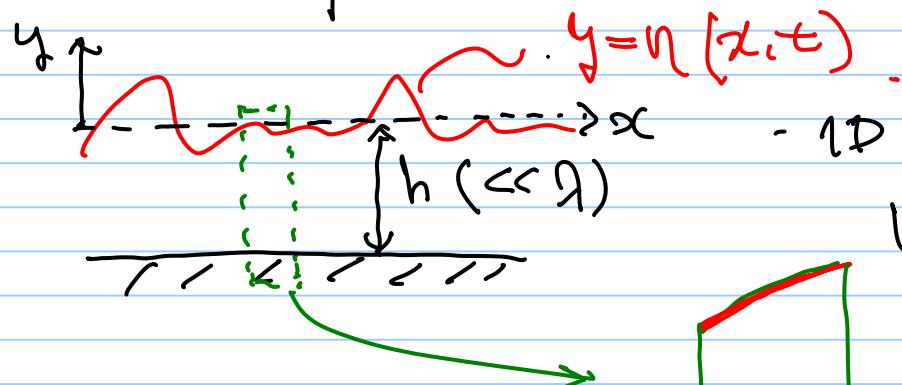


* Shallow liquid waves of arbitrary form.



- 1D approximation.

$$u = u(x), \nabla v = 0.$$

- mass conservation

$$\frac{\partial \eta}{\partial t} + h \frac{\partial u}{\partial x} = 0$$

- momentum conservation

$$\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} = 0.$$

$$\left. \begin{aligned} \frac{\partial^2 u}{\partial t^2} - g h \frac{\partial^2 u}{\partial x^2} &= 0 \\ \frac{\partial^2 \eta}{\partial t^2} - g h \frac{\partial^2 \eta}{\partial x^2} &= 0 \end{aligned} \right\} : 1D \text{ wave equation.}$$

↳ general solution of

$$u(x,t) = f_1(x - \sqrt{gh}t) + g_1(x + \sqrt{gh}t)$$

$$\eta(x,t) = f_2(x - \sqrt{gh}t) + g_2(x + \sqrt{gh}t)$$

(f_1, f_2, g_1, g_2 : any differentiable functions)

: single propagation speed.

(arbitrary wave travelling along the surface

of shallow liquid does not disperse!)
 → shape does not change.

* Complex potential for traveling waves.

$$\eta(x, t) = \varepsilon \sin \frac{2\pi}{\lambda} (x - ct), \quad \varepsilon \ll 1, \text{ arbitrary depth.}$$

$$\phi(x, y, t) = C_2 \cos \frac{2\pi}{\lambda} (x - ct) \left(\tanh \frac{2\pi h}{\lambda} \sinh \frac{2\pi y}{\lambda} + \cosh \frac{2\pi y}{\lambda} \right).$$

(apply static BC to determine C_2)

$$\hookrightarrow \frac{\partial \eta}{\partial t}(x, t) = \frac{\partial \phi}{\partial y}(x, 0, t).$$

$$\rightarrow C_2 = - \frac{ce}{\tanh \frac{2\pi h}{\lambda}}.$$

$$\therefore \phi(x, y, t) = -CE \cos \frac{2\pi}{\lambda} (x - ct) \left(\sinh \frac{2\pi y}{\lambda} + \coth \frac{2\pi h}{\lambda} \cosh \frac{2\pi y}{\lambda} \right).$$

$$U = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}, \quad V = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}.$$

$$\rightarrow \psi(x, y, t) = CE \sin \frac{2\pi}{\lambda} (x - ct) \left(\cosh \frac{2\pi y}{\lambda} + \coth \frac{2\pi h}{\lambda} \sinh \frac{2\pi y}{\lambda} \right).$$

$$\rightarrow F(z, t) = \phi + iV.$$

$$(z = x + iy) = - \frac{CE}{\sinh \frac{2\pi h}{\lambda}} \cos \frac{2\pi}{\lambda} (z - ct + ih),$$

* Standing Waves. (remain stationary, moves vertically)

$$\eta_1(x,t) = \frac{1}{2} \varepsilon \sin \frac{2\pi}{\lambda} (x - ct), \quad \eta_2(x,t) = \frac{1}{2} \varepsilon \sin \frac{2\pi}{\lambda} (x + ct).$$

$$\begin{aligned}\rightarrow \eta = \eta_1 + \eta_2 &= \frac{1}{2} \varepsilon \left[\sin \frac{2\pi}{\lambda} (x - ct) + \sin \frac{2\pi}{\lambda} (x + ct) \right] \\ &= \frac{1}{2} \varepsilon \cdot \sin \frac{2\pi x}{\lambda} \cdot \cos \frac{2\pi ct}{\lambda}.\end{aligned}$$

• complex potential.

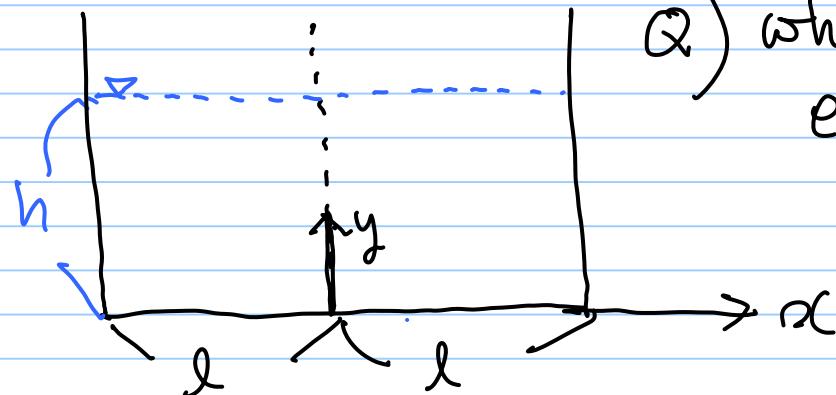
$$F(z,t) = \frac{c\varepsilon/2}{\sinh \frac{2\pi h}{\lambda}} \left[-\cos \frac{2\pi}{\lambda} (z - ct + ih) + \cos \frac{2\pi}{\lambda} (z + ct + ih) \right]$$

infinite space.

$$= - \frac{c\varepsilon}{\sinh \frac{2\pi h}{\lambda}} \cdot \sin \frac{2\pi}{\lambda} (z + ih) \sin \frac{2\pi ct}{\lambda}.$$

→ practically, confined space condition is more relevant.

* Waves in 2D rectangular vessels.



Q) what types of standing waves do exist, if any?

$$-\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \quad (1)$$

- kinematic B.C : (X)

- pressure condition: $\frac{\partial^2 \phi}{\partial t^2}(x, h, t) + g \frac{\partial^2 \phi}{\partial y^2}(x, h, t) = 0 \quad (2)$

- $\frac{\partial \phi}{\partial y}(x, 0, t) = 0, \quad \frac{\partial \phi}{\partial x}(\pm l, y, t) = 0. \quad (3) \quad (4)$

$$\rightarrow \phi(x,y,t) = \left(A_1 \sin \frac{2\pi x}{\lambda} + A_2 \cos \frac{2\pi x}{\lambda} \right) \left(B_1 \sinh \frac{2\pi y}{\lambda} + B_2 \cosh \frac{2\pi y}{\lambda} \right) \sin \frac{2\pi ct}{\lambda}.$$

trigonometric in "x"

due to side wall

@ $x = \pm l$.

hyperbolic in "y"

to satisfy
Laplace eq.

↑
standing
-wave
type.

$$\phi(x, y, t) = \left(A_1 \sin \frac{2\pi x}{\lambda} + A_2 \cos \frac{2\pi x}{\lambda} \right) \left(B_1 \sinh \frac{2\pi y}{\lambda} + B_2 \cosh \frac{2\pi y}{\lambda} \right) \sin \frac{2\pi ct}{\lambda}.$$

• apply BC ③. $\left(\frac{\partial \phi}{\partial y}(x, 0, t) = 0 \right) \Rightarrow B_1 = 0.$

• apply pressure condition ② :

$$\frac{C^2}{gh} = \tanh \frac{2\pi h}{\lambda}$$

• apply BC ④ $\left(\frac{\partial \phi}{\partial x}(\pm l, y, t) = 0 \right)$

$$\rightarrow P_1 \cos \frac{2\pi l}{\lambda} = \pm P_2 \sin \frac{2\pi l}{\lambda}, \text{ if } P_1 = P_2 = 0 \text{ (trivial)}$$

i) let $P_1 \neq 0$ and $P_2 = 0$: $\cos \frac{2\pi l}{\lambda} = 0 \rightarrow \lambda_n = \frac{4l}{2n+1}$

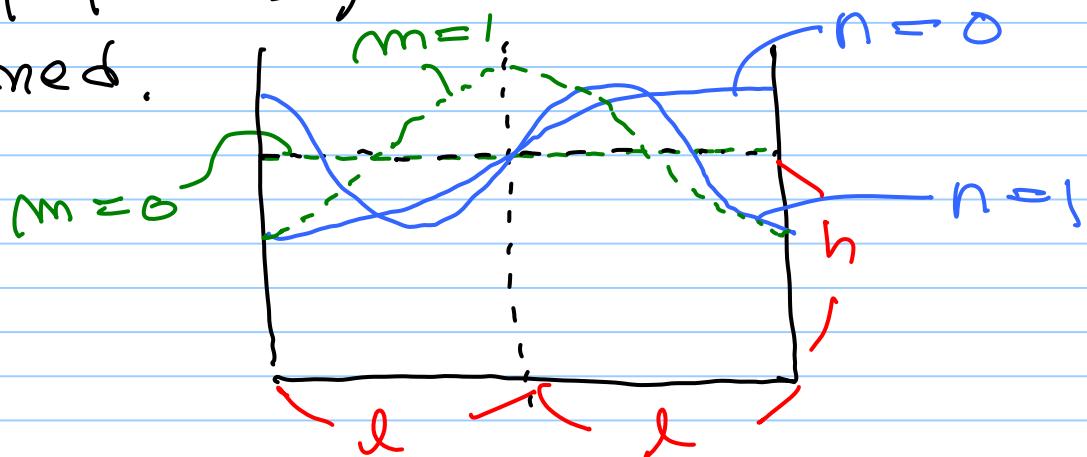
$$\Rightarrow \phi_n(x, y, t) = P_{in} \cdot \sin \frac{(2n+1)\pi x}{2l} \cosh \frac{(2n+1)\pi y}{2l} \sin \frac{(2n+1)\pi ct}{2l} \quad C_n = f(2n)$$

ii) let $D_2 \neq 0$ and $D_1 = 0$: $\sin \frac{2\pi l}{\lambda} = 0 \rightarrow \lambda_m = \frac{2l}{m}$

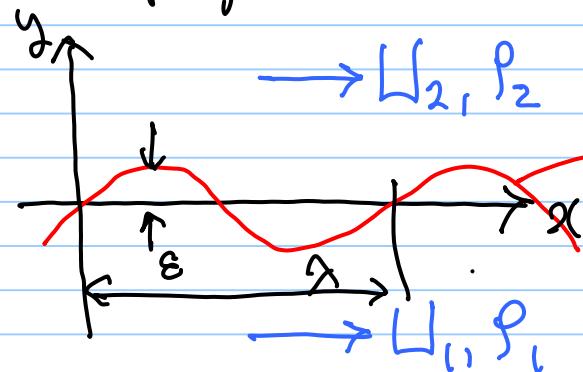
$$\Rightarrow \phi_{lm}(x, y, t) = D_{2lm} \cos \frac{m\pi x}{l} \cosh \frac{m\pi y}{l} \sin \frac{m\pi Cmt}{l} \quad C_m = f(\lambda_m)$$

- only a certain discrete waveforms exist!

↪ by superposition, general waveform can be obtained.



* Propagation of waves at an interface
(between two dissimilar fluids)



$$y = \eta(x, t) = \epsilon e^{i \frac{2\pi}{\lambda} (x - \sigma t)}$$

wave propagation speed.
 σ : real # : wave is traveling in x -direction.

σ : imaginary #

$\sigma/\imath < 0$: decaying

"unstable," $\sigma/\imath > 0$: growing

- To linearize the boundary conditions,

$$: U_i = \underbrace{U_i \hat{e}_x}_{(i=1,2)} + \underbrace{\nabla \phi_i}$$

\downarrow

unit flow.
(un-disturbed flow)

perturbation velocity potential.
(caused by the waves
at the interface)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_i \cdot \nabla = \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x} + \nabla \phi_i \cdot \nabla .$$

\Rightarrow Derive new BC's.

. Governing eqns and BC's.

$$\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial y^2} = 0, \quad |\nabla \phi_i| = \text{finite} .$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0, \quad |\nabla \phi_2| = \text{finite}.$$

• $\frac{\partial \phi_i}{\partial y}(x, 0, t) = \frac{\partial \eta}{\partial t}(x, t) + U_i \frac{\partial \eta}{\partial x}(x, t)$: kinematic BC.

• $\rho_i \frac{\partial \phi_i}{\partial t}(x, 0, t) + \rho_i U_i \frac{\partial \phi_i}{\partial x}(x, 0, t) + \rho_i g \eta(x, t) = \text{const}$: dynamic BC.

• separation of variable + kinematic BC.

$$\Rightarrow \phi_1(x, y, t) = -i\varepsilon(\sigma - U_1) e^{\frac{2\pi}{\lambda} y} \cdot e^{i \frac{2\pi}{\lambda} (x - \sigma t)}$$

$$\phi_2(x, y, t) = i\varepsilon(\sigma - U_2) e^{-\frac{2\pi}{\lambda} y} \cdot e^{i \frac{2\pi}{\lambda} (x - \sigma t)}$$

• apply Dynamic BC. \uparrow

$$\therefore \rho_1 \frac{2\pi}{g} (\sigma - U_1)^2 + \rho_1 g = \rho_2 \frac{2\pi}{g} (\sigma - U_2)^2 + \rho_2 g.$$

$$\hookrightarrow \sigma = \frac{\rho_2 U_2 + \rho_1 U_1}{\rho_2 + \rho_1} \pm \sqrt{\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) \frac{g \lambda}{2\pi} - \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2} (U_2 - U_1)^2}$$

i) $U_1 = U_2 = 0, \rho_2 = 0$

(stationary gas over stationary liquid)

$$\sigma = \pm \sqrt{\frac{g \lambda}{2\pi}} \quad \text{for deep liquids.}$$

$$U_2 = 0, \rho_2 \ll 1$$

$$U_1 = 0, \rho_1 \gg 1$$

: real #.; wave at the interface will propagate
 \rightarrow surface of separation will remain intact!
 (stable)

ii) $\rho_2 = 0$. (gas blowing over a liquid surface)

$$\Rightarrow \sigma = U_1 \pm \sqrt{\frac{g\lambda}{2\pi}}$$

(stable)

$$U_2 \rightarrow 0, \rho_2 \ll 1$$



$$U_1 \rightarrow 0, \rho_1 \gg 1.$$

iii) $\rho_1 = \rho_2, U_1 \neq U_2$.

(homogeneous fluid) shear.

$$\sigma = \frac{U_1 + U_2}{2} \pm i \frac{|U_2 - U_1|}{2}$$

Interfacial wave grows exponentially w/ time.

→ unstable = Helmholtz (Rayleigh) instability.

iv) $\omega_1 = \omega_2 = 0$, $\rho_1 \neq \rho_2$.

$$\sigma = \pm \sqrt{\frac{g\lambda}{2\pi} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)}$$

$\left. \begin{array}{l} \rho_1 > \rho_2 : \text{stable} \\ \quad (\text{heavier fluid at} \\ \quad \text{the bottom}) \end{array} \right\}$

$\left. \begin{array}{l} \rho_1 < \rho_2 : \text{unstable} \\ \quad (\text{Taylor instability}) \end{array} \right\}$

in density stratification.