# Ch. 17. High-Frequency Waves in Magnetized Plasmas

⊛ In the presence of a strong equilibrium magnetic field, $\vec{B}_0$,

the medium is anisotropic.

## $\boxed{17.1.}$ High-Frequency EM waves propagating $\perp$ to $\vec{B}_0$

⊛ $\vec{K} \perp \vec{B}_0$ :

1. O-waves : $\vec{E}_1 \parallel \vec{B}_0$ $\rightarrow$ $\vec{B}_0$ plays no role in wave dynamics.

"ordinary" —

$\Rightarrow$ we can apply the results for unmagnetized plasma with $\vec{u}_1 \parallel \vec{E}_1$.

$\vec{u}_1 \times \vec{B}_0 = 0$ and O-mode (O-wave) never notices $\vec{B}_0$.

$\Rightarrow$ Recall that EM waves in unmagnetized plasma is "transverse":

" $\vec{E}_1 \perp \vec{K}$ " , With $\perp$ propagation ( $\vec{K} \perp \vec{B}_0$ ) :

(*) $\vec{k} \perp \vec{B}_0$ :

(2). X-waves : $\vec{E}_1 \perp \vec{B}_0$

extra-ordinary ...

- In general, $\vec{E}_1$ of this X-wave has a component along $\vec{k}$ ($\perp$ to $\vec{B}_0$) and also a component $\perp$ to both $\vec{k}$ and $\vec{B}_0$.

(*) For $\vec{B}_0 = B_0 \hat{z}$ and $\vec{k} = k\hat{x}$,

$$\vec{E}_1 = E_{1x}\hat{x} + E_{1y}\hat{y}.$$

- For this hi-freq. wave, we will take the ions to be stationary.

- In addition, we will neglect the electron pressure assuming a cold plasma with $T_i \simeq T_e \cong 0$.

(*) Recall the wave equation, (16.27):

$$k^2 \vec{E}_1 - \vec{k}(\vec{k}\cdot\vec{E}_1) = \left(\frac{\omega}{c}\right)^2 \left(\vec{E}_1 + i\vec{j}_1/\epsilon_0\omega\right), \qquad (17.4)$$

where $\vec{j}_1 = -n_0 e \vec{u}_1$ carried by electrons.

* $\vec{u}_1$ can be obtained from the linearized electron fluid eqn of motion including the Lorentz force, $-e(\vec{u}_1 \times \vec{B}_0)$,

$$-i\omega m\, u_{x1} = -e\,(E_{x1} + u_{y1} B_0) \qquad (17.1)$$

$$-i\omega m\, u_{y1} = -e\,(E_{y1} - u_{x1} B_0) \qquad \cancel{(17.2)}$$

* This can be solved for $u_{x1}$ and $u_{y1}$, which will be used in Eq. (17.4)...

$$u_{x1} = \frac{\left(\frac{e}{m}\right)(i\omega E_{x1} + \omega_c E_{y1})}{\omega_c^2 - \omega^2} \qquad (17.3)$$

$$u_{y1} = \frac{\left(\frac{e}{m}\right)(i\omega E_{y1} - \omega_c E_{x1})}{\omega_c^2 - \omega^2},$$

where $\omega_c = \frac{eB_0}{m}$ is the electron cyclotron freq.

* From this procedure, we can obtain two coupled eqns for $E_{x1}$ and $E_{y1}$. Determinant of 2×2 matrix (of coefficients) $= 0$

$\Rightarrow$ after some arrange-ments ;

$\perp$ propⁿ, X-mode

$$\boxed{\frac{c^2k^2}{\omega^2} = \frac{c^2}{v_p^2} = 1 - \frac{\omega_p^2\,(\omega^2 - \omega_p^2)}{\omega^2\,(\omega^2 - \omega_h^2)}} \tag{17.12}$$

where $\omega_h^2 \equiv \omega_p^2 + \omega_c^2$, "Upper-hybrid" frequency

* Resonance :

- At $\omega = \omega_h$, $k \to \infty$ $(\lambda \to 0)$, $v_p \to 0$ and wave fronts pile up.

- We can also check that $E_{y1}/E_{x1} \to 0$ at $\omega = \omega_h$, $\vec{E}_1 = E_{x1}\hat{x} \parallel \vec{k} = k\hat{x}$ at resonance, and wave is electrostatic at resonance.

⊛ Cutoffs ; where $k \to 0$ $(\lambda \to \infty)$.

From Eq. (17.12), numerator on RHS $= 0$ for $k = 0$,

$\Rightarrow$

$$\omega = \left[ (\omega_c^2 + 4\omega_p^2)^{1/2} \pm \omega_c \right] \equiv \begin{cases} \omega_R \\ \omega_L \end{cases} \qquad (17.19)$$

⊛ Examination of Eq. (17.12); $\Rightarrow$

Waves can propagate for $\omega > \omega_R$ and $\omega_h > \omega > \omega_L$,

but cannot 〃 $\omega_R > \omega > \omega_L$ and $\omega_L > \omega$.

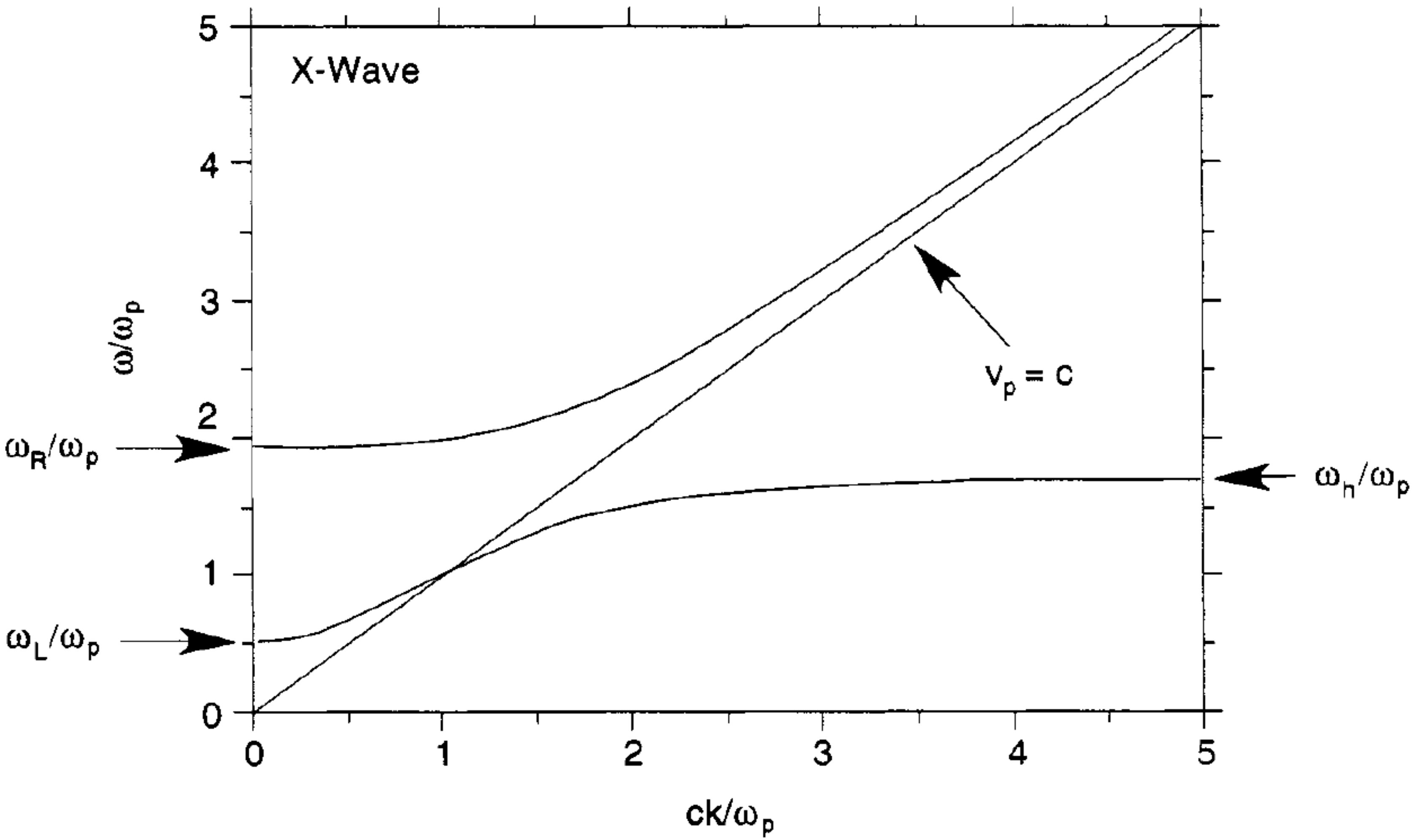


**Figure 17.1.** Dispersion relation for the extraordinary wave propagating perpendicular to **B** in a magnetized plasma, with $\omega_c^2$ chosen to be equal to $2\omega_p^2$.

(*) Discussion related to Experiments:

- In general, a fixed-frequency wave is driven by a generator.

- It is much easier to arrange a wave propagating up a density gradient, since the RF source is usually located outside of the plasma.

- It is possible to arrange a wave propagating down a $\vec{B}_o$ gradient (high-field side launch in tokamak).

Recall that $\omega_p^2 \propto n_e$ and $\omega_c \propto |\vec{B}_o|$.

$\Rightarrow$ It's easy to arrange a wave to propagate and reach $\omega_R$-cutoff.

$\Rightarrow$ To make a wave propagate and reach $\omega_h$-resonance, one should rely on $|\vec{B}_o|$ variation in space.

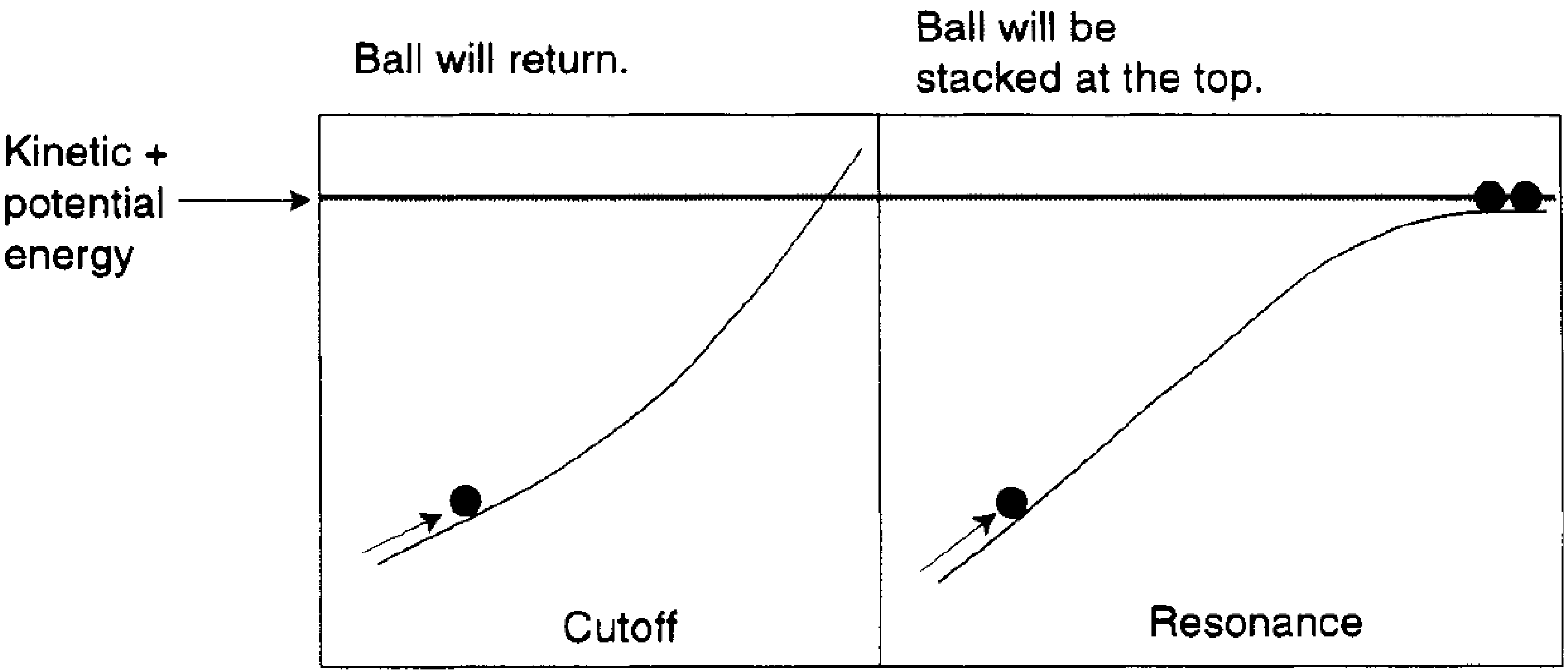


**Figure 17.2.** Mechanical analog to wave cutoffs and resonances.

(*) Within the context of $\perp$ propagation of EM waves based on linear cold-plasma theory, the wave amplitude must grow steadily at the upper-hybrid resonance layer when we pump energy from the outside.

(*) On the other hand, it can be shown that the wave energy is reflected at the cut-off (accelerating the wave back out of the plasma) at $\omega = \omega_p$ in an unmagnetized plasma.

(by using an EM wave example)

| Homework : Problem 17.2 on page 277 |
|---|

# * 17.2. Hi-Freq. EM waves propagating // $\vec{B}_0$

(*) Consider $\vec{k} \parallel \vec{B}_0$ and high frequency limit such that ions can be considered to be stationary.

Once again, we use the wave eqn.

$$k^2 \vec{E}_1 - \vec{k}(\vec{k}\cdot\vec{E}_1) = \left(\frac{\omega}{c}\right)^2 \left[\vec{E}_1 + i\,\underline{\vec{j}_1}/\epsilon_0\omega\right]. \quad (17.27)$$

(*) Since a longitudinal mode ($\vec{E}_1 \parallel \vec{k}$) corresponds to the electrostatic Langmuir wave, we consider a new <u>EM wave with $\vec{k}\cdot\vec{E}_1 = 0$ (transverse wave)</u>.

(*) Once again, we can solve the linearized electron fluid eqn of motion, to get ~~ex~~ expressions for $u_{x1}$ and $u_{y1}$ in terms of $E_{x1}$ and $E_{y1}$ and substitute to $\underline{\vec{j}_1 = -n_e e \vec{u}_1}$.

(*) Once again, we get a coupled set of eqns for $E_{x1}$ and $E_{y1}$,

$$\begin{bmatrix} \text{matrix} \end{bmatrix} \begin{bmatrix} E_{x1} \\ E_{y1} \end{bmatrix} = 0 . \qquad \text{Det} \begin{bmatrix} \text{matrix} \end{bmatrix} = 0 \quad \text{yields},$$

$$\boxed{\tilde{n}^2 \equiv \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_c)}} \qquad (17.35)$$

$\tilde{n}$ : the index of refraction

- "+" and "−" signs correspond to "L-wave" and "R-wave" respectively.
- Both correpond to "circularly polarized waves."
- $E_{x1}$ and $E_{y1}$ are $\pi/2$ out of phase.

[ L: / R: ] wave's $\vec{E}_1$ vector rotates according to a [ left-hand / right-hand ] rule.

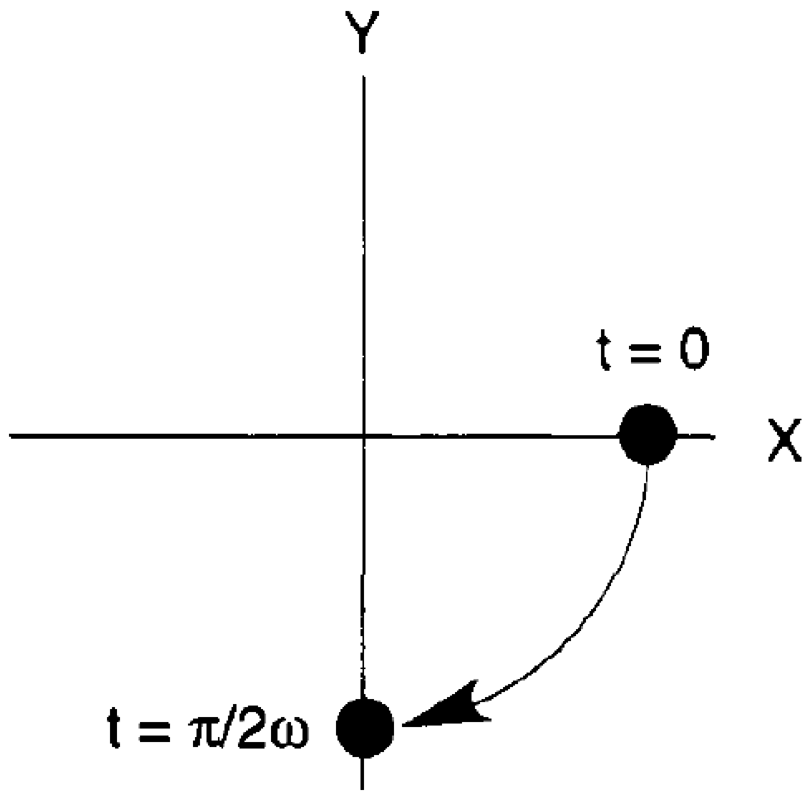


**Figure 17.3.** Time progression of the **E**-field vector for a left-hand circularly polarized wave with $\mathbf{B}_0$ along the *z* direction, out of the page.

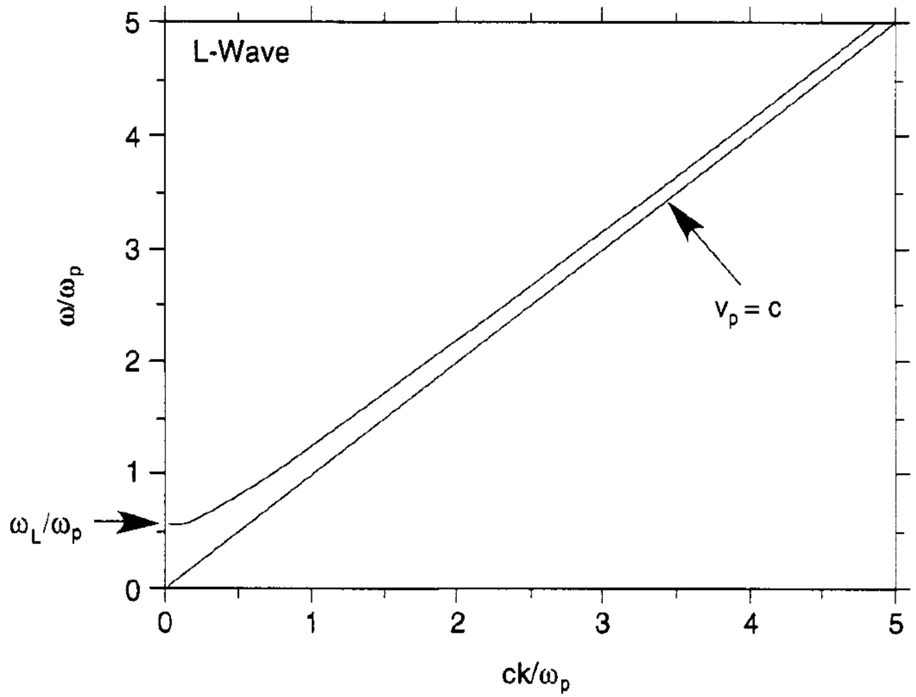


**Figure 17.4.** Left-hand circularly polarized electromagnetic wave propagating parallel to $\mathbf{B}_0$ in a magnetized plasma, with $\omega_c^2$ chosen to equal $2\omega_p^2$.

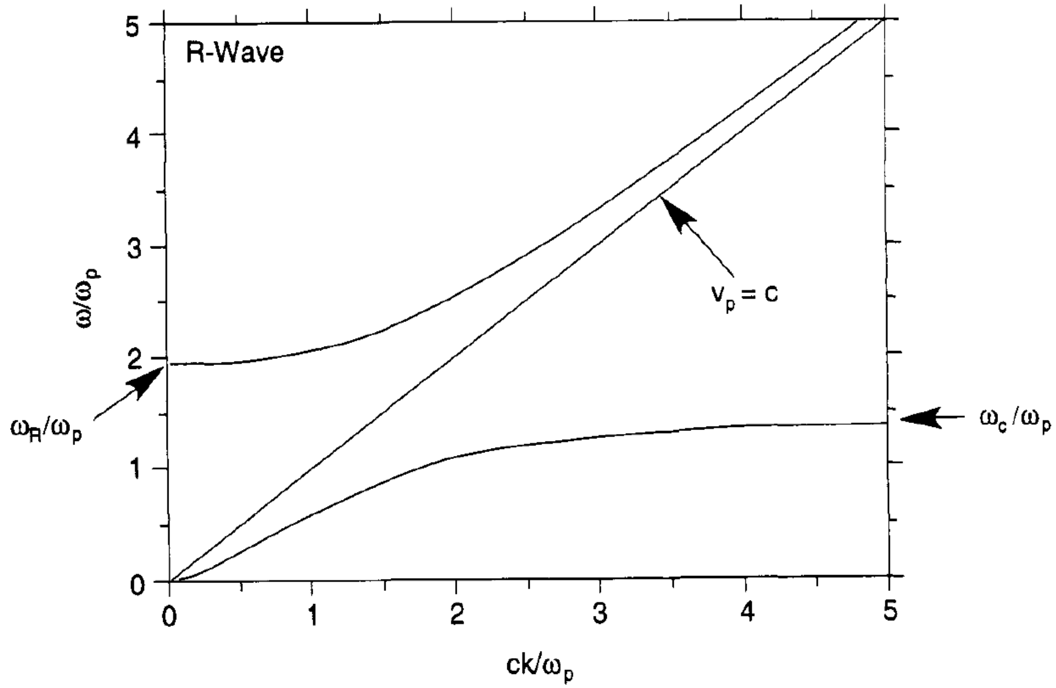


**Figure 17.5.** Right-hand circularly polarized electromagnetic wave propagating parallel to $\mathbf{B}_0$ in a magnetized plasma, with $\omega_c^2$ chosen to equal $2\omega_p^2$.

## ⊛ $\boxed{\text{Whistler wave}}$:

⊛ R-wave in the region of $\omega < \omega_c$.

( lower frequency range )

⊛ One can show that $v_g = \frac{\partial \omega}{\partial k} \nearrow$ as $\omega \nearrow$.

→ White radio noise generated in a burst in the ionosphere due to lightning flashes, and propagating as a whistler wave will travel faster at high frequencies than a low.

⇒ Ground-based receiver will then hear a "whistle" going from hi-freq. to low due to lighting flashes.

# ⊛ Faraday Rotation :

⊛ For the same freq. "$\omega$", the upper band of the R-wave ($\omega > \omega_R$) has a higher phase velocity $v_p$ than the corresponding L-wave at the same "$\omega$".

⟹ When a linearly polarized EM wave propagates // to $\vec{B}_0$, the angle of polarization of the wave rotates as it travels.

⊛ This is illustrated on page 283 of G and R.

⊛ Since the resulting difference in the phase depends on $\omega_c$ and $\omega_p$, one can determine the magnetic field in plasma if density of plasma is known from other means.

For details, try Problem 17.1 and 17.4.

# Ch. 18. Low-frequency Waves in Magnetized Plasmas.

## 18.1. Dielectric Tensor

* For low frequency waves, ion dynamics has to be considered.
* We will also consider finite pressure and arbitrary angle of propagation
* Take $\vec{B}_0 = B_0 \hat{z}$, $\vec{k} = k_x \hat{x} + k_z \hat{z}$, $= k(\sin\theta\, \hat{x} + \cos\theta\, \hat{z})$

(*) For each species, we have

$$m n_0 \frac{\partial}{\partial t} \vec{u}_1 = q n_0 (\vec{E}_1 + \vec{u}_1 \times \vec{B}_0) - \gamma T \vec{\nabla} n_1 \qquad (18.1)$$

and

$$\vec{\nabla} \cdot (n_0 \vec{u}_1) = -\frac{\partial}{\partial t} n_1 \qquad (18,5)$$

* $\vec{J}_1 = \sum_{sp} n_0 q \, \vec{u}_1 \equiv \overleftrightarrow{\sigma} \cdot \vec{E}_1$ (18.9)

* $\overleftrightarrow{\sigma}$ : a complex frequency-dependent tensor electrical conductivity.

* Wave Eqn :

$$k^2 \vec{E}_1 - \vec{k}(\vec{k} \cdot \vec{E}_1) = \left(\frac{\omega}{c}\right)^2 \left(\vec{E}_1 + i\vec{j}_1/\epsilon_0\omega\right)$$

$$\Downarrow$$

$$\left(\frac{\omega}{c}\right)^2 \mu_0 \, \overleftrightarrow{\epsilon} \cdot \vec{E}_1$$

* Di-electric tensor :

$$\overleftrightarrow{\epsilon} = \epsilon_0 \left( \overleftrightarrow{I} + i\overleftrightarrow{\sigma}/\epsilon_0\omega \right) \quad (18.13)$$

By defining $\overleftrightarrow{X} \equiv \overleftrightarrow{I} - \vec{k}\vec{k}/k^2$, the wave equation can be written in a compact form,

$$\boxed{\left(\omega^2 \mu_0 \overleftrightarrow{\epsilon} - k^2 \overleftrightarrow{X}\right) \cdot \vec{E}_1 = 0} \quad (18.15)$$

* Eq (18.15) leads to the "warm" plasma dispersion relation.

## 18.2. The Cold-Plasma Dispersion Relation,

* Take $T=0$ and define,

- $\tilde{n} \equiv ck/\omega = c/v_p$, $\rightarrow$ $c\vec{k}/\omega \equiv \tilde{n}(\sin\theta\,\hat{x} + \cos\theta\,\hat{z})$

- $R \equiv 1 - \dfrac{\omega_p^2}{\omega(\omega-\omega_c)} - \dfrac{\Omega_p^2}{\omega(\omega+\Omega_c)}$

- $L \equiv 1 - \dfrac{\omega_p^2}{\omega(\omega+\omega_c)} - \dfrac{\Omega_p^2}{\omega(\omega-\Omega_c)}$

- $S \equiv \dfrac{R+L}{2}$

- $D \equiv \dfrac{R-L}{2}$

- $P \equiv 1 - \dfrac{\omega_p^2}{\omega^2} - \dfrac{\Omega_p^2}{\omega^2}$

$\omega_p, \Omega_p$ : electron and ion plasma freq's

$\omega_c, \Omega_c$ : " cyclotron " .

(*) Then, the wave equation (18.15) can be written as

$$\begin{bmatrix} \hat{x}\hat{x}(S-\tilde{n}^2\cos^2\theta) & -\hat{x}\hat{y}\, iD & +\hat{x}\hat{z}\tilde{n}^2\sin\theta\cos\theta \\ +\hat{y}\hat{x}\, iD & +\hat{y}\hat{y}(S-\tilde{n}^2) & +0 \\ +\hat{z}\hat{x}\,\tilde{n}^2\sin\theta\cos\theta & +0 & +\hat{z}\hat{z}(P-\tilde{n}^2\sin^2\theta) \end{bmatrix} \cdot \vec{E}_1 = 0 \qquad (18.16)$$

* $\boxed{\text{Homework: Problem 18.1 on page 288.}}$

Determinant of (18.16) $=0$

$$\Rightarrow (S^2P - D^2P) - \underline{\tilde{n}^2}(SP\cos^2\theta + SP + S^2\sin^2\theta - D^2\sin^2\theta)$$

$$+\underline{\tilde{n}^4}(P\cos^2\theta + S\sin^2\theta) = 0.$$

$\rightarrow$ Two branches of Dispersion Relation

$$\Rightarrow \quad \boxed{\tan^2\theta = \frac{-P(\tilde{n}^2 - R)(\tilde{n}^2 - L)}{(S\tilde{n}^2 - RL)(\tilde{n}^2 - P)}} \tag{18.24}$$

A useful form of cold-plasma D.R.

(*) For // propagation, $\theta = 0$ by definition.

We have two solutions, $\tilde{n}^2 = R$ and $\tilde{n}^2 = L$ which are the right- and left-circularly polarized waves,

(*) For $\perp$ propagation, $\tan^2\theta \to \infty$.

We have two solutions, $\tilde{n}^2 = P$ and $\tilde{n}^2 = \frac{RL}{S}$.

O-wave X-wave

Note that ion dynamics are included via the definitions of R, L, S and P.
contributions from

(*) Resonances $\Rightarrow$ $\tilde{n} \to \infty$ ($k \to \infty$, $\lambda \to 0$),

$\Rightarrow$ $\tan^2\theta = -P/S$, i.e., resonance freqs. vary with the angle of propagation.

* For $\theta = 0$, $P = 0$ corresponds to $\omega = \omega_p$ resonance.

$S \to \infty$ correspond to $\begin{cases} R \to \infty, & \omega = \omega_c \text{ resonance}, \\ \text{and} & \\ L \to \infty, & \omega = \Omega_c \text{ ″} \end{cases}$,

* For $\theta = \pi/2$,

$S \to 0$ $\to$ upper and lower-hybrid resonances, including ion dynamics.

(*) Cutoffs $\Rightarrow$ We must go back to Eq. (18.18), to get

$PRL = 0$,

$P = 0$ $\to$ $\omega_p$ cutoff of O-wave and Langmuir wave

$R = 0$ $\to$ $\omega_R$
$L = 0$ $\to$ $\omega_L$ cutoffs with ion dynamics included,