Ch.4. Particle Drifts in time-dependent Fields

4.1. Time-varying B field :

Consider a slowly time-varying B field with $\frac{\partial}{\partial t} \sim \omega \ll \omega_c$.

(Similar to weakly varying B field in space with $\frac{\partial}{\partial x} \sim k \ll \frac{1}{r_L}$).

* Faraday's Law : $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \xrightarrow{\text{Stokes' theorem}} \oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{S}$

* If we consider charged particles gyrating along $d\vec{l}$, from the right-hand rule, a particle will be accelerated for $\frac{\partial}{\partial t}\vec{B} > 0$ regardless of the sign of its charge.

* Indeed, $\frac{d}{dt}\langle W_\perp \rangle = q\langle \vec{v} \cdot \vec{E} \rangle = |q|\, v_\perp \frac{\pi (r_L)^2}{2\pi r_L}\frac{\partial B}{\partial t} = \frac{W_\perp}{B}\frac{\partial B}{\partial t} = \mu \frac{\partial B}{\partial t}$ (4.3)

∴ $W_\perp$ grows steadily as B increases.

* $$\boxed{\frac{d\mu}{dt} = \frac{1}{B}\frac{dW_\perp}{dt} - \frac{W_\perp}{B^2}\frac{\partial B}{\partial t} = 0}$$

(Note that, Flux $= \pi r_L^2 B = \frac{2\pi m}{q^2} \cdot \mu$) (4.4).

4.2. Adiabatic Compression

* A changing magnetic field will heat (or cool) a plasma as a consequence of magnetic moment "μ".

* For a cylindrical plasma in a solenoidal B field, if $\frac{\partial B_z}{\partial t} > 0$, $\frac{dW_\perp}{dt} > 0$, and plasma will be driven in towards the center of the solenoid (i.e., compressed away from the coils).

* Faraday's law: →

$$2\pi r E_\theta = -\pi r^2 \frac{\partial B_z}{\partial t}. \qquad (4.9)$$

$$\therefore \frac{dr}{dt} = \vec{V}_E \cdot \hat{r} = \frac{E_\theta}{B_z} = -\frac{r}{2B_z}\frac{\partial B_z}{\partial t}. \qquad (4.10)$$

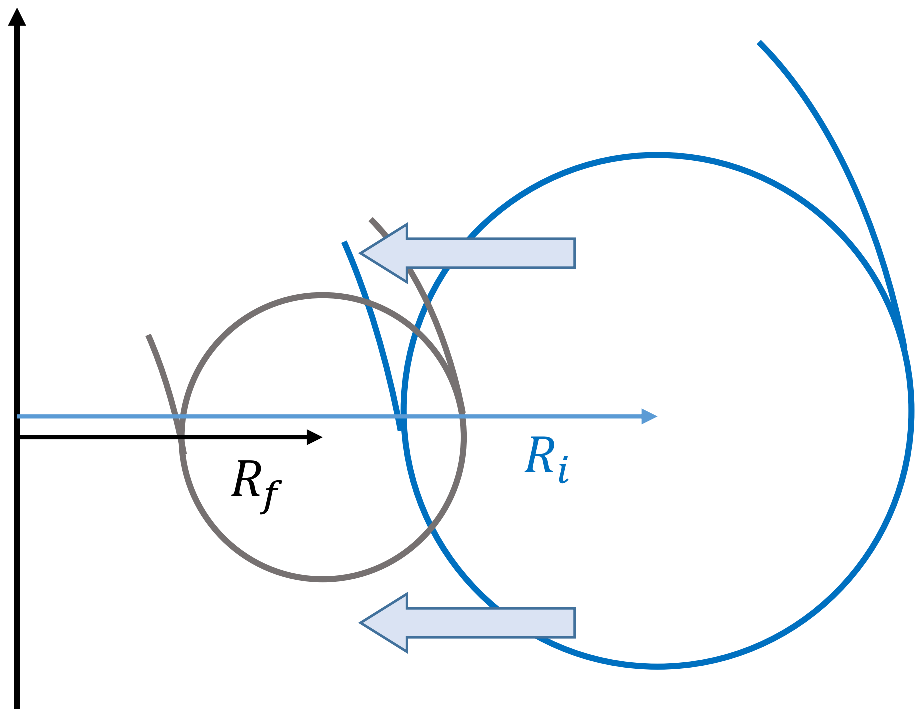
Frozen-in-flux!

↗

* During this "adiabatic" compression ("$\omega \ll \omega_c$"), the amount of magnetic flux enclosed by the annulus is conserved.

$$\frac{d}{dt}(\pi r^2 B_z) = 2\pi r B_z \frac{dr}{dt} + \pi r^2 \frac{\partial B_z}{\partial t} = 0. \qquad (4.11)$$

Plasma Heating by Adiabatic Compression



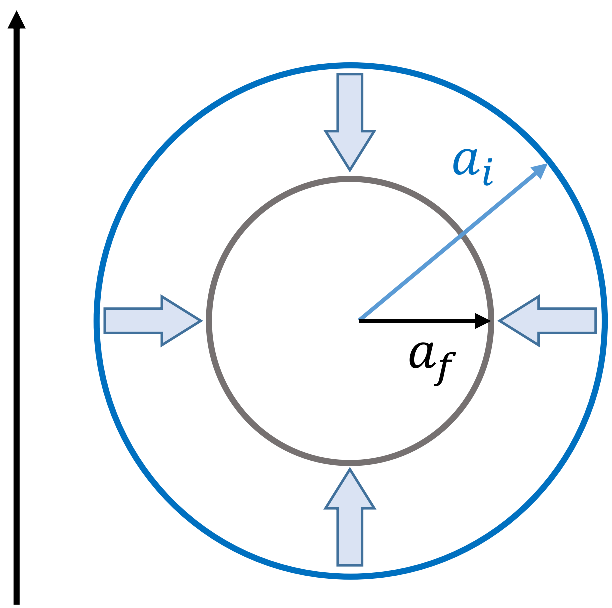
- By suddenly increasing the vertical magnetic field B_v, tokamak plasma can be compressed in major radius.

- As a result, plasma moves toward stronger B field region ($B \propto \frac{1}{R}$) and volume shrinks.

→ Both temperature and density increase!

Plasma Heating by Adiabatic Compression

- This has been demonstrated in Adiabatic Toroidal Compressor (ATC) in Princeton Plasma Physics Laboratory (1972-1975).
- Compression in minor radius without shift in major radius is also possible by changing B_{Tor}.



- This has been demonstrated in TUMAN experiment in USSR around the same period.

4.3. Polarization Drift.

* Consider a uniform and strong $\vec{B}$ field and slowly time-varying $\vec{E}$ field $\perp \vec{B}$ with $\frac{\partial}{\partial t} \sim \omega \ll \omega_c$.

* $$m \frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (1)$$

We learned that the lowest order guiding-center drift is given by $\vec{E}\times\vec{B}$ drift, i.e.,

$$\vec{V}^{(0)} = \vec{V}_E = \frac{\vec{E}\times\vec{B}}{B^2} \quad (2)$$

This makes $\vec{E}_\perp + \vec{v}^{(0)} \times \vec{B} = 0$.

(with a small parameter, $\omega/\omega_c \ll 1$)

* Writing Eq. (1) at each order, after formally expanding $\vec{V} = \vec{V}^{(0)} + \vec{V}^{(1)} + \cdots$,

0-th order : $$0 = q(\vec{E}_\perp + \vec{v}^{(0)} \times \vec{B}) = 0. \quad (3)$$

1-st order : $$m \frac{d}{dt}\vec{v}^{(0)} = q(0 + \vec{v}^{(1)} \times \vec{B}), \quad (4)$$

* Inverting Eq. (4) by applying $\vec{B} \times (\cdots)$, we obtain the Polarization Drift.

$$\boxed{\vec{V}_\perp^{(1)} = \frac{m}{qB^2}\vec{B} \times \frac{d}{dt}\vec{v}^{(0)} = \frac{m}{qB^2}\frac{d\vec{E}_\perp}{dt}} \quad (4.15)$$

4.1. Review of Single Particle Motion in a Strong Magnetic Field

4.1.1. Adiabatic Invariant

When magnetic field varies in space smoothly (i.e. $\rho_i \ll L_B \equiv |\nabla \ln B|^{-1}$), we can identify approximate constants of motion.
"Adiabatic" in here means slow variation in time and space. This is well illustrated from the point of view of Quantum Mechanics (QM).
Let's consider a Simple Harmonic Oscillator (SHO): Schrödinger Equation is

$$H\psi(x) = \Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_0^2 x^2\Big)\psi(x) = E\psi(x)$$

Here the eigenvalues are

$$E = \hbar\omega_0\Big(N + \frac{1}{2}\Big)$$

where N is the quantum number ($N = 0, 1, 2, \dots$).

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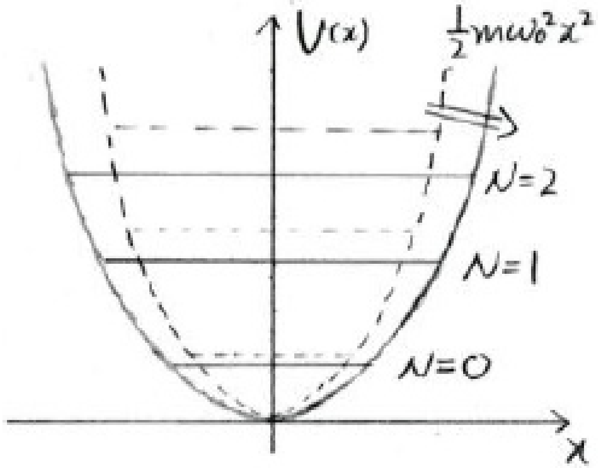


Suppose that potential well characterized by $\omega_0(t)$ is changed very slowly in the time with a scale τ ($\tau \gg 1/\omega_0$).
In this adiabatic process, what remains constant is "N" (eigenstates are preserved).
While the energy (eigenvalue) changes in time. "N" is an example of adiabatic invariant.

In classical limit, $N = E/\omega_0$.

4.1.2. Adiabatic Invariant in Classical Limit

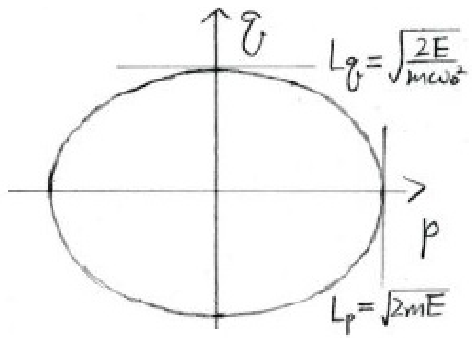
In Classical Mechanics (CM), the Hamiltonian of SHO is given by

$$H(p,q) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 = E$$

Adiabatic invariant in CM is related to the conservation of the volume in the phase space for appropriate action-angle variables.

$$I = \oint dq\, p$$

This is also called "Action Invariant".



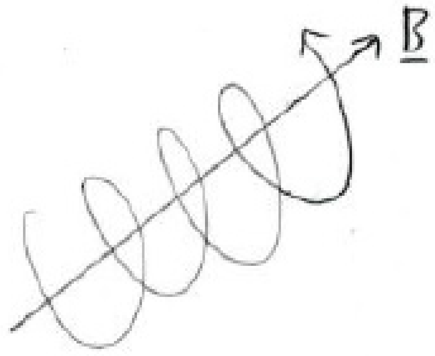
For SHO the action invariant is

$$I = \pi L_q L_p = \pi\sqrt{2mE\frac{2E}{m\omega_0^2}} = 2\pi\frac{E}{\omega_0}$$

Thus except for a numerical factor 2π, we recover $N = E/\omega_0$ from SHO in QM. (The useful formula $N = E/\omega_0$ represents the "Duality of Wave and Particles".) This illustration of geometric meaning of action invariant can be extended to quasi-periodic motion (recall $\tau \gg 1/\omega_0$).

4.1.3. Gyromotion in Slowly Varying Magnetic Field

Consider the gyrating motion of charged particles in slowly varying magnetic field in time ($1/\omega$) and space (L_B).



For this gyration, the corresponding action invariant is the magnetic moment (the 1st adiabatic invariant).

- Energy corresponding to gyration: $E_\perp = mv_\perp^2/2$
- Frequency corresponding to gyration: $\Omega_c = eB/mc$

$$\Rightarrow \mu \propto \frac{1}{2}mv_\perp^2/\Big(\frac{eB}{mc}\Big) \propto \Big(\frac{v_\perp^2}{2B}\Big)$$ This is not an exact constant

What is the error or precision of the statement?

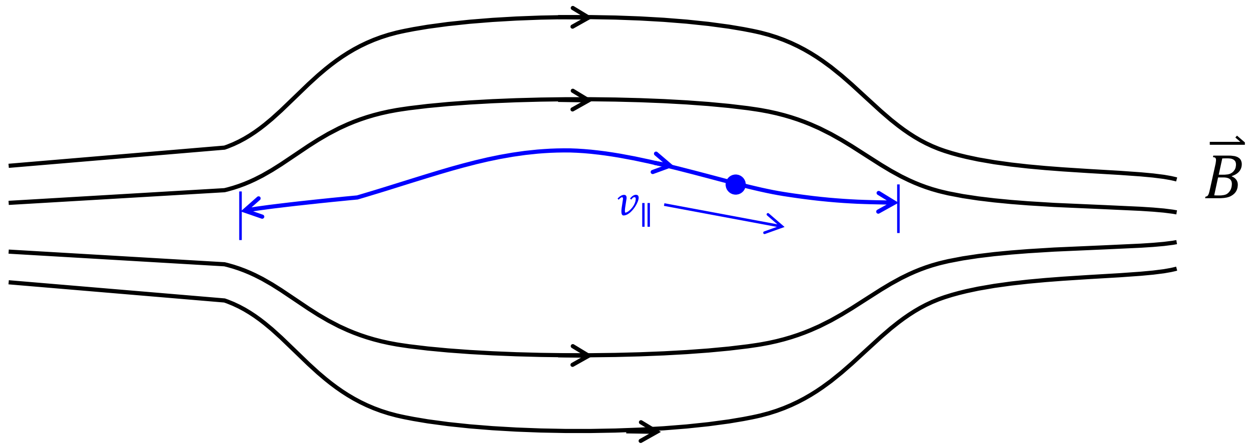
What is the expansion parameter or smallness parameter describing this motion?

If $\epsilon_\omega \equiv \omega/\Omega_c \ll 1$ and $\epsilon_B \equiv \rho_i/L_B \ll 1$,

the adiabatic invariant is good up to any order!

$$\text{Error} = \mathcal{O}\Big(\exp\big(-\frac{\text{const}}{\epsilon_B}\big),\ \exp\big(-\frac{\text{const}}{\epsilon_\omega}\big)\Big)$$

4.5. Second Adiabatic Invariant : J



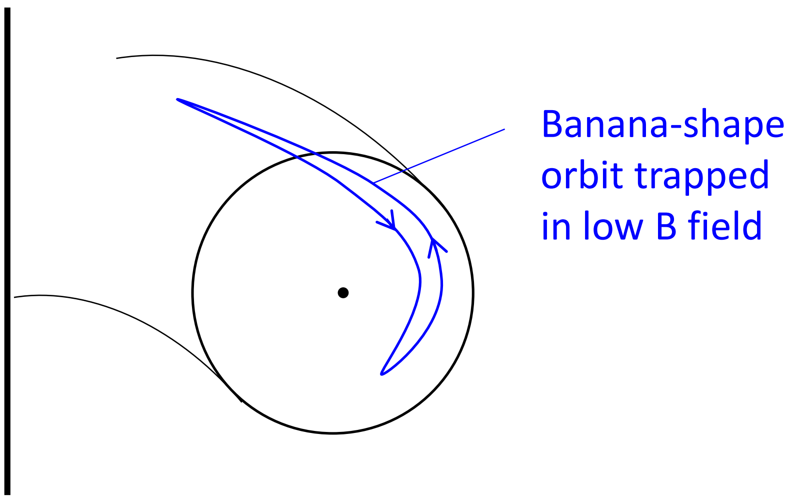
- Guiding center motion of mirror-trapped particles:

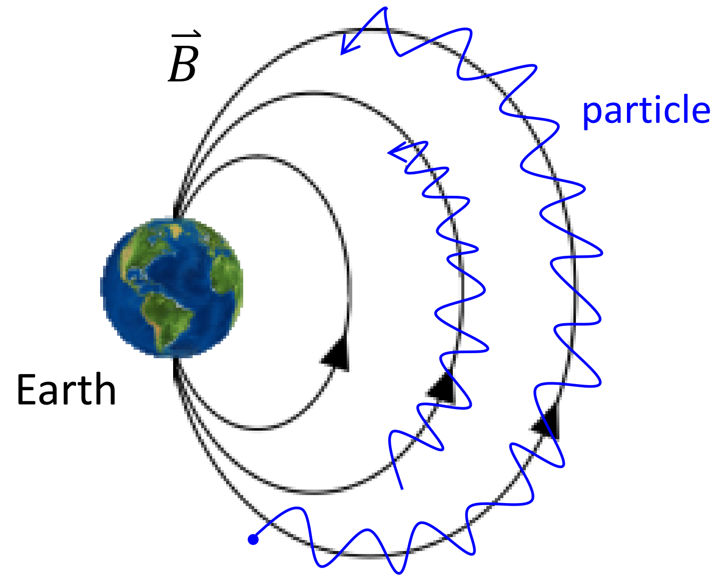
  This is also periodic in nature.

$J \equiv \oint v_\parallel \cdot ds$: the loop integral of the parallel velocity along a trapped particle trajectory. (4.27)

where $v_\parallel = \sqrt{\frac{2(E-\mu B)}{m}}$

- This 2nd adiabatic invariant (J conservation) is not only useful in magnetic mirror geometry, but also in tokamak configuration and in the earth's magnetic field (dipole field).





- Note that this is also an "Action":

$$J \equiv \oint v_{\parallel} \cdot ds = \frac{1}{m} \oint p_{\parallel} \, dq_{\parallel}$$

$$p_{\parallel} \equiv m v_{\parallel} - \frac{e}{c} A_{\parallel}$$

No contribution to integral