Unit 2 : Plasmas as Fluids

Ch. 6. Fluid Equations for a Plasma

* Consider the behavior of "an ensemble of charged particles,"
  rather than " of individual charged particles.

6.1. Continuity Equation :

$$\frac{\partial}{\partial t} n + \vec{\nabla} \cdot (n \vec{u}) = S \qquad (6.3)$$

S: volume source rate of particles :

>0 for ionization

<0 for recombination

for instance.

"Conservation of particle numbers" for $S = 0$

6.2. Momentum Balance Equation :

* Consider the rate of change of momentum density in a differential element of volume.

* $\frac{\partial}{\partial t}(nm\vec{u}) = nq(\vec{E} + \vec{u}\times\vec{B}) + \cdots$ (6.5)

Macroscopic Forces on the element, where $\vec{u} = \langle\vec{v}\rangle$.

Sum of forces on individual particles.

?

Contribution from "particles' flux into/out of the vol. element carry momentum."

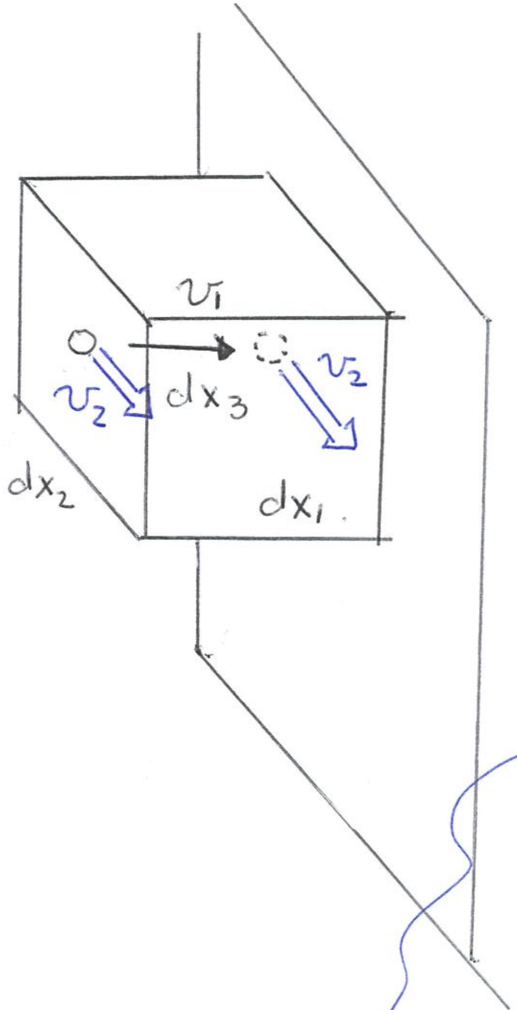
* $f(\vec{x},\vec{v})$: distribution function,

the relative probability of finding a particle with $\vec{v}$ at $\vec{x}$.

→ differential number of ptls in a phase space elt $d^3v\,d^3x$ located at $(\vec{x},\vec{v})$ is " $f(\vec{x},\vec{v})\,d^3v\,d^3x$ "

* differential number of ptls carried per unit time across the surface of constant x_1 by this elt in phase space is

" $f(\vec{x},\vec{v})\,d^3v\,d^3x/dt = v_1 f\,d^3v\,dx_2\,dx_3$, $\because dt = \frac{dx_1}{v_1}$."



* Specific ptls "o" in Fig. each carry x_2-directed momentum "mv_2".

→

* differential amount of momentum in this direction per unit time across a surface of constant x_1 by this elt. of phase space is

"$m v_2 v_1 f(\vec{x}, \vec{v}) \, d^3v \, dx_2 dx_3$

∴ The rate of change of x_2-directed momentum, averaged over all the particles, is expressed in terms of the divergence of fluxes of momentum across the various surfaces:

$$\frac{\partial}{\partial t}(nmu_2) = -\frac{\partial}{\partial x_1}(mn\langle v_2 v_1\rangle) - \frac{\partial}{\partial x_2}(mn\langle v_2 v_2\rangle) - \frac{\partial}{\partial x_3}(mn\langle v_2 v_3\rangle)$$

(const x_1, const x_2, and " x_3 respectively)

(6.6).

Momentum Balance Equation

* $$\frac{\partial}{\partial t}(nm\,u_i) = -\sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(mn\langle v_i v_j\rangle\right) \qquad (6.6).$$

* Pressure tensor :

$$P_{ij} \equiv mn\left(\langle v_i - u_i\rangle\langle v_j - u_j\rangle\right)$$
$$= mn\left(\langle v_i v_j\rangle - u_i u_j\right). \qquad (6.7)$$

$$\therefore\ mn\langle v_i v_j\rangle = P_{ij} + mn\,u_i u_j, \quad \text{where} \quad u_i \equiv \langle v_i\rangle.$$

* For drifting Maxwellian distribution function (i.e., $f(\vec{v}-\vec{u})$ is Maxwellian),

$$P_{ij} = nT \ \text{for } i=j, \ \text{and} \ P_{ij} = 0 \ \text{for } i \neq j$$

* For drifting bi-Maxwellian f with different temperatures,

$$P_{ij} = \begin{bmatrix} nT_\perp & 0 & 0 \\ 0 & nT_\perp & 0 \\ 0 & 0 & nT_\parallel \end{bmatrix}. \qquad (6.8).$$

* $\frac{\partial}{\partial t}(nm u_i) = - \sum_j' \frac{\partial P_{ij}}{\partial x_j} - m \sum_j \frac{\partial}{\partial x_j}(n u_i u_j)$, (6.10).

or adding macroscopic Force

* $$\boxed{\frac{\partial}{\partial t}(mn\vec{u}) = nq(\vec{E} + \vec{u} \times \vec{B}) - \vec{\nabla}\cdot\overleftrightarrow{P} - \vec{\nabla}\cdot(mn\vec{u}\vec{u}),} \quad (6.11).$$

or

using an identity : $\vec{\nabla}\cdot(n\vec{u}\vec{u}) = \vec{u}\,\vec{\nabla}\cdot(n\vec{u}) + n(\vec{u}\cdot\vec{\nabla})\vec{u}$,

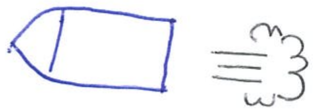
and the continuity equation ,

* $$\boxed{mn\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{u} = nq(\vec{E} + \vec{u}\times\vec{B}) - \vec{\nabla}\cdot\overleftrightarrow{P} - mS\vec{u}} \quad (6.12)$$

↓

convective derivative

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right).$$



Rocket consuming fuels.

* For relatively simple phenomena, simple form of $\overleftrightarrow{P}$ can be used.
* But the form of $\overleftrightarrow{P}$ gets more complicated for description of complex phenomena.

6.3. Equations of State

- $\frac{\partial}{\partial t} n = -\vec{\nabla}\cdot(n\,\vec{u})$
- $mn \frac{d}{dt}\vec{u} = -\vec{\nabla}\cdot\overleftrightarrow{P} + nq(\vec{E} + \vec{u}\times\vec{B})$
- $\frac{d}{dt}\overleftrightarrow{P} = \cdots\vec{\nabla}\cdot(\text{heat-flux})$

"Hierarchy of Fluid Equations."

$\downarrow$?

* To obtain a closed set of fluid equations, some approximation has to be made at some point. $\rightarrow$ "Fluid Closure."

* The simplest way to achieve the closure is to use the "equations of state" by relating $\overleftrightarrow{P}$ to already used variables such as "n" and "B".

Equations of State : cont'd.

* For scalar pressure "P" : $P = c\,n^{\gamma}$

γ : adiabatic exponent.

- $\gamma = 1$ for 'isothermal' compression ; for "slow" compression compared to thermal conduction.

- $\gamma = \dfrac{2+N}{N}$, "N" number of degrees of freedom for 'adiabatic' compression ; for 'fast'er than thermal conduction.

* If compression is slower than the collisional exchange of energy,
→ isotropic and $N = 3$ → $\gamma = 5/3$.

* If compression is faster than collisional exchange of energy,
→ anisotropic, $N = 1$ and $\gamma = 3$ for $\parallel$ direction,
$N = 2$ and $\gamma = 2$ $\perp$ "

"Double adiabatic" equations of state

* Compression that involves components both // and $\perp$ to magnetic field $\vec{B}$:

$\perp$: $$P_\perp = mn \left\langle \frac{v_\perp^2}{2} \right\rangle = n \langle \mu \rangle B \qquad (6.17)$$

If the compression is fast compared to collisions but slower than gyration of particles, μ is conserved,

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{P_\perp}{nB}\right) = 0} \qquad (6.18)$$

$\parallel$: $$P_\parallel = mn \langle v_\parallel^2 \rangle , \qquad J \propto v_\parallel L \qquad (6.19)$$

Using conservation of ptls : $nV = nAL = \text{const}$, $\Rightarrow L = \frac{V}{A} \propto \frac{B}{n}$.

" of magnetic flux : $BA = \text{const}$. (A = area).

Eq. (6.19) → ∴ $$\boxed{\frac{d}{dt}\left(\frac{P_\parallel B^2}{n^3}\right) = 0} \qquad (6.20),$$

* For pure perpendicular compression,

conservation of particles $\rightarrow$ $nA = \text{const}$

" magnetic flux $\rightarrow$ $BA = \text{const}$

$\therefore$ $\frac{d}{dt}\left(\frac{P_\perp}{nB}\right) = 0$ reduces to $\frac{d}{dt}\left(\frac{P_\perp}{n^2}\right) = 0$,

which is consistent with $\gamma = 2$ for $N = 2$.

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* For pure parallel compression in which B is unchanged,

$\frac{d}{dt}\left(\frac{P_\parallel B^2}{n^3}\right) = 0$ reduces to $\frac{d}{dt}\left(\frac{P_\parallel}{n^3}\right) = 0$,

which is consistent with $\gamma = 3$ for $N = 1$.

6.4. Two-fluid Equations

* There are always at least two species in any neutral plasma.

* In momentum balance equation for each species, transfer of momentum between different species due to collisions should be taken into account.

* Rate at which momentum per unit volume is gained by species "α" due to collisions with species β is given by

$$\boxed{\vec{R}_{\alpha\beta} = -m_\alpha n_\alpha \nu_{\alpha\beta} (\vec{u}_\alpha - \vec{u}_\beta)}. \qquad (6.21)$$

where $\nu_{\alpha\beta}$: collision frequency : the rate at which the momentum of species α is transferred to species β.

* Now, the momentum balance equation should be:

$$m_\alpha n_\alpha \left(\frac{\partial}{\partial t} \vec{u}_\alpha + (\vec{u}_\alpha \cdot \vec{\nabla}) \vec{u}_\alpha \right) = n_\alpha q_\alpha (\vec{E} + \vec{u}_\alpha \times \vec{B}) - \vec{\nabla} \cdot \overleftrightarrow{P}_\alpha - \sum_\beta \vec{R}_{\alpha\beta}$$

(6.22).

* From momentum conservation,

$$\vec{R}_{\beta\alpha} = -\vec{R}_{\alpha\beta},$$

This implies →

$$m_\alpha n_\alpha \nu_{\alpha\beta} = m_\beta n_\beta \nu_{\beta\alpha}.$$

6.5. Plasma Resistivity

* Collisions between electrons and ions in a plasma will impede the acceleration of electrons in response to an electric field applied along (or in the absence of) a magnetic field.

* Neglecting small electron inertia (m_e) and assuming homogeneous electron pressure and velocity,

"$0 = -n_e e E_{\parallel} + R_{ei\parallel}$", where $R_{ei\parallel} = -m_e n_e \nu_{ei} (u_{e\parallel} - u_{i\parallel})$

(6.24)

* Current density: $j_{\parallel} = -n_e e (u_{e\parallel} - u_{i\parallel})$. (6.26)

* From Eqs (6.24) and (6.25), we obtain

$$\boxed{E_{\parallel} = -\frac{m_e \nu_{ei}}{e}(u_{e\parallel} - u_{i\parallel}) = \frac{m_e \nu_{ei}}{n_e e^2} j_{\parallel} = \eta\, j_{\parallel}} \qquad (6.26)$$

Taking into account of velocity distribution of electrons,

$\nu_{ei} \rightarrow \langle \nu_{ei} \rangle$ and

$$\boxed{\eta = \frac{m_e \langle \nu_{ei} \rangle}{n_e e^2}} \qquad (6.27)$$

Then,

$$\vec{R}_{ei} = -m_e n_e \langle \nu_{ei} \rangle (\vec{u}_e - \vec{u}_i) = \eta\, n_e e\, \vec{j} \qquad (6.28)$$

* Note that Eq (6.28) can also be applied to the direction $\perp \vec{B}$:

$\eta_{\perp} \simeq 2\eta_{\parallel}$

because electron distribution can be distorted significantly from Maxwellian due to $E_{\parallel}$ in $\parallel$ direction to $\vec{B}$.

* Resistivity of fusion plasma is extremely low, lower than that of copper!