

$$\therefore \frac{\partial \eta}{\partial t}(x, t) = \frac{\partial \phi}{\partial y}(x, 0, t).$$

- Dynamic condition:

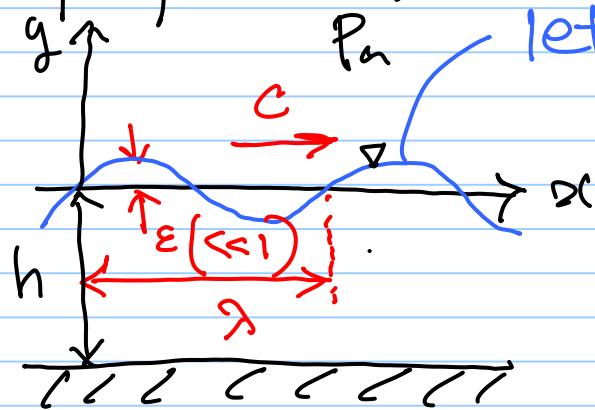
- $\nabla \phi \cdot \nabla \phi = \bar{u} \cdot \bar{u} \sim 0$ (small amplitude)
- $\frac{\partial \phi}{\partial t}(x, \eta, t) \approx \frac{\partial \phi}{\partial t}(x, 0, t)$ (Taylor Series Exp.)
- $F(t) \rightarrow$ included info $\phi(x, y, t)$.

$$\underbrace{\frac{\partial \phi}{\partial t}(x, 0, t) + \frac{1}{\rho} p(x, t) + g \cdot \eta(x, t)}_{F(x, y, t)} = 0.$$

$$\frac{\partial}{\partial t} \underbrace{\frac{\partial^2 \phi}{\partial t^2}(x, 0, t) + \frac{1}{\rho} \frac{\partial p}{\partial t}(x, t) + g \cdot \frac{\partial \eta}{\partial t}(x, t)}_{= 0} = 0.$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial t^2}(x_0, t) + \frac{1}{g} \frac{\partial p}{\partial t}(x_0, t) + g \frac{\partial^2 \phi}{\partial y^2}(x_0, t) = 0.$$

* Propagation of surface waves (travelling wave)



let's assume

$$\eta(x, t) = \epsilon \cdot \sin \frac{2\pi}{\lambda} (x - ct)$$

and

$$p = \text{constant } (= P_a)$$

(↑ i.e., ignore surf.-tension)

$$\cdot \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$|\cdot \frac{\partial \phi}{\partial y}(x_0, 0, t) = \frac{\partial \eta}{\partial t}(x_0, t) = -\epsilon \frac{2\pi c}{\lambda} \cdot \cos \frac{2\pi}{\lambda} (x_0 - ct) \quad \because \text{static}$$

$$\left\{ \begin{array}{l} \cdot \frac{\partial^2 \phi}{\partial t^2}(x, 0, t) + g \frac{\partial \phi}{\partial y}(x, 0, t) = 0 : \text{dynamic.} \\ \cdot \frac{\partial \phi}{\partial y}(x, -h, t) = 0. \end{array} \right.$$

→ separation of variables.

\nwarrow trigonometric in "x".
 \searrow exponential

or hyperbolic in "y"

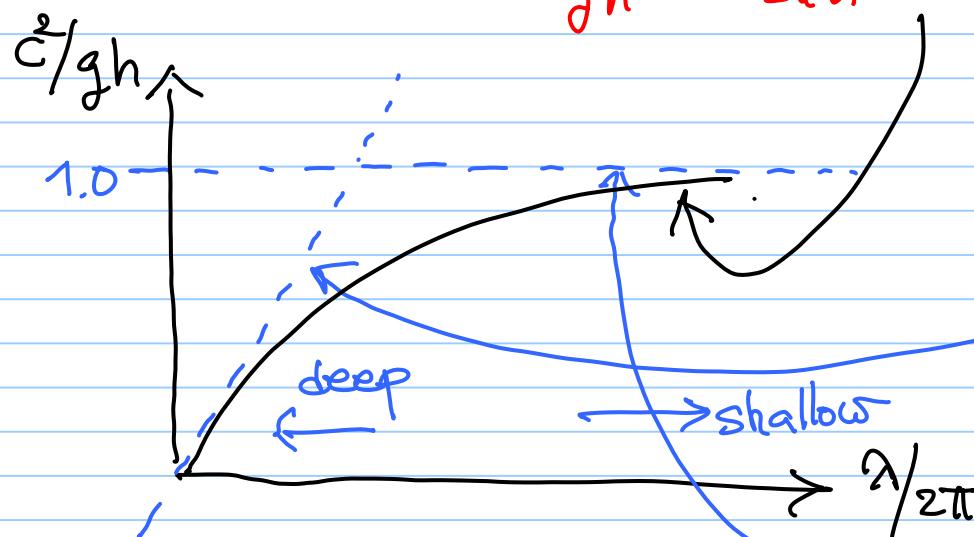
$$\Rightarrow \phi(x, y, t) = \cos \frac{\Sigma \pi l}{\lambda} (x - ct) \left[C_1 \sinh \frac{\Sigma \pi l y}{\lambda} + C_2 \cosh \frac{\Sigma \pi l y}{\lambda} \right].$$

- apply BC @ $y = -h$, $\therefore C_1 = C_2 \cdot \tanh \frac{\Sigma \pi l h}{\lambda}$

$$\phi = C_2 \cdot \cos \frac{\Sigma \pi l}{\lambda} (x - ct) \left[\tanh \frac{\Sigma \pi l h}{\lambda} \sin \frac{\Sigma \pi l y}{\lambda} + \cosh \frac{\Sigma \pi l y}{\lambda} \right].$$

• apply dynamic BC

$$\Rightarrow \frac{c^2}{gh} = \frac{\gamma}{2\pi h} \cdot \tanh \frac{2\pi h}{\gamma} \quad (\varepsilon \ll \lambda \text{ and } \varepsilon \ll h)$$



↓ limiting cases.

① $h \gg \lambda$ (deep liquids)

$$\frac{c^2}{gh} \approx \frac{\gamma}{2\pi h}$$

② $h \ll \lambda$ (shallow liquids)

$$\frac{c^2}{gh} \approx 1 \cdot f(\lambda)$$

• arbitrary shaped waves.

 ↳ Fourier decomposition.

$$\hookrightarrow \sum_{n=0}^{\infty} (\sin \omega_n)$$

• if not a shallow liquids

$$\hookrightarrow c_n = f(\lambda_n)$$

⇒ every component of the wave

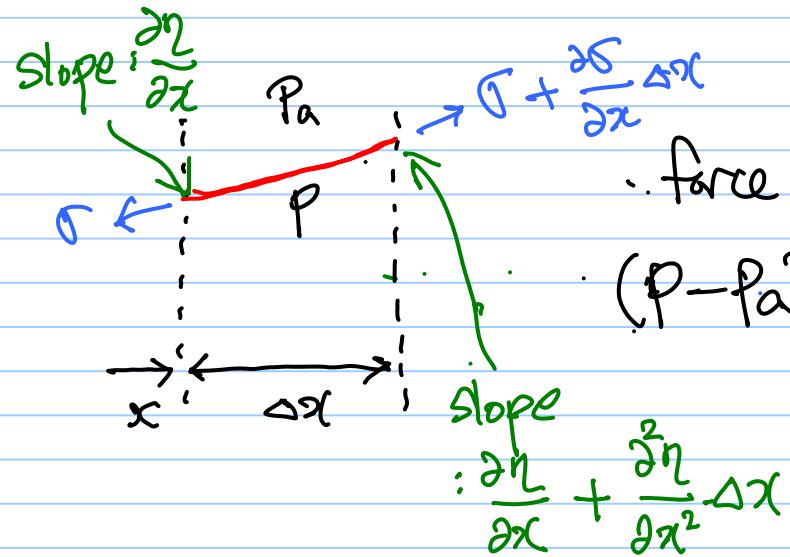
will travel at different speeds

→ wave form is continuously changing

→ "dispersion" (propagation in a disturbed way).

* Effect of surface tension. ($P \neq P_a$ @ interface)

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force balance in vertical direction.

$$(P - P_a) \Delta x + \left(\sigma + \frac{\partial \sigma}{\partial x} \Delta x \right) \cdot \left(\frac{\partial \eta}{\partial x} + \frac{\partial^2 \eta}{\partial x^2} \Delta x \right)$$

$$= \sigma \cdot \frac{\partial \eta}{\partial x}, \quad (\sigma = \text{const.})$$

$$(P - P_a) + \sigma \frac{\partial \eta}{\partial x^2} + \cancel{\frac{\partial \sigma}{\partial x} \cdot \frac{\partial \eta}{\partial x}} = 0$$

$$\Rightarrow P(x, t) = P_a - \sigma \frac{\partial \eta}{\partial x^2}$$

$$\rightarrow \frac{\partial p}{\partial t} = -\sigma \frac{\partial^2}{\partial x^2} \left(\frac{\partial \eta}{\partial t} \right) = -\sigma \frac{\partial^3 \phi}{\partial x^2 \partial y} (x, 0, t).$$

\nwarrow static BC.

then, dynamic BC: $\frac{\partial \phi}{\partial t} (x, 0, t) + \frac{\sigma}{\rho} \frac{\partial^3 \phi}{\partial x^2 \partial y} (x, 0, t) + g \frac{\partial \phi}{\partial y} (x, 0, t) = 0$

revisit traveling wave:

$$\phi(x, y, t) = C_2 \cos \frac{2\pi}{\lambda} (x - ct) \left(\tanh \frac{2\pi h}{\lambda} \sinh \frac{2\pi y}{\lambda} + \cosh \frac{2\pi y}{\lambda} \right)$$

$$\hookrightarrow \frac{c^2}{gh} = \frac{1}{2\pi h} \left[1 + \underbrace{\frac{\sigma}{\rho g} \left(\frac{2\pi}{\lambda} \right)^2}_{\text{effect of surface tension.}} \right] \tanh \frac{2\pi h}{\lambda}$$

for deep liquids ($h \gg \lambda$)

$$\frac{c^2}{gh} \approx \frac{\lambda}{2\pi h} \left(1 + \frac{\sigma}{\rho g} \left(\frac{2\pi}{\lambda} \right)^2 \right)$$

if $\frac{\sigma}{\rho h} \left(\frac{2\pi}{\lambda} \right)^2 \gg 1 \rightarrow \frac{c^2}{gh} \approx \frac{2\pi\sigma}{\rho g \lambda h}$, capillary wave.

($c \sim f(\lambda)$ only)

