# Ch. 7. Relation between fluid equations and g.c. drifts

## 7.1 Diamagnetic Drift.

* Ignoring inertia and collisions, the momentum balance relation is

$$\boxed{nq(\vec{E}+\vec{u}\times\vec{B}) \approx \vec{\nabla}\cdot\overleftrightarrow{P}} \qquad (7.2)$$

By taking $\vec{B}\times$ of Eq. (7.2) and expanding the triple product, we obtain

$$\vec{u}_\perp = \frac{\vec{E}\times\vec{B}}{B^2} + \frac{\vec{B}\times(\vec{\nabla}\cdot\overleftrightarrow{P})}{nqB^2} \qquad (7.5)$$

ExB drift

- Diamagnetic Drift
- ions and electrons in opposite direction

* We did NOT find a 'diamagnetic' guiding-center drift.

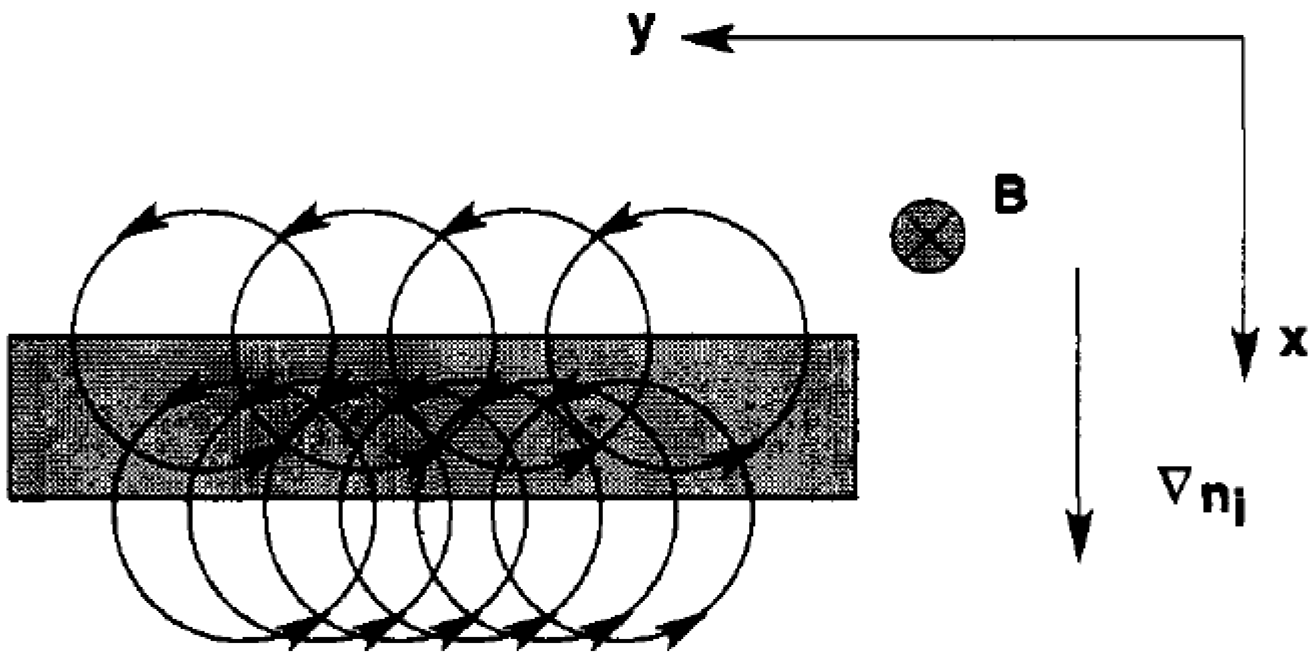
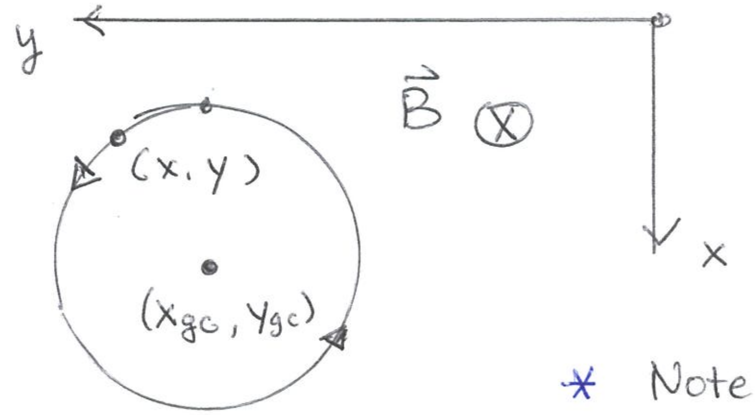



**Figure 7.1.** Larmor orbits of ions in the presence of a density gradient. In the shaded region there is a net current to the left, even though the guiding centers have no net motion.




* Note that particle position and guiding center position are related by

$$x = x_{gc} - \frac{v_y}{\omega_c} \qquad (7.7)$$

* The mean y-directed (fluid) drift, $u_y$, at $x$ due to particles in $dx$ is

$$\boxed{n u_y \, dx = \int v_y f(x, \vec{v}) \, d^3v \, dx} \qquad (7.6)$$

* We can also define a distribution function of guiding centers $\underline{f_{gc}(x_{gc}, \vec{v})}$, where $\vec{v}$ remains a particle velocity.

$$* \quad f(x, \vec{v})\, dx = f_{gc}(x_{gc}, \vec{v})\, dx_{gc}$$

$$= f_{gc}\left(x + \frac{v_y}{\omega_c}, \vec{v}\right) \underbrace{\left(\frac{dx_{gc}}{dx}\right)}_{=1 \text{ for uniform } \vec{B}.} dx \qquad (7.8)$$

$$= f_{gc}\left(x + \frac{v_y}{\omega_c}, \vec{v}\right).$$

$$\therefore \quad u_y = \frac{1}{n} \int v_y\, f_{gc}\left(x + \underbrace{\frac{v_y}{\omega_c}}_{\text{Taylor expand around } x.}, \vec{v}\right) d^3v$$

$$\approx \frac{1}{n} \int v_y \left( f_{gc}(x, \vec{v}) + \frac{v_y}{\omega_c} \frac{\partial f_{gc}(x, \vec{v})}{\partial x} \right) d^3v \qquad (7.10)$$

$$\rightarrow \quad \boxed{u_y = \frac{1}{nqB} \frac{dP}{dx}} \qquad (7.11)$$

# 7.4. Diamagnetic Drift in Non-uniform B Fields

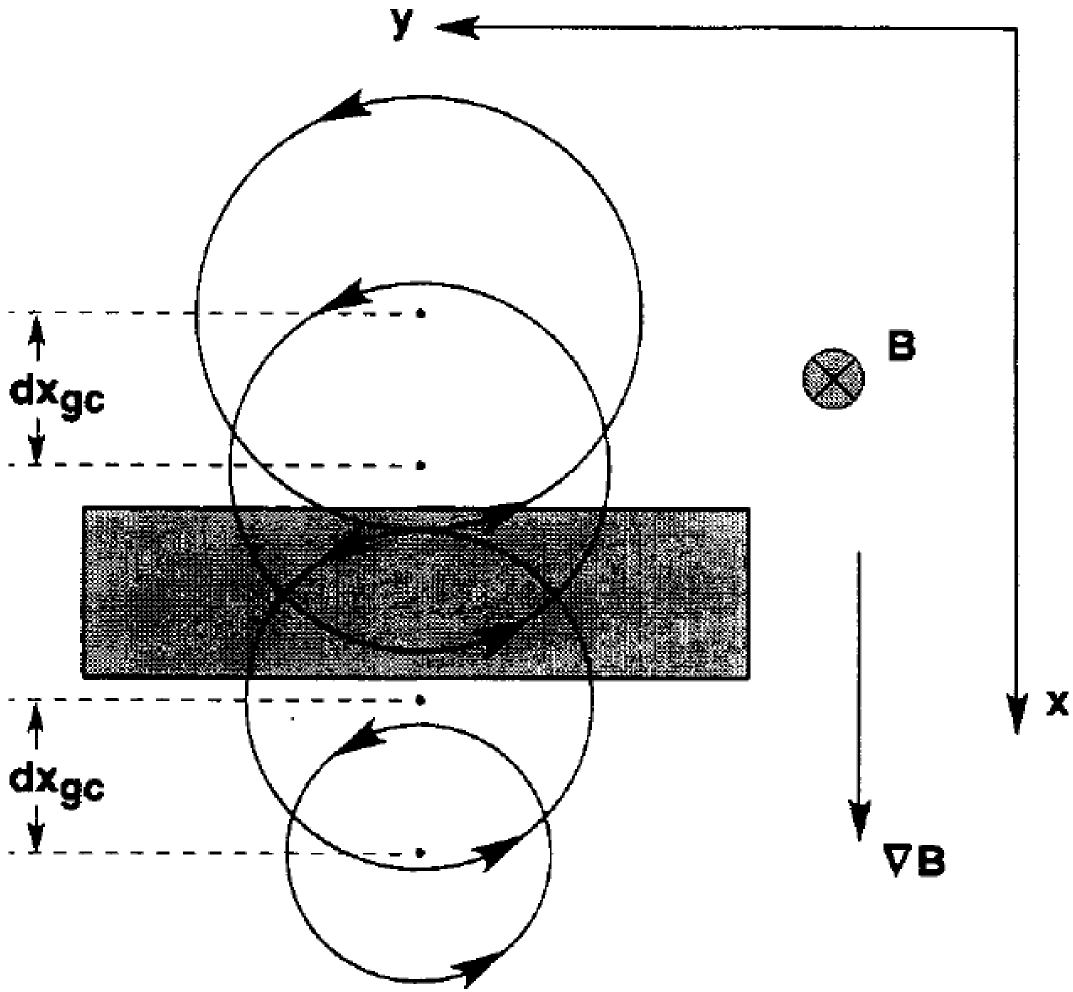



**Figure 7.3.** Larmor orbits of ions in the presence of a field gradient. For guiding centers with equal spacing $dx_{gc}$, there are more particles with $v_y < 0$ falling in the shaded region, and fewer particles with $v_y > 0$, leading to a net current to the right.

* Since $$x = x_{gc} - \frac{v_y}{\omega_c(x_{gc})} \qquad dx_{gc} \quad , \qquad (7.\cancel{24}\,7)$$

$$dx = \left(1 + \frac{v_y}{\omega_c}\frac{1}{B}\frac{dB}{dx}\right) dx_{gc} \qquad (7.24)$$

$$\therefore \; f(x, \vec{v}) = f_{gc}\left(x + \frac{v_y}{\omega_c}, \vec{v}\right) \frac{dx_{gc}}{dx}$$

$$= f_{gc}\left(x + \frac{v_y}{\omega_c}, \vec{v}\right)\left(1 + \frac{v_y}{\omega_c}\frac{1}{B}\frac{dB}{dx}\right)^{-1}$$

$$\approx \left(f_{gc} + \frac{v_y}{\omega_c}\frac{d}{dx} f_{gc}\right)\left(1 - \frac{v_y}{\omega_c}\frac{1}{B}\frac{dB}{dx}\right)$$

$$\approx f_{gc} + \frac{v_y}{\omega_c}\frac{df_{gc}}{dx} - \frac{v_y}{\omega_c}\frac{1}{B}\frac{dB}{dx} \cdot f_{gc} \qquad (7.25).$$

* Then, $$u_y = \frac{1}{n}\int v_y f(x, \vec{v})\, d^3v$$

$$\boxed{u_y = \frac{1}{nqB}\frac{dP}{dx} - \frac{I}{qB^2}\frac{dB}{dx}} \quad , \text{ for Maxwellian.} \qquad (7.27)$$

Average velocity in the frame in which guiding centers have no motion

* On the other hand, the average $\nabla B$ drift for a Maxwellian distribution function ( $\langle v_\perp^2/2 \rangle = T/m$ ) is

$$v_{Dy} = \frac{\langle v_\perp^2/2 \rangle}{\omega_c B} \frac{dB}{dx} = \frac{T}{qB^2} \frac{dB}{dx} \qquad (7.28)$$

* So the total fluid velocity in the laboratory frame can be obtained by adding Eq(7.27) to (7.28),

$$\boxed{u_y = \frac{1}{nqB} \frac{dP}{dx}} \qquad (7.29).$$

* A similar discussion regarding the curvature drift is given on pg. 108 – 109. Make sure you understand it.

$\Longrightarrow$ $\boxed{\text{Reading Assignment.}}$

## 7.6. Parallel Pressure Balance



* Since electrons can respond to a perturbation (say a "hump" in $n$) along the B field more quickly than ions do,

$$0 \approx e n_e \nabla_{\parallel} \phi - \nabla_{\parallel} P_e , \qquad (7.43)$$

which leads to

$$\boxed{n_e \propto \exp\left( e\phi / T_e \right)} \qquad (7.44).$$

"Boltzmann relation"

for uniform $T_e$.

* Then, electric field pulls the ions in the same direction as their own pressure gradient, i.e,

$$m_i n_i \dot{u}_{i\parallel} = -n_i e \nabla_{\parallel} \phi - \nabla_{\parallel} P_i \approx -(T_e + T_i) \nabla_{\parallel} n \qquad (7.46)$$

* Boltzmann relation is much more efficient than using Poisson's equation to find a relation between $n_e$ and $\phi$ in many cases.

# Single-fluid Magnetohydrodynamics

## 8.1. MHD equations:

* Consider a quasi-neutral ($n_e \simeq Z n_i$) two species plasma in the presence of a strong $\vec{B}$ field. Let's consider $Z=1$ for simplicity

* MHD model treats the plasma as a "single fluid" with

- $\rho = n_i M_i + n_e m_e \simeq n(M+m) \simeq nM$. mass density (8.1)
- $\sigma = (n_i - n_e) e$ charge " . (8.2)
- $\vec{u} = (n_i M \vec{u}_i + n_e m \vec{u}_e)/\rho \simeq \vec{u}_i + (\frac{m}{M}) \vec{u}_e$ mass velocity (8.3)
- $\vec{j} = e(n_i \vec{u}_i - n_e \vec{u}_e) \simeq n e (\vec{u}_i - \vec{u}_e)$ current density (8.4),

Eqs. (8.3) and (8.4) can be written as

$$\vec{u}_i \simeq \vec{u} + \frac{m}{M}\frac{\vec{j}}{ne}, \quad \vec{u}_e \simeq \vec{u} - \frac{\vec{j}}{ne} \tag{8.5}$$

* From $\frac{\partial n_i}{\partial t} + \vec{\nabla}\cdot(n_i \vec{u}_i) = 0$, Continuity eqn for ions,

and $\frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e \vec{u}_e) = 0$. " electrons

* By multiplying with $M$ and $m$ respectively, and adding

$$\boxed{\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0}$$ Mass Continuity Equation (8.7)

* By subtracting one from another,

$$\boxed{\frac{\partial}{\partial t}\sigma + \vec{\nabla}\cdot\vec{j} = 0}$$ Charge Continuity Equation (8.8)

* From

$$Mn_i \frac{d\vec{u}_i}{dt} = en_i(\vec{E} + \vec{u}_i \times \vec{B}) - \vec{\nabla} p_i + \vec{R}_{ie}$$ ion momentum balance

$$mn_e \frac{d\vec{u}_e}{dt} = -en_e(\vec{E} + \vec{u}_e \times \vec{B}) - \vec{\nabla} p_e + \vec{R}_{ei}$$ electron " (8.9)

* By adding two momentum balance equations, we obtain

$$\boxed{\rho \frac{d}{dt}\vec{u} = \rho\left(\frac{\partial}{\partial t}\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right) = \sigma\vec{E} + \vec{j}\times\vec{B} - \vec{\nabla}p} \tag{8.10}$$

cf. - Here, the pressure is defined w.r.t. the mass velocity $\vec{u}$,

unlike $p_i$ and $p_e$ w.r.t. $\vec{u}_i$ and $\vec{u}_e$ respectively.

- Typically, $\vec{u} \approx \vec{u}_i + \left(\frac{m}{M}\right)\vec{u}_e \approx \vec{u}_i$

* By ignoring electron inertia, and using $\vec{R}_{ei} = mn\langle\nu_{ei}\rangle(\vec{u}_i - \vec{u}_e) = \eta n e \vec{j}$,

the electron momentum balance equation can be written as

$$\vec{E} + \vec{u}_e\times\vec{B} = \eta\vec{j} - \frac{\vec{\nabla}p_e}{ne}. \tag{8.12}$$

or

$$\boxed{\vec{E} + \vec{u}\times\vec{B} = \eta\vec{j} + \frac{\vec{j}\times\vec{B} - \vec{\nabla}p_e}{ne}} \tag{8.12}$$

The generalized Ohm's law.

* The equation of State :

 - often adiabatic law $\boxed{\frac{d}{dt}\left(\frac{P}{\rho^{\gamma}}\right)=0}$ is used (8.14).

 - sometimes isothermal law with $T_e, T_i = $ constant is used

* The system of equations is closed by including the four Maxwell equations.

## 8.2. The quasi-neutrality Approximation

From
$$\rho\frac{d\vec{u}}{dt} = \rho\left(\frac{\partial}{\partial t}\vec{u} + \underbrace{(\vec{u}\cdot\vec{\nabla})\vec{u}}_{1}\right) = \underbrace{\sigma}_{2}\vec{E} + \vec{j}\times\vec{B} - \vec{\nabla}P,$$

$$2/1 \sim \frac{\sigma E}{\rho(\vec{u}\cdot\vec{\nabla})\vec{u}} \sim \frac{\epsilon_0 E^2/L}{\rho u^2/L} \sim \frac{\epsilon_0 E^2}{\rho u^2} \sim \frac{\epsilon_0 B^2}{\rho} \quad \underline{\ll 1} \qquad (8.19)$$

using Poisson's equ $\vec{\nabla}\cdot(\epsilon_0\vec{E}) = \sigma$, and $u \sim E/B$,

now $\epsilon_0 B^2/\rho$ is a very small quantity ($\sim 10^{-2}-10^{-3}$) $\therefore$ 2 is negligible. for almost all plasmas of interest.

* Comparing the two terms appearing in the charge continuity equation,

$$\frac{\partial\sigma/\partial t}{\vec{\nabla}\cdot\vec{j}} \sim \frac{\epsilon_0 E/L\tau}{j/L} \sim \frac{\epsilon_0 uB/\tau}{\rho u/B\tau} \sim \frac{\epsilon_0 B^2}{\rho} \ll 1 \qquad (8.20)$$

where $\vec{j}\times\vec{B} \sim \rho\frac{\partial}{\partial t}\vec{u}$ has been used to estimate $j$.

∘∘∘ $\sigma\vec{E}$ in momentum equation and $\frac{\partial\sigma}{\partial t}$ in charge continuity equation can be neglected. → "Quasi-neutrality approximation".

* But quasi-neutrality equation does NOT mean that $\sigma$ can be neglected in the Poisson equation $\vec{\nabla}\cdot(\epsilon_0\vec{E}) = \sigma$ !

* "$\sigma$" does not appear elsewhere in MHD equations, and Poisson equation can be dropped from the system of MHD equations.