\ \text{velocity at } x = x_i + \delta : & \frac{Q}{4\pi f^2} \hat{e}_x + \bar{u}_i + \delta \frac{\partial \bar{u}_i}{\partial x} + \dots \text{ H.o.T.} & \text{Taylor series exp.} \\ \text{from } \textcircled{*} : \bar{F} = & \begin{cases} \Re Q \left( \frac{Q}{4\pi f^2} \hat{e}_x + \bar{u}_i \right) & \text{from source} \\ -\Im Q \left( \frac{Q}{4\pi f^2} \hat{e}_x + \bar{u}_i + \delta \frac{\partial \bar{u}_i}{\partial x} \right) & \text{from sink} \end{cases} \end{aligned}$$

$$\therefore \bar{F} = -\rho Q \oint \frac{\partial \bar{U}_i}{\partial x} \quad \text{as } f \rightarrow 0. \text{ define } \lim_{f \rightarrow 0} Qf = \mu (\neq 0)$$

$$\therefore \bar{F} = -\rho \mu \frac{\partial \bar{U}_i}{\partial x}$$

ex) sphere in the source flow.

$\bar{U}_i = \frac{Q \cdot a/l}{4\pi} \cdot \frac{1}{(1-a^2/l)^2} \hat{e}_x$

$\bar{F} = \rho Q \bar{U}_i$

$Q^* = Q \cdot a^2/l = a \cdot Q/l$

$\phi = -\frac{Q}{4\pi r} \rightarrow U_r = \frac{Q}{4\pi r^2}$

due to  $Q^*$       due to  $q$ .

$$= \frac{Q-a/l}{4\pi} \cdot \frac{1}{(l-a^2/l)^2} \hat{e}_x - \frac{Q/l}{4\pi} \left[ \frac{1}{(l-a^2/l)} - \frac{1}{l} \right] \hat{e}_x$$

$$\Rightarrow \bar{F} = \rho Q \bar{U}_x = \frac{\rho Q^2 a^3}{4\pi l (l-a^2)^2} \hat{e}_x : \text{attracted to the } Q.$$

⑪ Kinetic energy of a moving fluid.  
(body moves in a fluid at rest)

