Ch. 15: Basic Concepts of small-amplitude Waves

* Relevant governing equations for plasma dynamics are nonlinear.

* Assume amplitudes of waves are small enough
  (infinitesimally small).

  $\Rightarrow$ Equations can be linearized.

  $\Rightarrow$ Fourier analysis becomes useful.

* 0-th order: equilibrium of the system.

* 1-st order: Solve for frequency "ω" $= \omega(\vec{k})$,
  and relative amplitudes and phases of various oscillating quantities.

* Exponential Notation:

For example, density perturbation:

$$\delta n = \underbrace{\delta \bar{n}}_{\text{amplitude (real number)}} \exp[i(\underbrace{\vec{k}\cdot\vec{x} - \omega t}_{\text{spatial, temporal dependences}} + \underbrace{\delta_n}_{\text{phase}})]$$

(cf: sometimes, $\delta\bar{n}\, e^{i\delta_n}$ is combined into a complex amplitude $\delta\tilde{n}$

So, understanding the notations depending on the context of discussion is useful, rather than being trapped in irrelevant details.)

Of course, $e^{iA} = \cos A + i \sin A$

is a useful identity.

Physical Quantities of Interests

* $\vec{k}$: wave vector

$\lambda = 2\pi/k = 2\pi/|\vec{k}|$: wavelength.

* ω : frequency

$$\boxed{\omega = \omega(\vec{k})}$$

Dispersion Relation

* **Phase Velocity** :

"an observer travelling with the phase velocity, $\vec{V}_p$, stays at a constant wave phase."

⇒ Demand $\frac{d}{dt}$ (total phase factor) = 0

(in $\exp[\underline{i(\vec{k}\cdot\vec{x} - \omega t + \delta_n)}]$)

* Phase Velocity :

From $\dfrac{d}{dt}(\vec{k}\cdot\vec{x} - \omega t) = \vec{k}\cdot\dfrac{d\vec{x}}{dt} - \omega = 0$,

$$\boxed{\vec{V}_p = \frac{\vec{k}}{|\vec{k}|^2}\,\omega} = \frac{\omega k_x}{k^2}\hat{x} + \frac{\omega k_y}{k^2}\hat{y} + \frac{\omega k_z}{k^2}\hat{z}$$

in Cartesian coordinates

$$\neq \frac{\omega}{k_x}\hat{x} + \frac{\omega}{k_y}\hat{y} + \frac{\omega}{k_z}\hat{z}.$$

not

* If we consider a wave-packet consisting of many components with different $\vec{k}$'s (and ω's), $\vec{V}_p$ is the velocity at which individual crests within the packet travel.

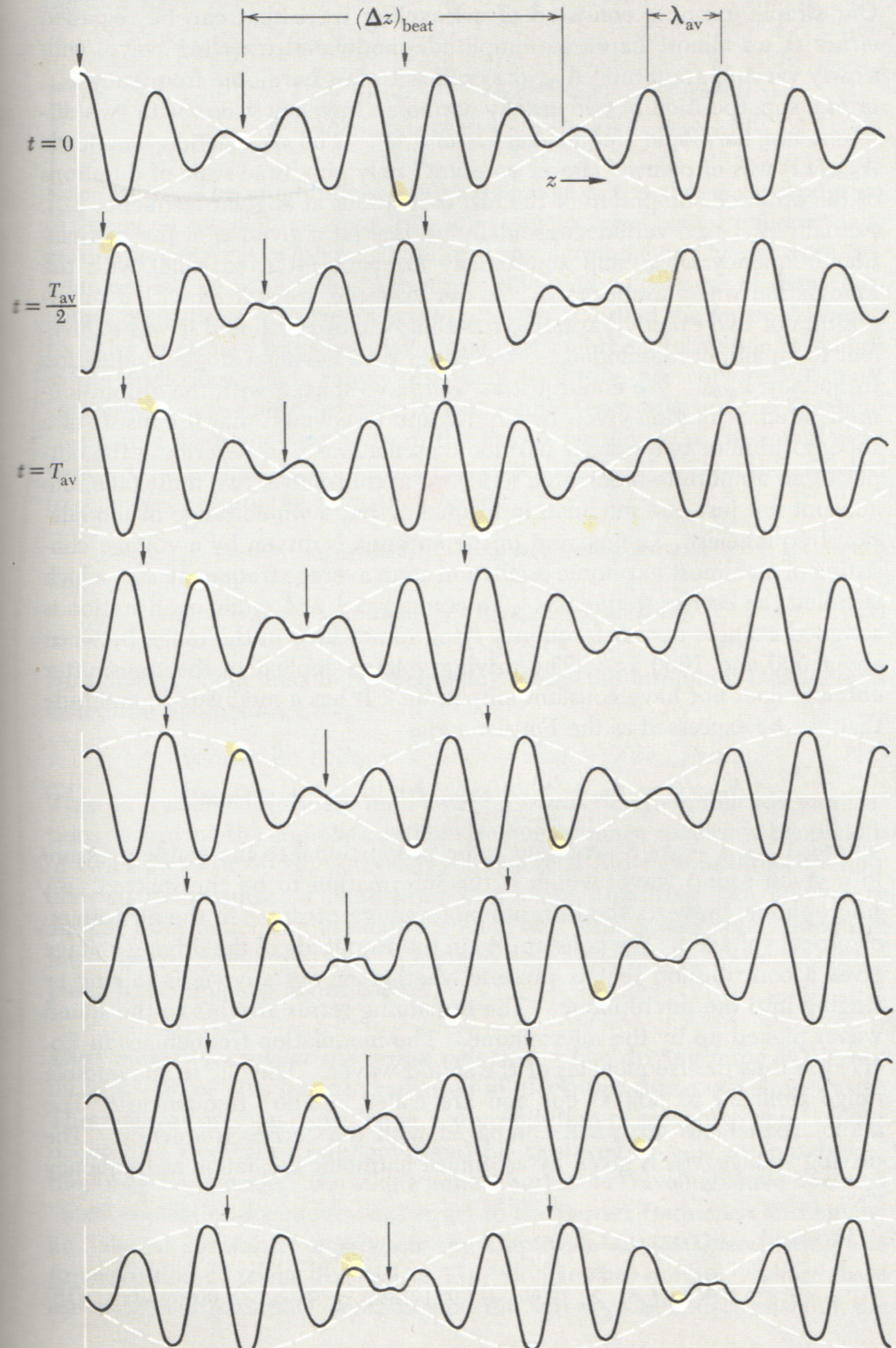


Fig. 6.1 **Group velocity.** *The arrows follow the beats, which travel at the group velocity v_g. The white circles follow individual wave crests, which travel at the average phase velocity v_{av}.*

draw this of 'envelope'

I.-7.

of ~~s~~crests, troughs

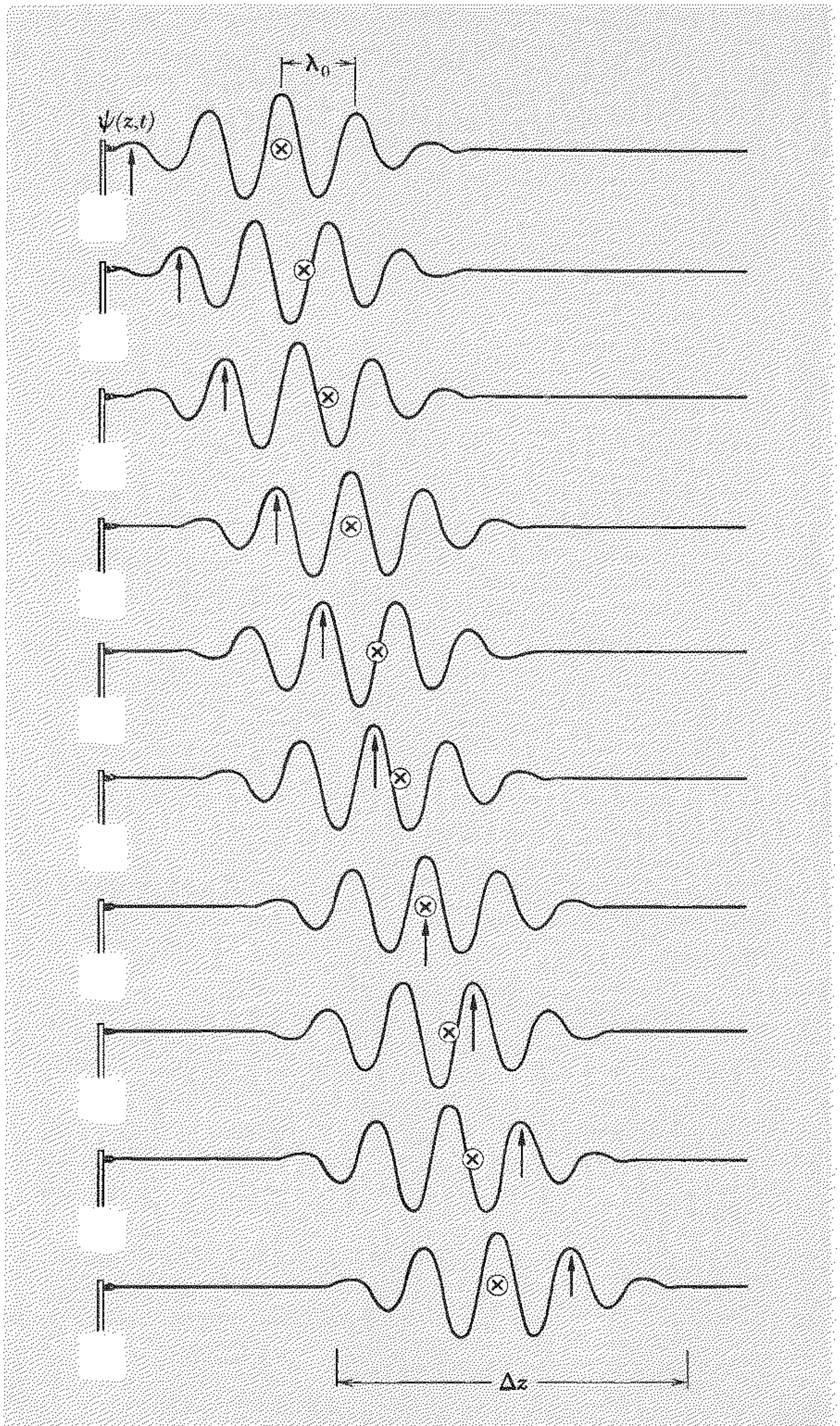
$\frac{1}{4}$ wl.

then packet prop. by

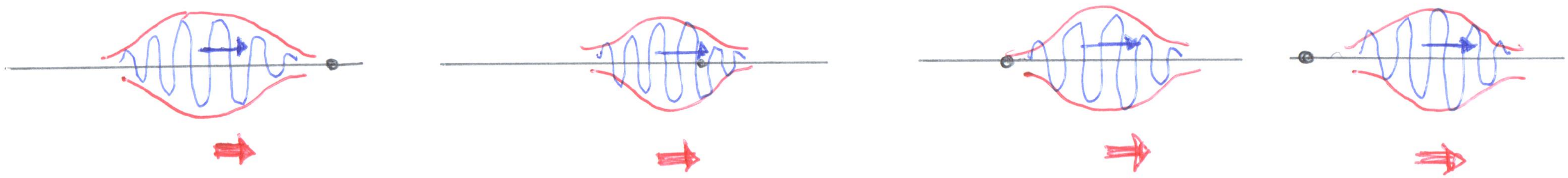
$\frac{1}{4}$ wl.

I.-7. a)

Fig. 6.7 *Wave packet with phase velocity twice the group velocity. The arrow travels at the phase velocity, following a point of constant phase for the dominant wavelength. The cross travels at the group velocity with the packet as a whole.*



15.2. Group Velocities



* $\vec{v}_p$: phase velocity of individual crests within the packet.

  (cf. it can be in oppositive direction to ⇒ of packet)

* $\vec{v}_g$: group velocity of the wave packet

  (~ envelope containing many wave crests).

* In general, $\vec{v}_g \neq \vec{v}_p$.

* Energy and Information are carried at $\vec{v}_g$!

* Packet of oscillations with Gaussian envelope;

- $A(x) = \mathrm{Re}\left[\exp\left(-\frac{x^2}{2\sigma^2}\right)\exp(ik_0 x)\right]$ (15.8)

See Fig 15.1 in G&R ; $k_0\sigma >> 1$ (typo in Fig. caption)

It can be shown that,

- $A(x) = \mathrm{Re}\left(\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \exp(ikx)\exp\left[-\frac{\sigma^2(k-k_0)^2}{2}\right]\right)$ (15.9)

*** Home work: Problem 15.2 on page 253

This can be viewed as t=0 snap shot (free-frames) of a set of propagating waves.

* Ensuing time-evolution of this system is then,

$$A(x,t) = \mathrm{Re}\left(\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \exp[i(kx-\omega(k)t)]\, e^{-\frac{\sigma^2(k-k_0)^2}{2}}\right) \tag{15.10}$$

Note:

- Dependences of Gaussian envelope on σ:
  - large σ → narrow (localized) packet in k-space
    → wide packet in x-space.

    (and vice versa for small σ)

    ~ uncertainty principle in quantum mechanics.

approximate;
$$\omega(k) = \omega(k_0) + \left(\frac{\partial \omega}{\partial k}\right)_{k_0}(k-k_0) + \cdots$$

$\Rightarrow$

$$A(x,t) = \mathrm{Re}\Big[\exp\Big[i\Big\{ k_0 \left(\frac{\partial \omega}{\partial k}\right)_{k_0} - \omega(k_0) \Big\} t \Big]$$

$$\text{''} \qquad\qquad \text{''}$$

$$\times \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \exp\Big\{ i \Big[kx - k\left(\frac{\partial \omega}{\partial k}\right)_{k_0} t \Big] \Big\} e^{-\frac{\sigma^2 (k-k_0)^2}{2}} \Big] \qquad (15.11)$$

Comparing this expression to $A(x, t=0)$ in Eq (15.9),

the second line "$\sim$" is exactly "$A\left(x - \left(\frac{\partial \omega}{\partial k}\right)_{k_0} t, 0\right)$"

i.e., a translation of the original $t=0$ freeze-frame

at velocity $\left(\frac{\partial \omega}{\partial k}\right)_{k_0}$!

15.3. Ray-tracing Equations :

* Consider an inhomogeneous plasma in 3d.

Without loss of generality, take $\vec{k}_0 = k_0 \hat{x}$.

Then, 3d-extension of Eq. (15.8) is ;

$$A(x) = \mathrm{Re}\left[\exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2} \right] \exp(i k_0 x) \right] \quad (15.12)$$

which can be re-expressed through the Fourier transform,

$$A(x) = \mathrm{Re}\left[\frac{\sigma_x \sigma_y \sigma_z}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3\vec{k} \exp(i \vec{k}\cdot\vec{x}) \exp\left[-\sigma_x^2 (k_x - k_0)^2/2 - \sigma_y^2 k_y^2/2 - \frac{\sigma_z^2 k_z^2}{2} \right] \right] \quad (15.13)$$

* As done in 1d before, expand

$$\omega(\vec{k}) \simeq \omega(\vec{k}_0) + (\vec{k} - \vec{k}_0) \cdot \vec{\nabla}_{\vec{k}}\, \omega \Big|_{\vec{k}_0} + \cdots \tag{15.14}$$

where $\vec{\nabla}_k \equiv \hat{x} \dfrac{\partial}{\partial k_x} + \hat{y} \dfrac{\partial}{\partial k_y} + \hat{z} \dfrac{\partial}{\partial k_z}$. (15.15)

* A similar analysis as Eqs (15.9) – (15.11) in 3d will lead to our 'freeze-frame' $A(\vec{x})$ translating at a vector group velocity given by $(A(\vec{x} - \vec{v}_g t))$,

$$\boxed{\vec{v}_g = \frac{\partial \omega}{\partial \vec{k}} \quad \left(= \nabla_{\vec{k}}\, \omega \right)}$$

* Note that $\vec{v}_g$ may not only have a different magnitude from $\vec{v}_p$, but even a different direction.

* As a packet propagates in an inhomogeneous plasma, the peak of $\vec{k}$ spectrum, $\vec{k}_o$ can change.

* On the other hand, if the medium (background), is by hypothesis linear and time-independent, $\omega(\vec{k}_o)$ should be constant.

$\Rightarrow$ Total derivative of ω, moving with the wave-packet must vanish. i.e,

$$\boxed{\frac{d}{dt}\,\omega(\vec{k},\vec{x}) = 0}$$

along the ray.

* Using the chain-rule,

$$\frac{d}{dt}\omega(\vec{k},\vec{x}) = \frac{d\vec{x}_0}{dt}\cdot\frac{\partial}{\partial\vec{x}}\Big|_{\vec{k}}\omega + \frac{d\vec{k}_0}{dt}\cdot\frac{\partial}{\partial\vec{k}}\Big|_{\vec{x}}\omega = 0 \qquad (15.17)$$

∴ we can identify the relation;

$$\boxed{\begin{aligned} &\bullet\ \frac{d\vec{k}_0}{dt} = -\frac{\partial\omega}{\partial\vec{x}}\Big|_{\vec{k}} \\ &\bullet\ \frac{d\vec{x}_0}{dt} = \frac{\partial\omega}{\partial\vec{k}}\Big|_{\vec{x}} \end{aligned}} \qquad (15.18)$$

* Ray-tracing equations for wave-packet.

* As the wave-packet propagates it maintains the peak of its frequency spectrum, while $\vec{k}$-spectrum transforms.

* Ray-tracing Equation for Wave-Packet : ← Wave-Particle duality → Hamiltonian Mechanics for Particle

$$\boxed{\begin{aligned} \frac{d\vec{k}_0}{dt} &= -\frac{\partial}{\partial \vec{x}}\,\omega \\ \frac{d\vec{x}_0}{dt} &= \frac{\partial}{\partial \vec{k}}\,\omega \end{aligned}}$$

$\hbar\omega = E$

$\hbar\vec{k} = \vec{p}$

in Q.M.

$$\boxed{\begin{aligned} \frac{d\vec{p}}{dt} &= -\frac{\partial}{\partial \vec{q}}\,H(\vec{p},\vec{q}) \\ \frac{d\vec{q}}{dt} &= \frac{\partial}{\partial \vec{p}}\,H(\vec{p},\vec{q}) \end{aligned}}$$

* $\vec{V}_g$ of de Broglie wave $\sim$ $\vec{V}_{particle}$.

* Non conservation of $\vec{k}_0$ in an inhomogeneous medium is related to the non-conservation of momentum

* Conservation of ω in time-independent medium is related to the conservation of energy.

Ch. 16. Waves in an unmagnetized plasma

16.1. Langmuir Waves

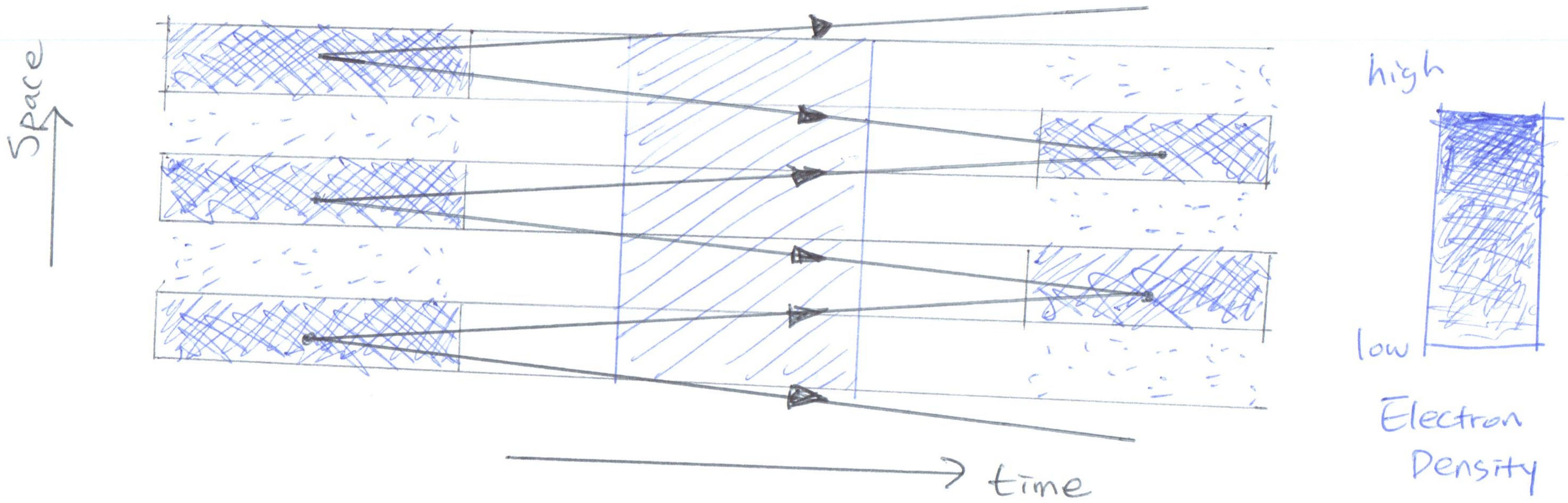


Fig 16.1.

* Initial displacements of electrons from equilibrium position
→ charge separation → electric field generation
→ pulling electrons back → overshoot → oscillations.

* This oscillation happens in very Short time scale

$\Rightarrow$ ions with heavy mass cannot respond, and can be considered as stationary.

* **Electron Dynamics :**

— $m n_e [\dot{\vec{u}}_e + (\vec{u}_e \cdot \vec{\nabla}) \vec{u}_e] = -e n_e \vec{E} - \vec{\nabla} p_e$ (16.1)

— $\dot{n}_e + \vec{\nabla} \cdot (n_e \vec{u}_e) = 0$ (16.2)

— $\epsilon_0 \vec{\nabla} \cdot \vec{E} = e(n_i - n_e)$ (16.3)

* Consider electrons moving in x direction and waves propagating " .

$\Rightarrow$ perturbed quantities vary as "$\exp[i(kx - \omega t)]$"

$\boxed{\frac{\partial}{\partial x} \to ik}$ and $\boxed{\frac{\partial}{\partial t} = -i\omega}$

* Linearize Eqs. (16.1)-(16.3) i.e., keep only the 1st order quantities:

In addition, relate p_1 to n_1 by using the equation of state for 1-dimensional compression faster than the thermal conduction, i.e., $p \propto n^{\gamma}$, where $\gamma = \frac{2+N}{N} = \frac{2+1}{1} = 3$.

$$\Rightarrow \left[\begin{array}{ll} i\omega m n_0 u_1 = e n_0 E_1 + 3ikTn_1 & (16.6) \\ -i\omega n_1 + ikn_0 u_1 = 0. & (16.7) \\ ik\epsilon_0 E_1 = -en_1 & (16.8) \end{array}\right.$$

3 equations for 3 perturbed quantities, u_1, n_1 and E_1.

* [Determinant of 3x3 matrix.] = 0 $\Rightarrow$ Dispersion Relation (between "ω" and "k")

* $$\boxed{\omega^2 = \omega_{pe}^2 + 3k^2 v_{T.e}^2}$$

where $\omega_{pe}^2 \equiv n_e e^2 / \epsilon_0 m_e$

electron plasma frequency

$v_{T.e}^2 = (T_e / m_e)$.

electron thermal velocity.

Bohm-Gross dispersion relation [PR 1949].

* Fig 16.2 illustrates the behavior at different k regimes.

* At long wavelength (low k); $\omega^2 \approx \omega_{pe}^2$

~ non-propagating "plasma oscillation".

(Langmuir)

* At short wavelength (high k); $\omega^2 \approx 3k^2 v_{Te}^2$

~ Propagating "electron sound wave."

16.2. Ion Sound Waves

* Take a look at another electrostatic longitudinal wave ($\vec{K} \parallel \vec{E}_1$) which oscillates in longer time scale so that ions dynamics plays an essential role.

→ Electrons can move fast enough to establish nearly exact force balance (between pressure gradient and electric field), i.e., a Boltzmann response.

* Follow a similar derivation as the one for Langmuir wave.

- Ion Fluid momentum eqn. (16.16)
- Poisson eqn. (16.19)
- Ion Continuity eqn (16.21)

* A crucial difference $\left(\begin{array}{l}\text{from the Langmuir wave where} \\ \qquad n_{i1} = 0\end{array}\right)$

is that

$$n_{e1} = n_e - n_{eo} = n_{eo}\left(\exp\left(e\phi_1/T_e\right) - 1\right)$$

$$\simeq n_{eo}\, e\phi_1/T_e \qquad (16.18)$$

i.e. Boltzmann response (or adiabatic response),

* Linearization → solve ~~for~~ coupled eqns for n_{i1}, ϕ_1, and u_{i1} → to get:

$$\boxed{\left(\omega/k\right)^2 = \frac{T_e/M}{1 + k^2\lambda_D^2} + \gamma_i T_i/M} \qquad (16.23)$$

$$\underline{\lambda_D}^2 \equiv \epsilon_0 T_e / n_e e^2 = v_{Te}^2/\omega_p^2.$$

Debye length.

* In the long wavelengh limit (~~sm~~ low k) ;

$$\bigstar \quad \boxed{(\omega/k)^2 \approx \frac{T_e + \gamma_i T_i}{M}.}$$

Sound wave

- Both electron and ion pressure contribute, but only ions provide "mass".

- In collisionless plasma, Eq. ✩ is valid for $T_e \gg T_i$ because one needs to consider wave-particle resonant ~~ex~~ interaction (Landau damping) for $\omega/k \sim v_{Ti}$ using kinetic theory.

* In the short wavelength limit (high k) ;

$$\omega^2 \approx \frac{T_e/M}{\lambda_D^2} \approx \left(\frac{m}{M}\right)\omega_p^2 \equiv \Omega_p^2 :$$ ion plasma frequency.

non-propagating wave.

16.3. High-Frequency Electromagnetic Waves in an Unmagnetized plasma.

* Now, we consider magnetic field perturbation $\vec{B}_1$ in addition to $\vec{E}_1$. Then, we ~~ea~~ need to consider Ampère's law and Faraday's law. With $\vec{\nabla} \Rightarrow i\vec{k}$,

- $i\vec{k} \times \vec{B}_1 = \mu_0 \vec{j}_1 - i\omega \vec{E}_1 / c^2$ (16.24)
- $i\vec{k} \times \vec{E}_1 = i\omega \vec{B}_1$ (16.25)

* Taking $\vec{k} \times$ Eq. (16.25) and expanding the triple vector product,

$$\boxed{k^2 \vec{E}_1 - \vec{k}(\vec{k} \cdot \vec{E}_1) = \left(\frac{\omega}{c}\right)^2 \left(\vec{E}_1 + i\vec{j}_1 / \epsilon_0 \omega\right)} \qquad (16.27)$$

* For longitudinal waves with $\vec{k} \parallel \vec{E}_1$, LHS $= 0$ and we have an electrostatic wave in which Poisson eqn plays an important role.

* We consider here a transverse wave with $\vec{k} \perp \vec{E}_1$ (i.e., $\vec{k} \cdot \vec{E}_1 = 0$). $\Rightarrow$ no need to consider Poisson eqn or density perturbation or continuity eqn.

* $$\boxed{\vec{j}_1 = -n_0 e \vec{u}_1} \quad (16.28)$$

(assume stationary ions for high frequency waves as before).

where

$$\boxed{-i\omega m \vec{u}_1 = -e \vec{E}_1} \quad (16.29)$$

$\Rightarrow$ Eq. (16.27) becomes,

$$(c^2 k^2 - \omega^2) \vec{E}_1 = i\omega \vec{j}_1 / \epsilon_0 = -\left(\frac{n_0 e^2}{m \epsilon_0}\right) \vec{E}_1 \quad (16.31)$$

$$\boxed{\omega^2 = c^2 k^2 + \omega_p^2} \qquad (16.32)$$

Dispersion Relation for "EM wave" in unmagnetized plasma.

(*) Note that:

$$v_p \equiv \frac{\omega}{k} = c\left(1 + \frac{\omega_p^2}{c^2 k^2}\right)^{1/2} \underline{> c}.$$

while

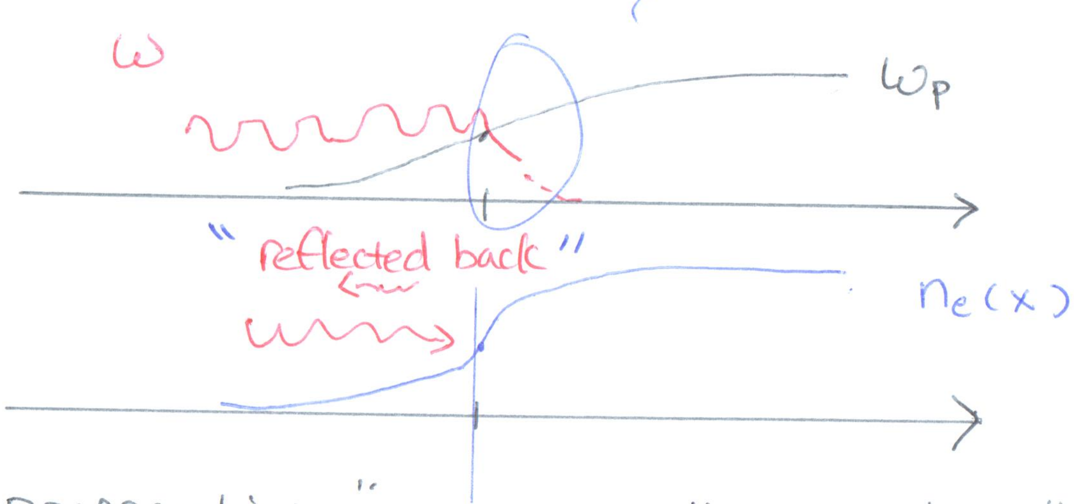
$$v_g \equiv \frac{\partial \omega}{\partial k} = \frac{c^2}{v_p} < c.$$

This is NOT a violation of the special relativity because the information and energy propagate with v_g not v_p.

- low k ; ⇝ constant freq. non-propagating ~~osc~~ oscillation
- hi k ; ⇝ EM waves in vacuum propagating with "c".

(*) Homework   Problem 16.2 on page 266.

(*) EM waves cannot propagate in a plasma with $\omega_p > \omega$.



"propagation" with $\omega > \omega_p$

Cut-off layer

"propagation" not possible

with cut-off density

$$n_c = m_e \epsilon_0 \omega^2 / e^2 .$$

For $\omega_p > \omega$,

D.R. admits "k" which is imaginary,

i.e.,

$$k = (\omega^2 - \omega_p^2)^{1/2}/c$$

$$= \pm i\,(\omega_p^2 - \omega^2)^{1/2}/c$$

(16.37)

- $\exp(ikx) = \exp\left(-x \dfrac{(\omega_p^2 - \omega^2)^{1/2}}{c}\right)$

  evanescent solution

- Collisionless Skin depth:

  $\equiv$ "c/ω_p"