# Markov Process 

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## Markovian Property

- Markovian Property
- Given past states and present state, conditional distribution of any future state is independent of past states and depends only on the present state.
- Markov Process
- A stochastic process that satisfies the Markovian property.
- Types
- Discrete time Markov chain (DTMC)
- Continuous time Markov chain (CTMC)
- Embedded Markov chain

| Distribution <br> of state duration | DTMC <br> geometric | $\subseteq$ | CTMC <br> exponential | $\subseteq$ | Embedded MC <br> general |
| :---: | :---: | :---: | :---: | :---: | :---: |

## Discrete Time Markov Chain



- The state duration has a geometric distribution.
- $p_{i j}(m)=\operatorname{Pr}\left\{X_{m+1}=j \mid X_{0}=i_{1}, X_{1}=i_{2}, \ldots, X_{m}=i\right\}$

$$
=\operatorname{Pr}\left\{X_{m+1}=j \mid X_{m}=i\right\}
$$

- $p_{i j}(m)$ : one-step transition probability from state $i$ to state $j$ at the $m$-th time index


## Continuous Time Markov Chain



- The state duration has an exponential distribution.
- $p_{i j}(t, s)=\operatorname{Pr}\left\{X_{t+s}=j \mid X_{t}=i, X_{u}=x_{u}, 0 \leq u<t\right\}$

$$
=\operatorname{Pr}\left\{X_{t+s}=j \mid X_{t}=i\right\}
$$

## Embedded Markov Chain



- The state duration of original process has general distribution; not Markov process.
- When observing the system only at departure epochs, the process has Markovian property. Then, the process at observation times is called Embedded Markov chain.
- The original process and the embedded Markov chain have the same statistical properties.


## Mathematically analyzable process

- Markovian property
- Time homogeneity
- Ergodicity
- Irreducible
- Positive recurrent
- Aperiodic



## Time homogeneity (1)

- If the conditional probability, $\operatorname{Pr}\left\{X_{m+1}=j \mid X_{m}=i\right\}$, is independent of $m$, the DTMC is said to be homogeneous.

$-p_{i j}=\operatorname{Pr}\left\{X_{l+1}=j \mid X_{l}=i\right\}=\operatorname{Pr}\left\{X_{m+1}=j \mid X_{m}=i\right\}$
without respect to time index $l, m$
- The next state depends only on the current state and is independent of observation times.


## Time homogeneity (2)

- The homogeneous DTMC is described with the state space, $S$, and one-step transition probability matrix, $\mathbb{P}=\left[p_{i j}\right]$, or state transition probability diagram.
- Example
- State space: $S=\{1,2,3\}$
- State space: $S=\{1,2,3\}$
- One-step transition probability matrix: $\quad \mathbb{P}=\left[\begin{array}{ccc}0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \text { - State transition Probability diagram : } & & \\ \frac{1}{2} & 0 & \frac{1}{2}\end{array}\right]$

$$
\mathbb{P}=\left[\begin{array}{ccc}
0 & 1 & 0 \\
\frac{1}{1} & \frac{1}{2} & \frac{1}{2} \\
4 & 4 & 2 \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{array}\right]
$$



## Time homogeneity (3)

- One-step transition probability

$$
p_{i j}=\operatorname{Pr}\left\{X_{m+1}=j \mid X_{m}=i\right\}
$$

- $n$-step transition probability

$$
P_{i j}^{(n)}=\operatorname{Pr}\left\{X_{m+n}=j \mid X_{m}=i\right\}
$$

- Chapman-Kolmogorov equation

$$
\begin{aligned}
& P_{i j}^{(m+n)}=\sum_{k \in S} P_{i k}^{(m)} P_{k j}^{(n)} \\
& \mathbb{P}^{(m+n)}=\mathbb{P}^{(m)} \times \mathbb{P}^{(n)}
\end{aligned}
$$

## Ergodic Markov Chain (1)

- An ergodic Markov chain has a limiting distribution.
- State transition probability to state j converges to only one value without respect to an initial state.
$-\lim _{n \rightarrow \infty} P_{i j}^{(n)}=q_{j}$
- After a long period of time, an ergodic Markov chain has a distribution independent of the starting condition (limiting distribution).


$$
\mathbb{P}=\left[\begin{array}{ccc}
0 & 1 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{array}\right] \quad n \rightarrow \infty \quad\left[\begin{array}{ccc}
\frac{3}{11} & \frac{4}{11} & \frac{4}{11} \\
\frac{3}{11} & \frac{4}{11} & \frac{4}{11} \\
\frac{3}{11} & \frac{4}{11} & \frac{4}{11}
\end{array}\right]
$$

## Ergodic Markov Chain (2)

## $\checkmark$ Ensemble average distribution

- Let $\pi_{j}{ }^{(n)}$ be the unconditional probability that DTMC is in state $j$ at the $n$-th time index, i.e., $\pi_{j}^{(n)} \triangleq \operatorname{Pr}\left\{X_{n}=j\right\}$
- Let $I_{k}(j, n)$ be an indicator representing that DTMC is in state $j$ at the $n$-th time index for the $k$-th sample path
$\Rightarrow I_{k}(j, n)= \begin{cases}1 & \text { if being in state } j \\ 0 & \text { othewise }\end{cases}$
- $\pi_{j}{ }^{(n)}=\lim _{N \rightarrow \infty} \frac{\sum_{k=1}^{N} I_{k}(j, n)}{N}:$ the average for innumerable sample paths




## Ergodic Markov Chain (3)

- $\pi_{j}^{(n)}=\sum_{i \in S} \pi_{i}{ }^{(0)} P_{i j}{ }^{(n)}, \quad$ where $S$ is a state space
- Ensemble average distribution at time limit : $\lim _{n \rightarrow \infty} \pi_{j}{ }^{(n)}$

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \pi_{j}^{(n)} & =\lim _{n \rightarrow \infty} \sum_{i \in S} \pi_{i}^{(0)} P_{i j}^{(n)} \\
& =\sum_{i \in S} \pi_{i}^{(0)} \lim _{n \rightarrow \infty} P_{i j}^{(n)}
\end{aligned}
$$

- In an ergodic DTMC, since $\lim _{n \rightarrow \infty} P_{i j}^{(n)}=q_{j}$,

$$
\lim _{n \rightarrow \infty} \pi_{j}^{(n)}=q_{j} \sum_{i \in S} \pi_{i}^{(0)}=q_{j}
$$

- When $\pi_{j}$ denotes an ensemble average distribution of ergodic DTMC at time limit, i.e., $\pi_{j} \triangleq \lim _{n \rightarrow \infty} \pi_{j}{ }^{(n)}$,

$$
\underline{\underline{\pi_{j}}=q_{j}}
$$

Ensemble average distribution is the same as the limiting distribution

## Ergodic Markov Chain (4)

$$
-\pi_{j}^{(n)}=\sum_{i \in S} \pi_{i}^{(n-1)} P_{i j}
$$

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \pi_{j}^{(n)} & =\lim _{n \rightarrow \infty} \sum_{i \in S} \pi_{i}^{(n-1)} P_{i j} \\
& =\sum_{i \in S}\left(\lim _{n \rightarrow \infty} \pi_{i}^{(n-1)}\right) P_{i j} \quad \Rightarrow \underline{\underline{\pi_{j}=\sum_{i \in S} \pi_{i} P_{i j}}}
\end{aligned}
$$

- We can obtain the state distribution of ergodic DTMC, by solving (1) and (2).

$$
\begin{array}{ll}
-\pi_{i}=\sum_{j \in S} \pi_{j} P_{j i} & \text { for all } i \in S \\
-\sum_{i \in S} \pi_{i}=1 & \ldots \text { (2) }
\end{array}
$$

## Ergodic Markov Chain (5)

- Example
- State space $S=\{1,2,3\}$
- State transition Probability diagram :

$-\pi_{1}=\frac{1}{2} \pi_{1}+\frac{1}{4} \pi_{2}$

$$
\begin{aligned}
& \pi_{2}=\frac{1}{2} \pi_{1}+\frac{1}{2} \pi_{2}+\frac{1}{2} \pi_{3} \\
& \pi_{3}=\frac{1}{4} \pi_{2}+\frac{1}{2} \pi_{3} \\
& \pi_{1}+\pi_{2}+\pi_{3}=1
\end{aligned} \quad \Rightarrow \quad \pi_{1}=\frac{1}{4}, \quad \pi_{2}=\frac{1}{2}, \quad \pi_{3}=\frac{1}{4}
$$

## Ergodic Markov Chain (6)

$\checkmark$ Time average distribution

- Let $I(j, n)$ be an indicator representing that DTMC is in state $j$ at the $n$-th time index for any one sample path

$$
\begin{gathered}
\quad>I(j, n)= \begin{cases}1 & \text { if being in state } j \\
0 & \text { othewise }\end{cases} \\
-\pi_{j}=\lim _{T \rightarrow \infty} \frac{\sum_{n=0}^{T} I(j, n)}{T}
\end{gathered}
$$

any one sample path: $2 \rightarrow 2 \rightarrow 0 \rightarrow 1 \rightarrow \cdots \rightarrow 1 \rightarrow 0 \rightarrow 2 \rightarrow 0 \rightarrow \cdots \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 0 \rightarrow \cdots \rightarrow$

- In Ergodic DTMC, the ensemble average distribution is the same as the time average distribution.

$$
-\pi_{j}=\frac{\lim _{n \rightarrow \infty} \lim _{N \rightarrow \infty} \frac{\sum_{k=1}^{N} I_{k}(j, n)}{N}}{\text { ensemble average at time limit }}=\frac{\lim _{T \rightarrow \infty} \frac{\sum_{n=0}^{T} I(j, n)}{T}}{\text { time average }}
$$

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- the probability that, as the number of trials $n$ goes to infinity, the average of the observations converges to the expected value, is equal to one.


## Ergodic Markov Chain (7)

- $\pi_{j}$ can be interpreted as two aspects; one is the time average, and the other is the ensemble average.
- Time average
- $\pi_{i}$ is the long-run time proportion that the DTMC is in state $j$ on any sample path
- Ensemble average
- $\pi_{i}$ is the probability that the state of DTMC is $i$ in steady state.
- $\{X(t)\}$ is ergodic in the more general sense if all its measures can be determined or well approximated from a single realization of the process.
- It is often done in analyzing simulation outputs.


## Necessary Conditions for an Ergodic MC (1)

## Irreducible

- State $j$ is reachable from state $i$ if there is an integer $n \geq 1$ such that $P_{i j}{ }^{(n)}>0$.
- If state $i$ is reachable from state $j$ and state $j$ is reachable from state $i$, state $i$ and $j$ are said to communicate.
- If all states in the Markov chain communicate to each other, the Markov chain is called "irreducible".

reducible

irreducible


## Necessary Conditions for an Ergodic MC (2)

## Positive recurrent

- $f_{i j}$ : the probability of ever making a transition into state $j$, given that Markov chain is in state $i$.
- State $i$ is said to be recurrent if $f_{i i}=1$
- If the mean recurrent time is finite, state $i$ is a positive recurrent state.
- If all states in the Markov chain are positive recurrent, the Markov chain is called "positive recurrent".
- An irreducible Markov chain having the finite number of states is positive recurrent.


## Necessary Conditions for an Ergodic MC (3)

## * Aperiodic

- State $i$ is said to have a period of $d$, if $P_{i i}{ }^{(n)}=0$ whenever $n$ is not divided by $d$ and $d$ is the greatest integer with this property.
- A state with period 1 is an aperiodic state.
- If all states in the Markov chain are aperiodic, the Markov chain is called "aperiodic".


Periodic
$\operatorname{GCD}_{i}(2,4,6,8, \ldots)=2$
$\mathrm{GCD}_{j}(2)=2$
$\mathrm{GCD}_{k}(2,4,6,8, \ldots)=2$


Aperiodic

$$
\begin{aligned}
& \operatorname{GCD}_{i}(2,3,4,5, \ldots)=1 \\
& \operatorname{GCD}_{j}(2,3,4,5, \ldots)=1 \\
& \operatorname{GCD}_{k}(2,3,4,5, \ldots)=1
\end{aligned}
$$

aperiodic if there is at least one self-loop
$\operatorname{GCD}_{s}\left(n_{1}, n_{2}, \ldots\right)$ : the greatest common divisor of the state transition steps $\left(n_{1}, n_{2}, \ldots\right)$ for back to the state $s$.

## Stationary ergodic DTMC

- $\pi_{j}{ }^{(n)}=\sum_{i \in S} \pi_{i}{ }^{(n-1)} P_{i j} . \quad \Rightarrow \quad \Pi^{(n)}=\Pi^{(n-1)} \mathbb{P}$
- In an ergodic DTMC, if the initial state distribution $\Pi^{(0)}$ is set to the limiting distribution,

$$
\left.\begin{array}{rl}
\Pi^{(1)}= & \Pi^{(0)} \mathbb{P}=\Pi \mathbb{P}=\Pi \\
\Pi^{(2)}= & \Pi^{(1)} \mathbb{P}=\Pi \mathbb{P}=\Pi \\
& \ldots \\
\Pi^{(m)}= & \Pi^{(m-1)} \mathbb{P}=\Pi \mathbb{P}=\Pi
\end{array}\right) \Rightarrow \Pi^{(n)}=\Pi, \quad \text { for all } n
$$

The state distribution is invariant over time, $\pi_{i}=\operatorname{Pr}\left\{X_{n}=i\right\}$ for all $n$ $\Rightarrow$ stationary process

- In summary, an ergodic DTMC of which the initial state distribution is set to the limiting distribution is stationary, and then the limiting distribution is called the stationary distribution.


## Exercise1: DTMC

- 어떤 시스템에 메시지들이 베르누이 프로세스에 따라 단위시간 당 확률 9 로 도착한다.
- 버퍼의 크기는 $K$ 이다. 버퍼가 차있을 때 도착하는 메시지들은 상실된다.
- 메시지의 전송은 매 시간단위 (time slot) 시작점에서만 가능하다. 메시지 전송에는 한 단위시간이 걸린다. 한 번에 하나의 메시지만 전송할 수 있다.
- 메시지가 성공적으로 전송될 확률은 $\rho$ 이다. 잘못 전송된 메시지는 다음 단 위시간에 다시 전송하여야 한다.
- 도착하는 임의의 메시지가 상실될 확률을 구하여라

System state: the number of messages in the system at the beginning of each time slot


## Exercise2: DTMC

- A man goes for a run every morning. When he leaves his house for his run, he is equally likely to go out either the front or the back door. Similarly, when he returns home, he is equally likely to go to either the front or back door. The runner owns five pairs of running shoes that he takes off after the run at whichever door he happens to be. If there are no shoes at the door from which he leaves to go running, he runs barefooted.
(1) Set this up as a Markov chain. Give the states and the transition probabilities
(2) Determine the proportion of days that he runs barefooted

System state: the number of shoes in the front door at the start of a running
(f, f) (b,b): no change
(f, b): - 1
(b, f): +1

$$
\frac{1}{4} \pi_{i}=\frac{1}{4} \pi_{i+1} \quad \text { for } 0 \leq i \leq 4
$$

$$
\text { (1) } \pi_{0}=\pi_{1}=\pi_{2}=\pi_{3}=\pi_{4}=\pi_{5}=1 / 6
$$

$$
\text { (2) } \frac{1}{2} \pi_{0}+\frac{1}{2} \pi_{5}=\frac{1}{6}
$$

