

Preliminary Study

Probability Distribution

Wha Sook Jeon

Mobile Computing & Communications Lab.

Probability (1)

- Sample space
 - A mathematical abstraction of the collection of all possible experimental outcomes
 - Example
 - If the experiment consists of flipping two coins, the sample space denoted by S is as
$$S = \{ (H, H), (H, T), (T, H), (T, T) \}$$
- Sample point
 - A possible outcome of a real world experiment

Probability (2)

- Event
 - A set of sample points
 - A subset of the sample space
 - Example
 - In the experiment of flipping two coins, the event E that at least one tail occurs is as
$$E = \{ (H, T), (T, H), (T, T) \}$$
 - A simple event: an event with only single sample point

Probability (3)

- Probability
 - Assignment of a real number $P(E)$ to each event E of the sample space S , satisfying the following three axioms
 1. $0 \leq P(E) \leq 1$
 2. $P(S) = 1$
 3. For any sequence of events $E_1, E_2, \dots$ that are mutually exclusive,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

- We refer to $P(E)$ as the probability of the event E

Probability (4)

- Joint probability
 - A probability for the outcomes of combined experiments
 - The joint probability of events A and B: $P(A,B)$
- Conditional probability
 - $P(B|A) = P(A,B)/P(A)$
- Statistical independence
 - $P(A,B) = P(A) P(B)$ or $P(B|A) = P(B)$

Random Variable (1)

- Random variable
 - A real valued function defined on the sample space
 - Because the value of a random variable is determined by the outcome of the experiment, we may assign a probability to a possible value of the random variable
 - A random variable is characterized by a probability distribution function

Random Variable (2)

1. Discrete r.v.
 - Countable number of possible values
2. Continuous r.v.
 - Uncountable number of possible values

Discrete Random Variable (1)

- Discrete random variable, X

- Probability distribution

$$P_X(x) = \Pr\{X = x\}$$

- Cumulative distribution function (CDF)

$$F_X(x) = \sum_{x_i \leq x} P_X(x_i)$$

- Statistically independent random variables, X and Y

$$P_{XY}(x_i, y_i) = P_X(x_i)P_Y(y_i) \quad \text{for all values } (x_i, y_i)$$

$$F_{XY}(x, y) = \sum_{x_i \leq x} \sum_{y_i \leq y} P_X(x_i)P_Y(y_i)$$

Discrete Random Variable (2)

- Mean and variance of a discrete r.v., X

- Mean

$$E[X] = \sum_{\text{all } k} x_k P_X(x_k)$$

- n -th moment

$$E[X^n] = \sum_{\text{all } k} (x_k)^n P_X(x_k)$$

- Variance

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

Discrete Random Variable (3)

- Examples

- Uniform (n_1, n_2)

$$\Pr\{X = k\} = \frac{1}{n_2 - n_1 + 1}, \quad k = n_1, n_1 + 1, \dots, n_2 \quad (n_2 > n_1)$$

- Poisson (λ)

$$\Pr\{X = n\} = \frac{e^{-\lambda} \lambda^n}{n!}, \quad n = 0, 1, 2, \dots$$

- Bernoulli (p)

$$\Pr\{X = n\} = p^n (1 - p)^{1-n}, \quad n = 0, 1$$

Discrete Random Variable (4)

- Examples

- Binomial (n, p)

$$\Pr\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n$$

- Geometric (p)

$$\Pr\{X = k\} = (1-p)^{k-1} p, \quad k = 1, 2, \dots$$

- Negative binomial (k, p)

$$\Pr\{X = n\} = \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} p, \quad n = k, k+1, \dots$$

Continuous Random Variable (1)

- An uncountable number of possible values
- Probability density function: $f_X(x)$
- Cumulative distribution function: $F_X(x)$

$$F_X(x) = \int_{-\infty}^x f_X(x) dx$$

$$f_X(x) = \frac{dF}{dx}$$

$$\Pr\{a \leq X \leq b\} = \int_a^b f_X(x) dx$$

Continuous Random Variable (2)

- Examples

- Uniform (a, b)

$$f_X(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

- Exponential (λ)

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

- Normal (μ, σ^2)

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

Continuous Random Variable (3)

- Examples

- Erlang (n, λ) : n -stage

$$f_X(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{n-1}}{(n-1)!}, \quad x > 0, n \geq 2$$

- When $n=1$, exponential

- Gamma(α, λ)

$$f_X(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0$$

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

- When $\alpha = 1$, exponential
 - When α is a natural number k , Erlang- k
 - Generalized: Exponential or Erlang

Continuous Random Variable (4)

- Mean and variance of a continuous r.v., X

- Mean:

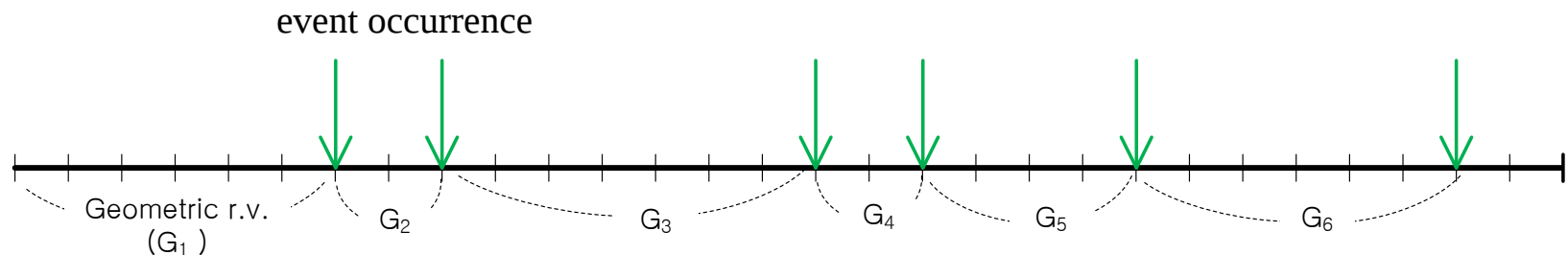
$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- The n th moment :

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

Geometric Distribution

- Bernoulli r.v., B
 - Success or Failure
 - $\Pr\{B = 0\} = 1 - p$
 - $\Pr\{B = 1\} = p$
- Geometric r.v., G
 - the number of Bernoulli trials until success (event occur)
 - $\Pr\{G = k\} = (1 - p)^{k-1}p$



✓ In discrete time domain, Geometric r.v. represents interevent time

Binomial Distribution

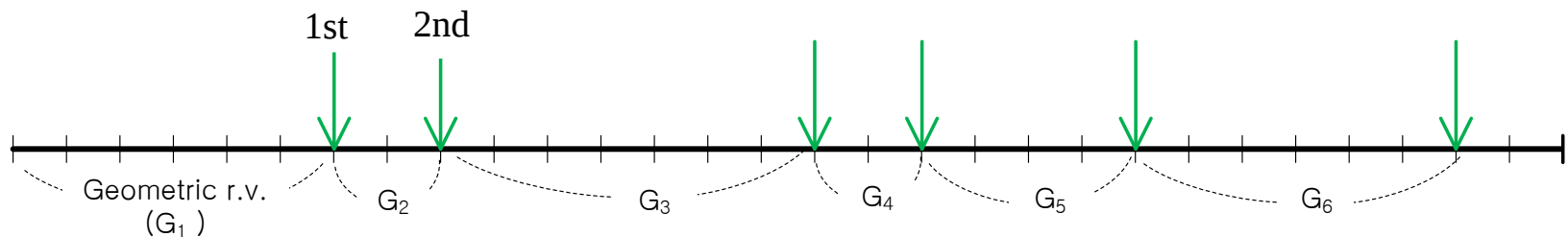
- Binomial r.v., X
 - the number of successes in n Bernoulli trials

$$\Pr\{X = k\} = \binom{n}{k} p^k (1 - p)^{n-k}$$

- $X = B_1 + B_2 + \cdots + B_n$ (B_i : Bernoulli r.v. ; 0 or 1)

Negative Binomial Distribution

- Negative Binomial r.v., X
 - the number of Bernoulli trials until the k -th success
 - $\Pr\{X = k\} = \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} p$
 - $X = G_1 + G_2 + \dots + G_k$ (G_i : Geometric r.v.)



Exponential Distribution

- Exponential r.v., Y
 - $\Pr\{Y > t\} = e^{-\lambda t}$
 - Cumulative distribution function (CDF): $F_Y(t) = \Pr\{Y \leq t\}$

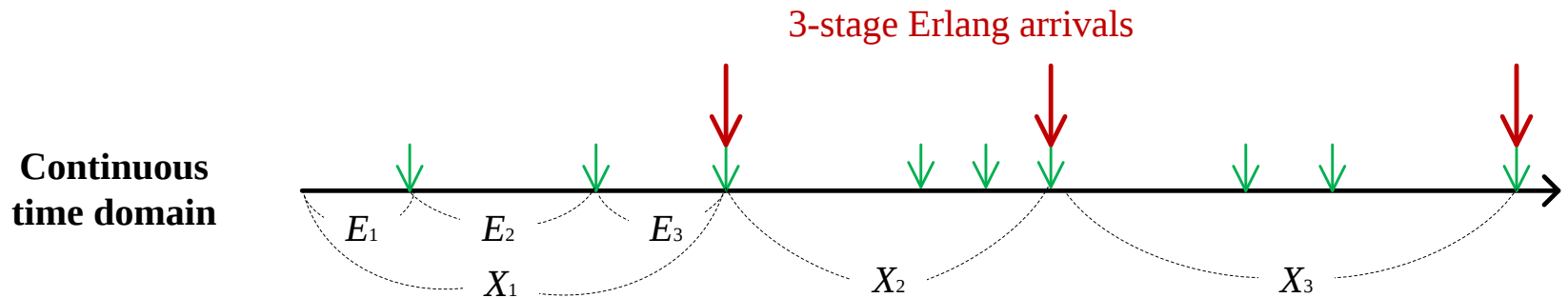
$$F_Y(t) = 1 - e^{-\lambda t}$$

- Probability distribution function (PDF): $f_Y(t) = \frac{d}{dt} F_Y(t)$

$$f_Y(t) = \lambda e^{-\lambda t}$$

k -stage Erlang Distribution (1)

- k -Erlang r.v., X
 - $X = E_1 + E_2 + \cdots + E_k$ (E : exponential r.v.)
 - k -fold convolution of exponential distribution
 - Hypo-exponential
 - An example of 3-stage Erlang distribution



- $\Pr\{X > t\} = \sum_{j=0}^{k-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!}$: the probability of events less than k during t

k -stage Erlang Distribution (2)

- k -stage Erlang r.v.
 - Cumulative distribution function (CDF): $F_X(t) = \Pr\{X \leq t\}$

$$\begin{aligned} F_X(t) &= 1 - \sum_{j=0}^{k-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!} \\ &= 1 - \left(e^{-\lambda t} + \sum_{j=1}^{k-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!} \right) \end{aligned}$$

- Probability distribution function (PDF): $f_X(t) = \frac{d}{dt} F_X(t)$

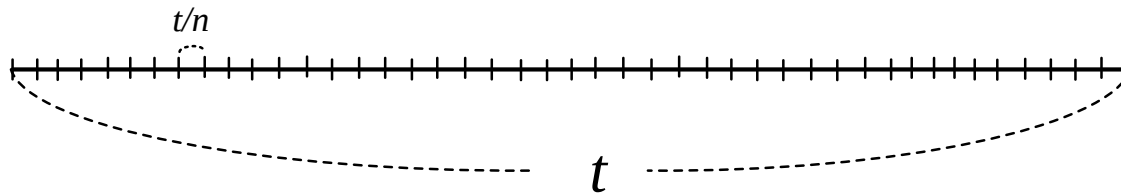
$$\begin{aligned} f_X(t) &= \lambda e^{-\lambda t} - \sum_{j=1}^{k-1} \left\{ \frac{\lambda(\lambda t)^{j-1} e^{-\lambda t}}{(j-1)!} - \frac{\lambda(\lambda t)^j e^{-\lambda t}}{j!} \right\} \\ &= \frac{\lambda(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} \end{aligned}$$

Relationship between random variables

Discrete Random Variable	Continuous Random Variable
Binomial	Poisson
Geometric	Exponential
Negative binomial	Erlang

Binomial $\rightarrow$ Poisson (1)

- Binomial r.v. in the discrete time domain can be represented as Poisson r.v. in the continuous time domain
- Proof



- An interval of duration t is divided into n sub-intervals, each of which has the length of t/n
- Bernoulli trial at each sub-interval
 - p : the success probability of a Bernoulli trial
- Average number of successes (events) during t
 - λt : Poisson ; np : Binomial
 - $\lambda t = np$

Binomial $\rightarrow$ Poisson (2)

- When a r.v. X represents the number of success events of n trials,

$$P_k \triangleq \Pr\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

- $P_0 = (1-p)^n = \left(1 - \frac{\lambda t}{n}\right)^n \xrightarrow{n \rightarrow \infty} \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda t}{n}\right)^n = e^{-\lambda t} \dots (1)$

- $$\begin{aligned} \frac{P_{k+1}}{P_k} &= \frac{\binom{n}{k+1} (1-p)^{n-k-1} p^{k+1}}{\binom{n}{k} (1-p)^{n-k} p^k} = \frac{(n-k)p}{(k+1)(1-p)} \\ &= \frac{np\left(1 - \frac{k}{n}\right)}{(k+1)(1-p)} = \frac{\lambda t\left(1 - \frac{k}{n}\right)}{(k+1)\left(1 - \frac{\lambda t}{n}\right)} \end{aligned}$$

$$\xrightarrow{n \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{P_{k+1}}{P_k} = \frac{\lambda t}{(k+1)} \dots (2)$$

Binomial $\rightarrow$ Poisson (3)

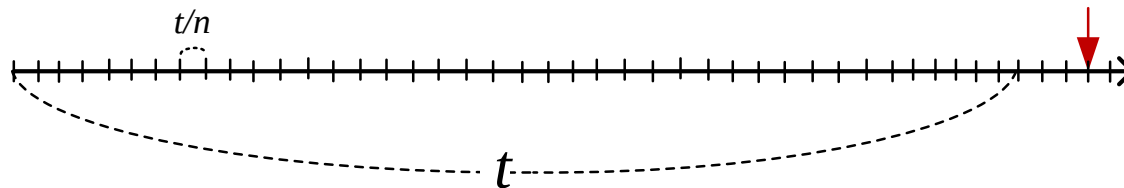
- When $n \rightarrow \infty$, from (1) and (2)

$$\begin{aligned} P_k &= \frac{\lambda t}{k} P_{k-1} = \frac{\lambda t}{k} \frac{\lambda t}{k-1} P_{k-2} = \cdots = \frac{(\lambda t)^k}{k!} P_0 \\ &= \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad : \text{Poisson distribution} \end{aligned}$$

- When $n \rightarrow \infty$, since the discrete time domain becomes continuous time domain, a binomial r.v. in the discrete time domain can be represented as Poisson r.v. in the continuous time domain

Geometric $\rightarrow$ Exponential (1)

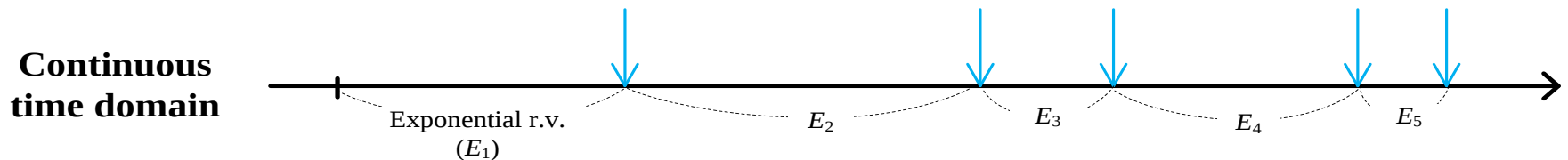
- Exponential r.v. can be obtained as a limiting form of Geometric r.v.
 - λ : the occurrence rate of events in continuous time domain
 - Y : exponential r.v.



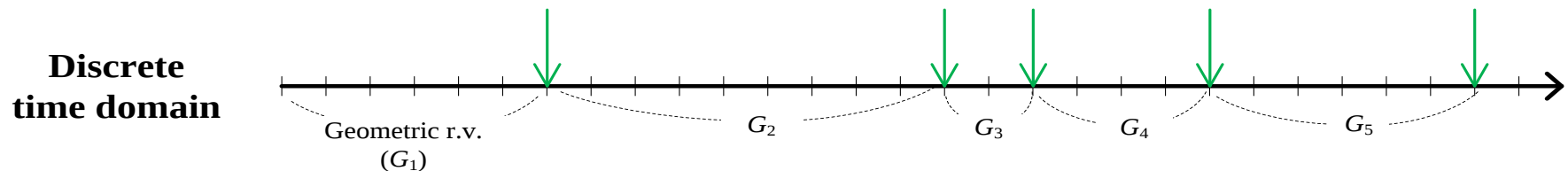
- Let X be a geometric r.v.: $\Pr\{X > n\} = (1 - p)^n$
- An average number of successes (events) for n trials: $np = \lambda t$
- $\Pr\{Y > t\} = \lim_{n \rightarrow \infty} \Pr\{X > n\} = \lim_{n \rightarrow \infty} (1 - p)^n$
$$= \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda t}{n}\right)^n = e^{-\lambda t}$$
- $\Pr\{Y \leq t\} = 1 - e^{-\lambda t}$: exponential distribution

Geometric $\rightarrow$ Exponential (2)

- Arrival events occur according to **Poisson** process



- Arrival events occur according to **Bernoulli** process



Memoryless Property (1)

- Memoryless property of geometric r.v. X
 - A geometric r.v. X has memoryless property if for all nonnegative integers n, m

$$\Pr\{X = n + m | X > n\} = \Pr\{X = m\}$$

< **Proof** >

$$\begin{aligned}\bullet \Pr\{X = n + m | X > n\} &= \frac{\Pr\{X=n+m\}}{\Pr\{X>n\}} \\ &= \frac{(1-p)^{n+m-1}p}{(1-p)^n} \\ &= (1-p)^{m-1}p \\ &= \Pr\{X = m\}\end{aligned}$$

∴ Geometric r.v. X has memoryless property

Memoryless Property (2)

- Memoryless property of exponential r.v. X
 - A exponential r.v. X has memoryless property if for all nonnegative t, x

$$\Pr\{X \leq t + x | X > t\} = \Pr\{X \leq x\}$$

< **Proof** >

$$\begin{aligned} \bullet \Pr\{X \leq t + x | X > t\} &= \frac{\Pr\{t < X \leq t + x\}}{\Pr\{X > t\}} \\ &= \frac{\Pr\{X \leq t + x\} - \Pr\{X \leq t\}}{\Pr\{X > t\}} \\ &= \frac{1 - e^{-\lambda(t+x)} - (1 - e^{-\lambda t})}{e^{-\lambda t}} \\ &= 1 - e^{-\lambda x} \\ &= \Pr\{X \leq x\} \end{aligned}$$

∴ Exponential r.v. X has memoryless property.