Birth and Death Process
Little’s Law

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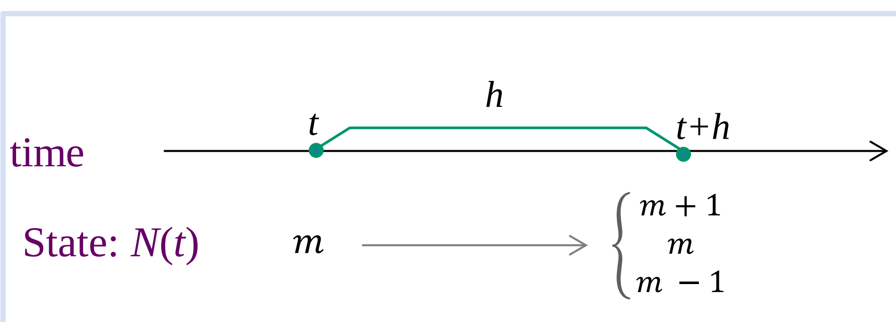
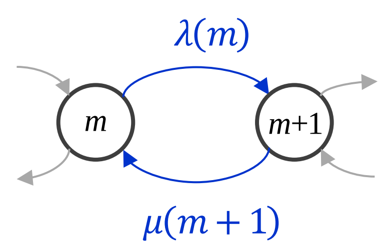
Birth and Death Process (1)

- Birth and death process is a special case of continuous time Markov chain.
- Definition
  - For a random process $\{N(t)\}$, let $q_{m,n}(h) \coloneqq \Pr\{N(t+h) = n \mid N(t) = m\}$,
  - A random process $\{N(t)\}$ is a birth and death process

if it satisfies
$$\begin{cases} q_{m,m+1}(h) = \lambda(m)h + o(h) \\ q_{m,m-1}(h) = \mu(m)h + o(h) \\ q_{m,m}(h) = 1 - \big(\lambda(m) + \mu(m)\big)h + o(h) \\ q_{m,n}(h) = o(h) \quad \text{for } |m-n| > 1, \quad \text{where } \lim_{h\to 0}\left(\frac{o(h)}{h}\right) = 0 \end{cases}$$





$\lambda(m)$: birth rate

$\mu(m)$: death rate

Birth and Death Process (2)

- We need $\pi_n = \lim_{t\to\infty} \Pr\{N(t) = n\}, \quad \forall n$

  π_n : steady state distribution (an ensemble average).

  – For $n \geq 1$

  $$\Pr\{N(t+h) = n\}$$

  $$= \Pr\{N(t) = n\} \Pr\{N(t+h) = n \mid N(t) = n\}$$

  $$+ \Pr\{N(t) = n-1\} \Pr\{N(t+h) = n \mid N(t) = n-1\}$$

  $$+ \Pr\{N(t) = n+1\} \Pr\{N(t+h) = n \mid N(t) = n+1\}$$

  $$+ \sum_{m\in S\setminus\{n,n-1,n+1\}} \Pr\{N(t) = m\} \Pr\{N(t+h) = n \mid N(t) = m\} \cdot$$

  – Let $P_n(t) := \Pr\{N(t) = n\}$. $q_{m,n}(h) := \Pr\{N(t+h) = n \mid N(t) = m\}$

  $$P_n(t+h) = P_n(t) \cdot q_{n,n}(h) + P_{n-1}(t) \cdot q_{n-1,n}(h)$$

  $$+ P_{n+1}(t) \cdot q_{n+1,n}(h) + \sum_{m\in S\setminus\{n,n-1,n+1\}} P_m(t) \cdot q_{m,n}(h)$$

Birth and Death Process (3)

– According to the definition of birth and death process,

$$
\begin{aligned}
P_n(t+h) = P_n(t)\{1 - \big(\lambda(n) + \mu(n)\big)h + o(h)\} \\
+ P_{n-1}(t)\{\lambda(n-1)h + o(h)\} \\
+ P_{n+1}(t)\{\mu(n+1)h + o(h)\} \\
+ \textstyle\sum_{m \in S \setminus \{n, n-1, n+1\}} P_m(t) \cdot o(h)
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) - P_n(t) = -\lambda(n)\, P_n(t)h \;-\; \mu(n)\, P_n(t)h \\
+ P_{n-1}(t)\, \lambda(n-1)h \\
+ P_{n+1}(t)\, \mu(n+1)h \\
+ o(h)\{P_n(t) + P_{n-1}(t) + P_{n+1}(t) + \textstyle\sum_{m \in S \setminus \{n, n-1, n+1\}} P_m(t)\}
\end{aligned}
$$

Birth and Death Process (4)

– Since $\lim_{h\to 0}\left(\frac{P_n(t+h)-P_n(t)}{h}\right) = 0$ in steady state and $\lim_{h\to 0}\left(\frac{o(h)}{h}\right) = 0$,

$$\lim_{h\to 0}\left(\frac{P_n(t+h)-P_n(t)}{h}\right)$$

$$= -\lambda(n)\,P_n(t) - \mu(n)\,P_n(t) + P_{n-1}(t)\,\lambda(n-1) + P_{n+1}(t)\,\mu(n+1)$$

$$+ \lim_{h\to 0}\left(\frac{o(h)}{h}\right)\sum_{m\in S} P_m(t)$$

$$= 0$$

$$\lambda(n)\,P_n(t) + \mu(n)\,P_n(t) = P_{n-1}(t)\,\lambda(n-1) + P_{n+1}(t)\,\mu(n+1)$$

$$\lim_{t\to\infty}\big(\lambda(n)\,P_n(t) + \mu(n)\,P_n(t)\big) = \lim_{t\to\infty}\left(P_{n-1}(t)\,\lambda(n-1) + P_{n+1}(t)\,\mu(n+1)\right)$$

$$\lambda(n)\lim_{t\to\infty} P_n(t) + \mu(n)\lim_{t\to\infty} P_n(t) = \lambda(n-1)\lim_{t\to\infty} P_{n-1}(t) + \mu(n+1)\lim_{t\to\infty} P_{n+1}(t)$$

– Since $\pi_n = \lim_{t\to\infty} P_n(t)$,

$$\lambda(n)\,\pi_n + \mu(n)\,\pi_n = \lambda(n-1)\,\pi_{n-1} + \mu(n+1)\,\pi_{n+1}\,, \quad \text{for } n \geq 1 \qquad \text{... (1)}$$

Birth and Death Process (5)

- When $n = 0$,

  $P_0(t+h) = P_0(t)\{1 - (\lambda(0) + \mu(0))h + o(h)\} + P_1(t)\{\mu(1)h + o(h)\} + o(h).$

  $\lim_{h\to 0}\left(\frac{P_n(t+h)-P_n(t)}{h}\right) = -\lambda(0)P_0(t) - \mu(0)P_0(t) + \mu(1)P_1(t) = 0.$

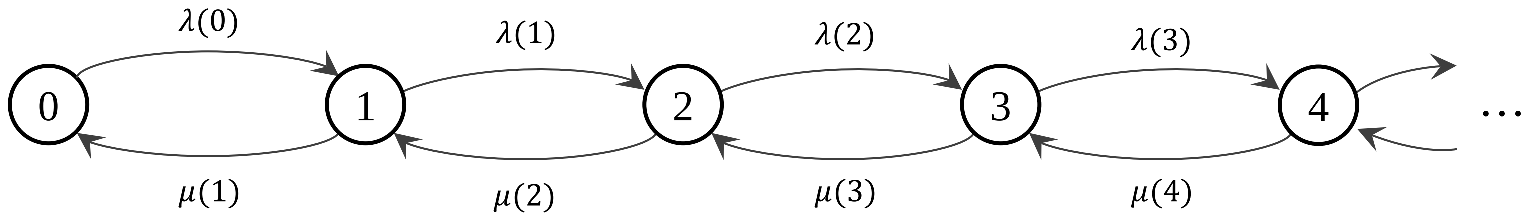
  Thus, $\lambda(0)\ \pi_0 = \mu(1)\ \pi_1 \quad (\because \mu(0) = 0)$ ..... (2)

- From (1)and (2),

  $\lambda(n)\ \pi_n = \mu(n+1)\ \pi_{n+1}, \quad n \geq 0$

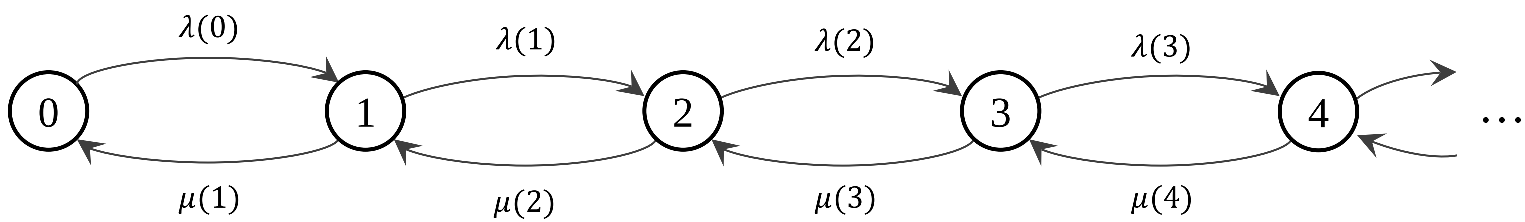
  $\Downarrow$

- State transition rate diagram



Birth and Death Process (6)

– State transition rate diagram



$$\lambda(0)\ \pi_0 = \mu(1)\ \pi_1$$

$$\lambda(1)\ \pi_1 = \mu(2)\ \pi_2$$

$$\cdots$$

$$\times)\ \lambda(n-1)\ \pi_{n-1} = \mu(n)\ \pi_n$$

$$\lambda(0)\lambda(1)\cdots\lambda(n-1)\pi_0\cancel{\pi_1}\cdots\cancel{\pi_{n-2}}\cancel{\pi_{n-1}}$$

$$= \mu(1)\ \mu(2)\cdots\mu(n)\cancel{\pi_1}\cancel{\pi_2}\cdots\cancel{\pi_{n-1}}\pi_n$$

$$\Rightarrow \quad \pi_n = \pi_0 \prod_{i=0}^{n-1} \frac{\lambda(i)}{\mu(i+1)}, \qquad n \geq 1$$

Birth and Death Process (7)

– We need π_0.

All the state probabilities can be expressed in terms of π_0

$$\pi_n = \pi_0 \prod_{i=0}^{n-1} \frac{\lambda(i)}{\mu(i+1)}, \; n \geq 1$$

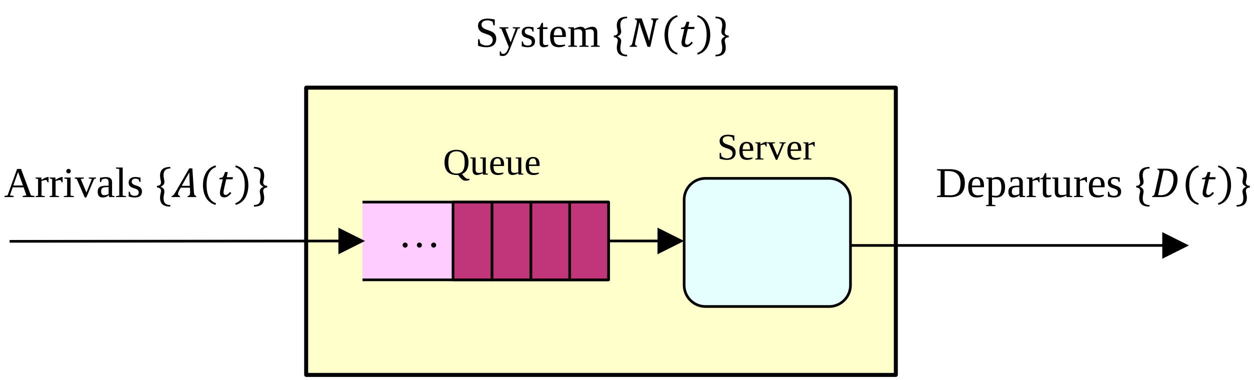
By using $\sum_{n=0}^{\infty} \pi_n = 1$,

$$\pi_0 + \sum_{n=1}^{\infty} \pi_n = \pi_0 + \pi_0 \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda(i)}{\mu(i+1)} = 1$$

$$\pi_0 = \left(1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda(i)}{\mu(i+1)}\right)^{-1}$$

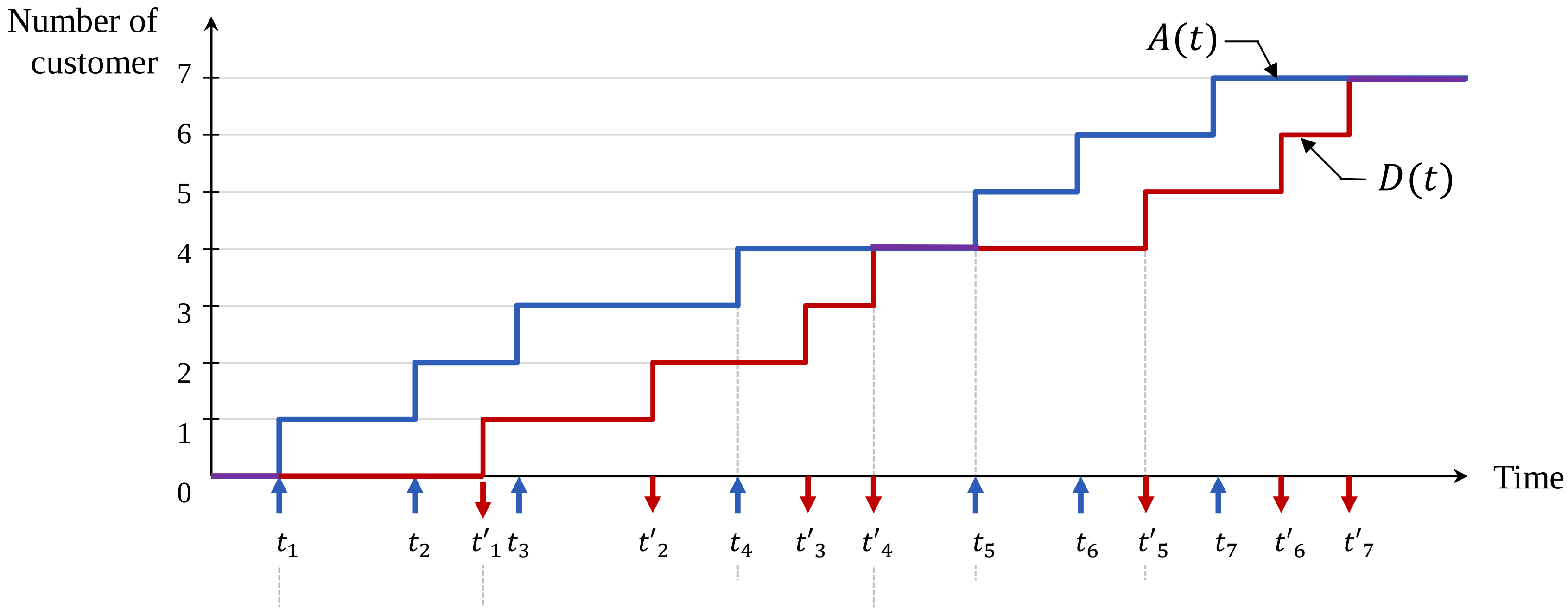
Little's Law (1)

- A single queueing system can be described as follows:
  - Customers arrive for a service, wait if the service cannot start immediately, and leave after being served.



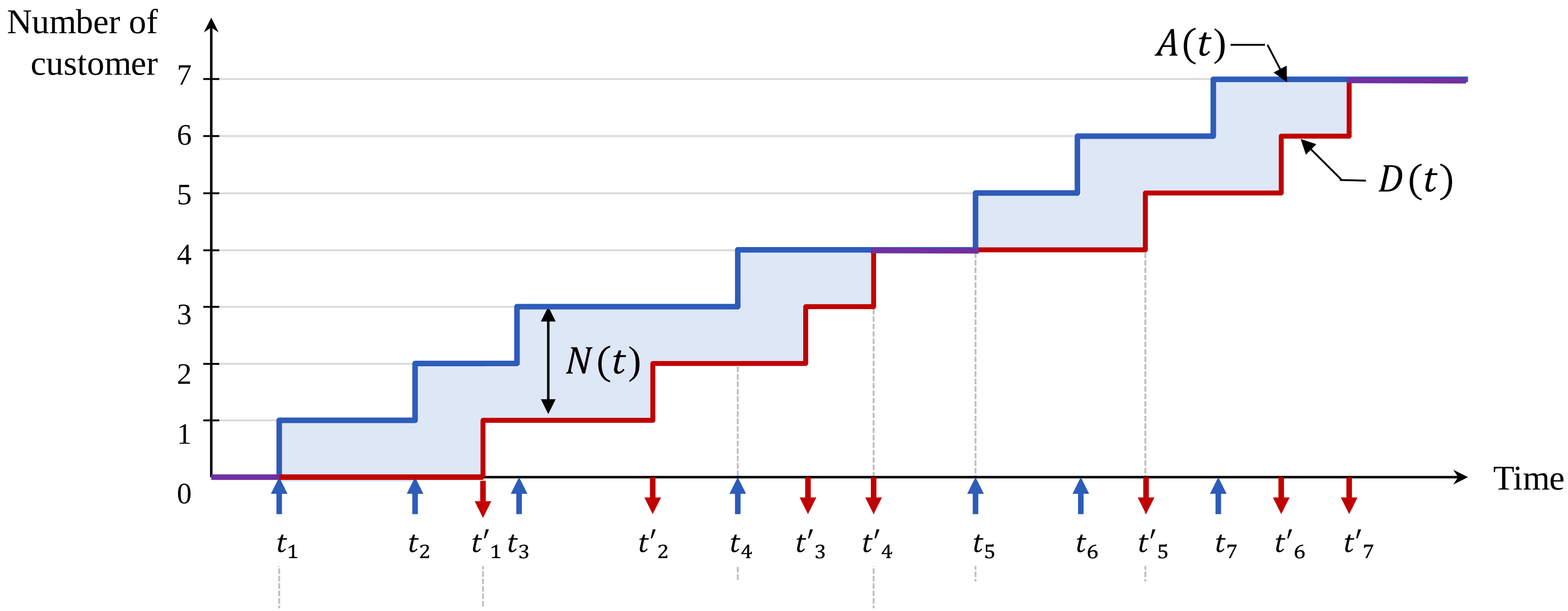
  - $A(t)$: the total number of arrivals until time t
  - $D(t)$: the total number of departures until time t
  - $N(t)$: the total number of customers in the system at time t
  - $N(t) = A(t) - D(t)$

Little's Law (2)



t_n : the n-th arrival time , t'_m: the m-th departure time

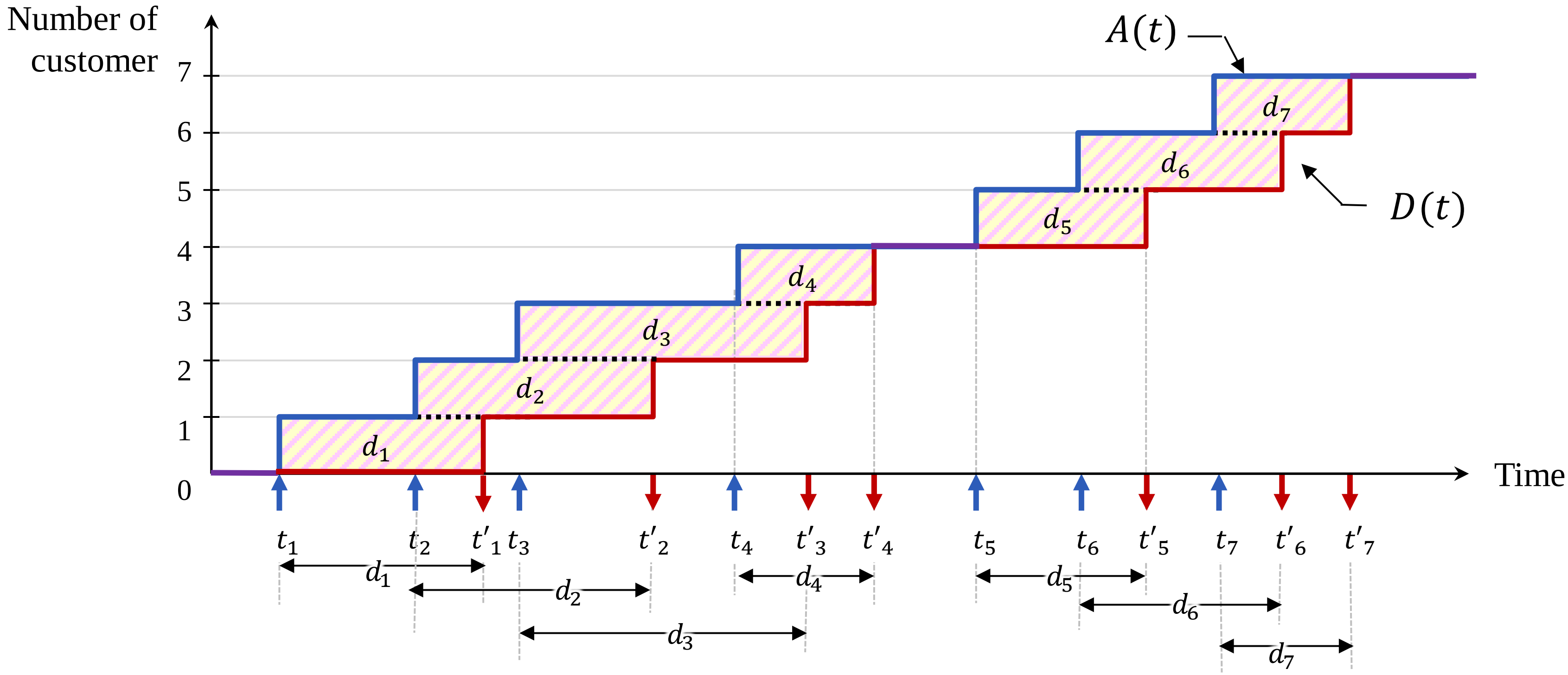
Little's Law (3)



– $\bar{N}$: the mean number of the customers in the system (the long-term average)

$$\bar{N} = \lim_{T\to\infty} \frac{1}{T} \int_0^T N(t)dt$$

Little's Law (4)



– Sojourn time of the customer i in the system: $t_i - t'_i = d_i$

$$\int_0^T N(t)dt = \sum_{i=1}^{M} (t'_i - t_i) = \sum_{i=1}^{M} d_i$$

where M is the total number of arrivals during T

Little's Law (5)

$$\bar{N} = \lim_{T\to\infty} \frac{1}{T}\int_0^T N(t)dt = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{M} d_i$$

– When λ is a customer arrival rate, the total number of arrived customers during time period T is $M = T\lambda$

$$\Rightarrow \frac{1}{T} = \frac{\lambda}{M}, \qquad T \to \infty \Rightarrow M \to \infty$$

$$\bar{N} = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{M} d_i = \lim_{M\to\infty} \frac{\lambda}{M}\sum_{i=1}^{M} d_i$$

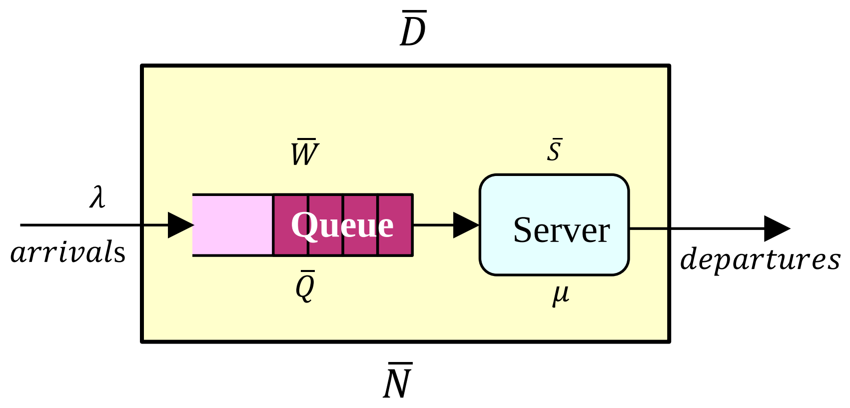
$$= \lambda \lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M} d_i \leftarrow \bar{D}$$

- $\bar{D}$: Mean sojourn time of a customer in the system

$$\Rightarrow \bar{N} = \lambda \bar{D}$$

Little's Law (6)

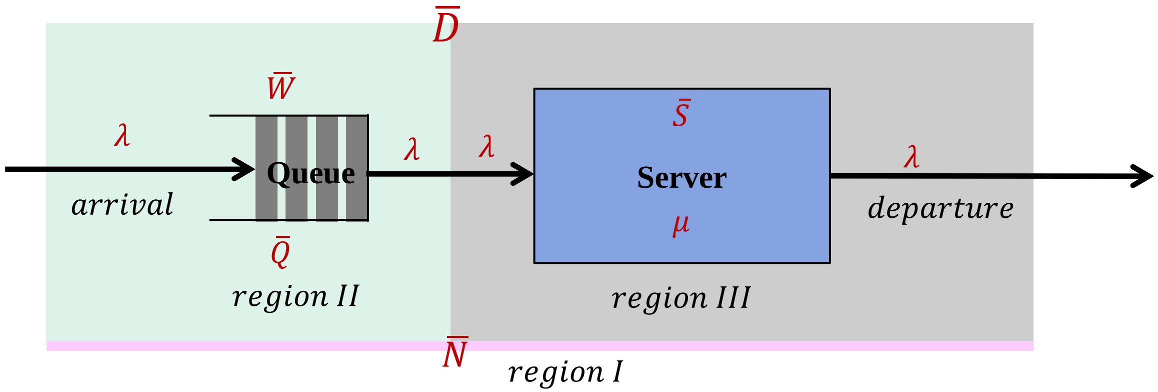
- In summary



  - $\bar{N}$: the mean number of jobs in system
  - $\bar{W}$: the mean waiting time in queue
  - $\bar{S}$: the mean service time
  - $\bar{D}$: the mean sojourn time in system
  - $\bar{Q}$: the mean number of jobs in queue
  - λ : the mean arrival rate
  - μ : the mean number of jobs in server

Little's Law (7)

- The system can be viewed from several regions

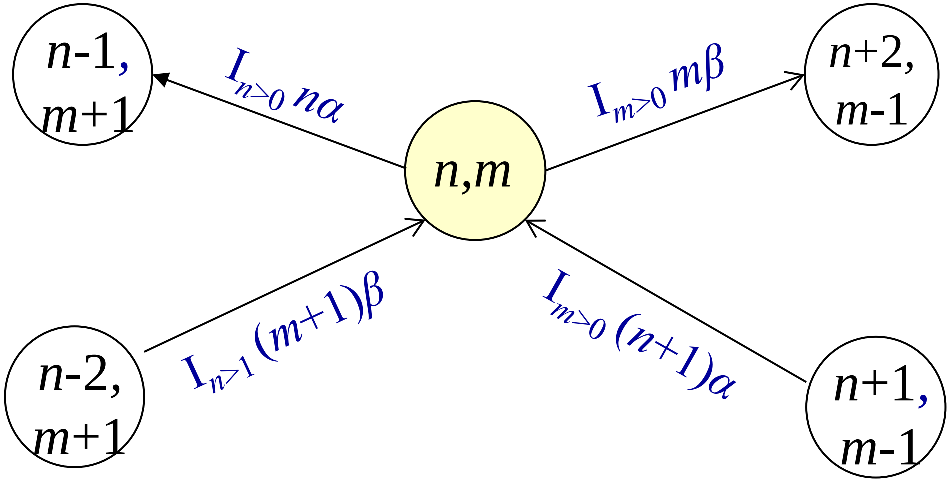


  - From region I : $\lambda\overline{D} = \overline{N}$
    - $\overline{N} = \bar{Q} + \mu$
    - $\overline{D} = \overline{W} + \bar{S}$
  - From region II : $\lambda\overline{W} = \bar{Q}$
  - From region III : $\lambda\bar{S} = \mu$

Exercise: BD Process

- Suppose that a one-celled organism can be in one of two states-either A or B.
- An individual in state A will change to state B at an exponential rate α
- An individual in state B divides into two new individuals of type A at an exponential rate β .
- Define an appropriate continuous-time Markov chain for a population of such organisms and determine the appropriate parameters for this model.

System state: (n, m) where n (m) is the number of state A (B) individuals.
The state (0,0) is not defined.
I_c is an indicator whose value is 1 if the condition is true; otherwise, $I_c = 0$.



$n \geq 0, m \geq 0$