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# Poisson Process

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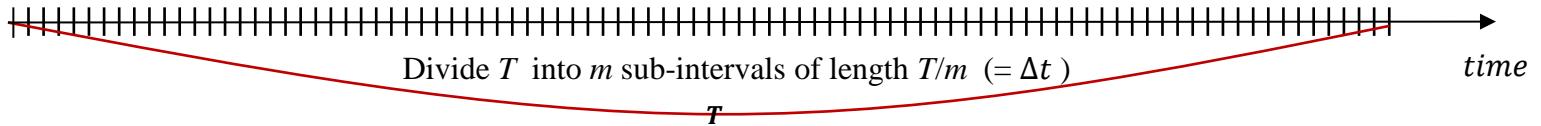
# Definition

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- (Homogeneous) Poisson Process
  - The counting process  $\{N(t), t \geq 0\}$  is Poisson process having rate  $\lambda$  ( $\lambda > 0$ ) if
    - 1)  $N(0) = 0$
    - 2) The events in  $(t, t + \Delta t]$  is independent of  $t$
    - 3) The probability of an event in  $(t, t + \Delta t]$  is  $\lambda\Delta t + o(\Delta t)$
    - 4) The probability of no event in  $(t, t + \Delta t]$  is  $1 - \lambda\Delta t + o(\Delta t)$   
where  $o(\Delta t)$  is any function such that  $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$
  - When  $\Delta t \rightarrow 0$ , it implies that
    - Probability of an event within  $\Delta t$  :  $\lambda\Delta t$
    - Probability of no event within  $\Delta t$  :  $1 - \lambda\Delta t$
    - Probability of more than one event within  $\Delta t \approx 0$
- ⇒ This can be seen as Bernoulli process

# Probability distribution

- Representation with Bernoulli process



- When  $\Delta t \rightarrow 0$ ,  $m \rightarrow \infty$
- $X$ : random variable representing the number of events during  $T$
- Since an event occurs at each subinterval according to Bernoulli process with probability  $\lambda\Delta t$ ,

$$\begin{aligned}\Pr\{X = n\} &= \lim_{m \rightarrow \infty} \binom{m}{n} (\lambda\Delta t)^n (1 - \lambda\Delta t)^{m-n} \\ &= \lim_{m \rightarrow \infty} \frac{m!}{n! (m-n)!} \frac{(\lambda T)^n}{m^n} \frac{\left(1 - \lambda \frac{T}{m}\right)^m}{\left(1 - \lambda \frac{T}{m}\right)^n} \\ &= \frac{(\lambda T)^n}{n!} \lim_{m \rightarrow \infty} \frac{m!}{(m-n)! m^n} \frac{\left(1 - \lambda \frac{T}{m}\right)^m}{\left(1 - \lambda \frac{T}{m}\right)^n} \\ &= \frac{(\lambda T)^n}{n!} \lim_{m \rightarrow \infty} \left(1 - \lambda \frac{T}{m}\right)^m = \frac{(\lambda T)^n}{n!} e^{-\lambda T}\end{aligned}$$

⇒  $\Pr\{X = n\} = \frac{(\lambda T)^n}{n!} e^{-\lambda T}$

# Exponential Distribution Property (1)

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## 1. The inter-event time is exponentially distributed

*<Proof>*

- Let  $A_c(t)$  denote the probability of no event in  $(t_0, t_0 + t]$
- $A_c(t + \Delta t) = Pr\{\text{no event in } (t_0, t_0 + t]\} \times Pr\{\text{no event in } (t_0 + t, t_0 + t + \Delta t]\}$

$$A_c(t + \Delta t) = A_c(t)(1 - \lambda\Delta t + o(\Delta t))$$

$$A_c(t + \Delta t) - A_c(t) = -\lambda\Delta t A_c(t) + o(\Delta t)A_c(t)$$

$$\lim_{\Delta t \rightarrow 0} \frac{A_c(t + \Delta t) - A_c(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} (-\lambda A_c(t) + \frac{o(\Delta t)A_c(t)}{\Delta t})$$

$$A'_c(t) = -\lambda A_c(t)$$

$$\frac{A'_c(t)}{A_c(t)} = -\lambda \quad \dots (1)$$

# Exponential Distribution Property (2)

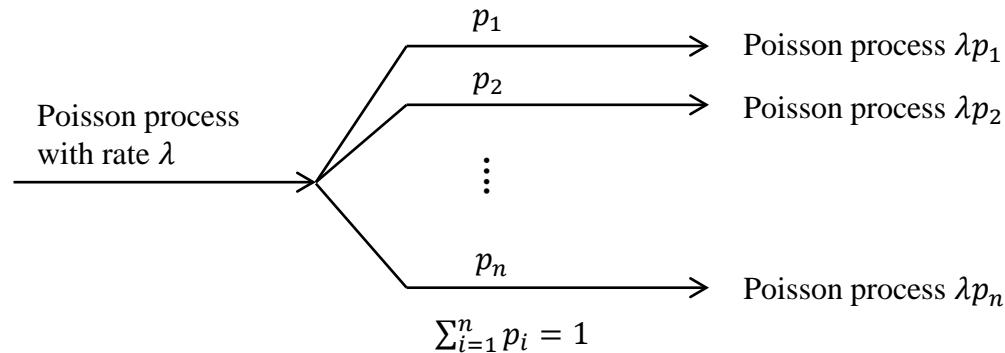
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- From (1)
  - $\int \frac{A'_c(u)}{A_c(u)} du = \ln A_c(t) = -\lambda t + k$
- When  $t = 0$ ,  $\ln A_c(0) = k$
- Since  $A_c(0) = 1$ ,  $k = 0$
- Thus,  $\ln A_c(t) = -\lambda t$ 
  - $A_c(t) = e^{-\lambda t}$
- If  $X$  is a random variable of inter-event time
  - $A_c(t) = Pr\{X > t\} = e^{-\lambda t}$
  - $Pr\{X \leq t\} = 1 - e^{-\lambda t}$

The last term implies that  $X$  is exponentially distributed.

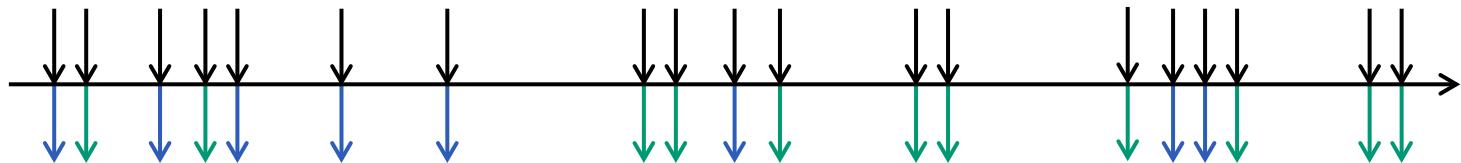
# Decomposition Property (1)

## 2. Decomposition



### – Example

- Event arrival follows Poisson process (black arrow)
- Each event is classified with probability
  - ✓ blue arrow with  $p_1$  and green arrow with  $(1 - p_1)$
- According to decomposition property, these are also respectively Poisson processes



# Decomposition Property (2)

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## 2. Decomposition

*< Proof >*

- Consider a Poisson process with rate  $\lambda$
- Without loss of generality, we can assume that each event is categorized into two types:
  - Type I with probability  $p$
  - Type II with probability  $(1 - p)$
- Let  $N_1(t)$  be the number of type I events in  $[0, t]$   
Let  $N_2(t)$  be the number of type II events in  $[0, t]$   
Let  $N(t)$  be the total number of events in  $[0, t]$ 
  - $N(t) = N_1(t) + N_2(t)$

# Decomposition Property (3)

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< Proof (cont.) >

$$Pr\{N_1(t) = n, N_2(t) = m\}$$

$$= Pr\{N(t) = n + m\} Pr\{N_1(t) = n, N_2(t) = m | N(t) = n + m\}$$

$$= \frac{(\lambda t)^{m+n}}{(m+n)!} e^{-\lambda t} \times \binom{m+n}{n} p^n (1-p)^m$$

$$= \frac{(\lambda pt)^n}{n!} \frac{(\lambda(1-p)t)^m}{m!} e^{-\lambda t}$$

$$P_r\{N_1(t) = n\} = \sum_{m=0}^{\infty} P_r\{N_1(t) = n, N_2(t) = m\}$$

$$= \sum_{m=0}^{\infty} \frac{(\lambda pt)^n}{n!} \frac{(\lambda(1-p)t)^m}{m!} e^{-\lambda t} = \frac{(\lambda pt)^n}{n!} e^{-\lambda t} \sum_{m=0}^{\infty} \frac{(\lambda(1-p)t)^m}{m!}$$

$$= \frac{(\lambda pt)^n}{n!} e^{-\lambda t} e^{\lambda(1-p)t}$$

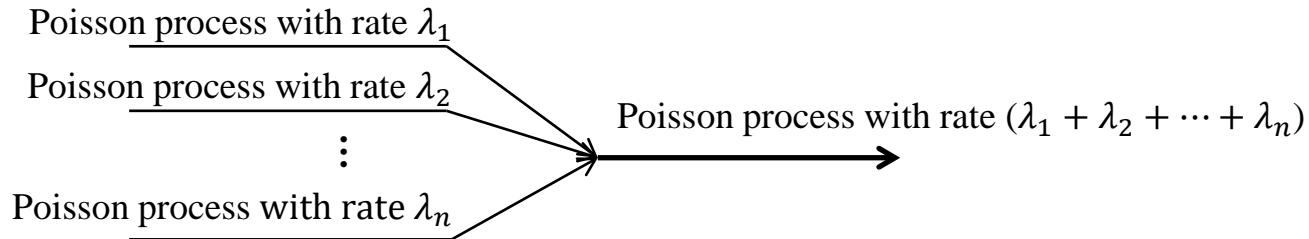
$$\text{Thus, } Pr\{N_1(t) = n\} = \frac{(\lambda pt)^n}{n!} e^{-\lambda pt}$$

$$\text{Similarly, } Pr\{N_2(t) = m\} = \frac{(\lambda(1-p)t)^m}{m!} e^{-\lambda(1-p)t}$$

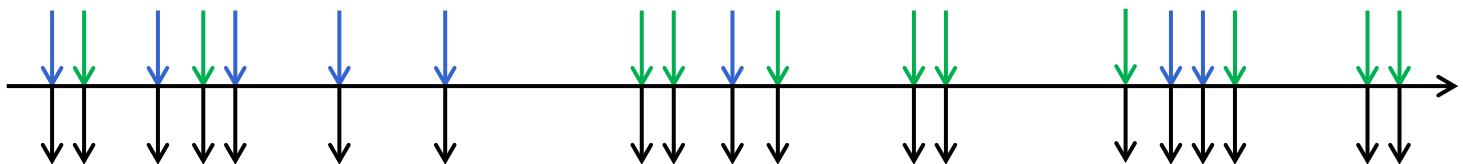
Poisson process

# Superposition Property (1)

## 3. Superposition



- Example
  - There are two types of events (blue and green), which respectively arrive according to Poisson process
  - The combination of those events (black arrows) also follows Poisson process



# Superposition Property (2)

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<Proof>

- Consider two Poisson processes with rate  $\lambda_1$  and  $\lambda_2$

$$\begin{aligned} Pr\{N(t) = n\} &= \sum_{k=0}^n Pr\{N_1(t) = k\} \times Pr\{N(t) = n | N_1(t) = k\} \\ &= \sum_{k=0}^n Pr\{N_1(t) = k\} \times Pr\{N_2(t) = n - k\} \\ &= \sum_{k=0}^n \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_1 t} \times \frac{(\lambda_2 t)^{n-k}}{(n-k)!} e^{-\lambda_2 t} \\ &= \frac{e^{-(\lambda_1 + \lambda_2)t}}{n!} \sum_{k=0}^n \frac{n!}{k! (n-k)!} (\lambda_1 t)^k (\lambda_2 t)^{n-k} \\ &= \frac{((\lambda_1 + \lambda_2)t)^n}{n!} e^{-(\lambda_1 + \lambda_2)t} \end{aligned}$$

- Therefore,  $\{N(t), t \geq 0\}$  is a Poisson process with rate  $(\lambda_1 + \lambda_2)$

# Non-Homogeneous Poisson Process (NHPP)

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- Definition
  - The counting process  $\{N(t), t \geq 0\}$  is NHPP with rate  $\lambda(t), t \geq 0$  if
    - $N(0) = 0$
    - $\Pr\{N(t + \Delta t) - N(t) = 0\} = 1 - \lambda(t)\Delta t + o(\Delta t)$
    - $\Pr\{N(t + \Delta t) - N(t) = 1\} = \lambda(t)\Delta t + o(\Delta t)$
    - $\Pr\{N(t + \Delta t) - N(t) \geq 2\} = o(\Delta t)$
- Derivation of  $\Pr\{N(t) = n\}$

$$\Pr\{N(t + \Delta t) = 0\}$$

$$= \Pr\{N(t) = 0\} \times \Pr\{N(t + \Delta t) - N(t) = 0\}$$

Let  $P_n(t) := \Pr\{N(t) = n\}$  for simple notation. Then

$$P_0(t + \Delta t) = P_0(t)(1 - \lambda(t)\Delta t + o(\Delta t)) \quad \dots \quad (1)$$

$$P_n(t + \Delta t) = \sum_{k=0}^n P_{n-k}(t) \Pr\{N(t + \Delta t) - N(t) = k\} \quad \dots \quad (2)$$

# NHPP (2)

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- From (1)

$$\frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -P_0(t)\lambda(t) + \frac{P_0(t)o(\Delta t)}{\Delta t}$$

$$\lim_{\Delta t \rightarrow 0} \frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -P_0(t)\lambda(t) + \lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} P_0(t)$$

$$\Rightarrow \quad \frac{dP_0(t)}{dt} = -P_0(t)\lambda(t) \quad \dots (3)$$

- From (2)

$$\begin{aligned} P_n(t + \Delta t) &= P_n(t) \Pr\{N(t + \Delta t) - N(t) = 0\} + P_{n-1}(t) \Pr\{N(t + \Delta t) - N(t) = 1\} \\ &\quad + \sum_{k=2}^n P_{n-k}(t) \Pr\{N(t + \Delta t) - N(t) = k\} \\ &= P_n(t)(1 - \lambda(t)\Delta t + o(\Delta t)) + P_{n-1}(t)(\lambda(t)\Delta t + o(\Delta t)) + o(\Delta t) \sum_{k=2}^n P_{n-k}(t) \end{aligned}$$

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} &= -\lambda(t)P_n(t) + \lambda(t)P_{n-1}(t) + \lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} \sum_{k=2}^n P_{n-k}(t) \\ \Rightarrow \quad \frac{dP_n(t)}{dt} &= -\lambda(t)P_n(t) + \lambda(t)P_{n-1}(t) \quad \dots (4) \end{aligned}$$

# NHPP (3)

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- Define PGF for random variable  $N(t)$ :  $\mathbb{P}(z, t) := \sum_{n=0}^{\infty} P_n(t) z^n$
- Differentiate:  $\frac{d\mathbb{P}(Z, t)}{dt} = \frac{dP_0(t)}{dt} + z \frac{dP_1(t)}{dt} + z^2 \frac{dP_2(t)}{dt} + \dots$
- From (3), (4)

$$\frac{d\mathbb{P}(Z, t)}{dt} = -\lambda(t)P_0(t)$$

$$- z\lambda(t)P_1(t) + z\lambda(t)P_0(t)$$

$$- z^2\lambda(t)P_2(t) + z^2\lambda(t)P_1(t)$$

....

$$= -\lambda(t)(P_0(t) + zP_1(t) + z^2P_2(t) + \dots)$$

$$+ z\lambda(t)(P_0(t) + zP_1(t) + z^2P_2(t) + \dots)$$

$$= -\lambda(t)\mathbb{P}(z, t) + z\lambda(t)\mathbb{P}(z, t)$$

$$\frac{d\mathbb{P}(z, t)}{dt} = -\lambda(t)(1 - z)\mathbb{P}(z, t)$$

–  $\left. \frac{d\mathbb{P}(z, t)}{dt} \right|_{\mathbb{P}(z, t)} = -\lambda(t)(1 - z) \quad \dots (5)$

# NHPP (4)

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- Integrating (5)

$$\int_0^t \frac{d\mathbb{P}(z, u)}{du} \Big/ \mathbb{P}(z, u) du = \int_0^t -\lambda(u)(1 - z) du + K(z)$$

$$\ln \mathbb{P}(z, t) = \int_0^t -\lambda(u)(1 - z) du + K(z)$$

- Since  $N(0) = 0$ ,  $P_0(0) = 1$ ,  $P_n(0) = 0$  for  $n \geq 1 \Rightarrow \mathbb{P}(z, 0) = 1$

$$\ln \mathbb{P}(z, 0) = \int_0^0 -\lambda(u)(1 - z) du + K(z) = 0 \Rightarrow K(z) = 0$$

- $\ln \mathbb{P}(z, t) = - \int_0^t \lambda(u) du + z \int_0^t \lambda(u) du$

$$\mathbb{P}(z, t) = e^{- \int_0^t \lambda(u) du} e^{z \int_0^t \lambda(u) du}$$

# NHPP (5)

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- Since  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  and  $\mathbb{P}(z, t) = e^{-\int_0^t \lambda(u)du} e^{z \int_0^t \lambda(u)du},$

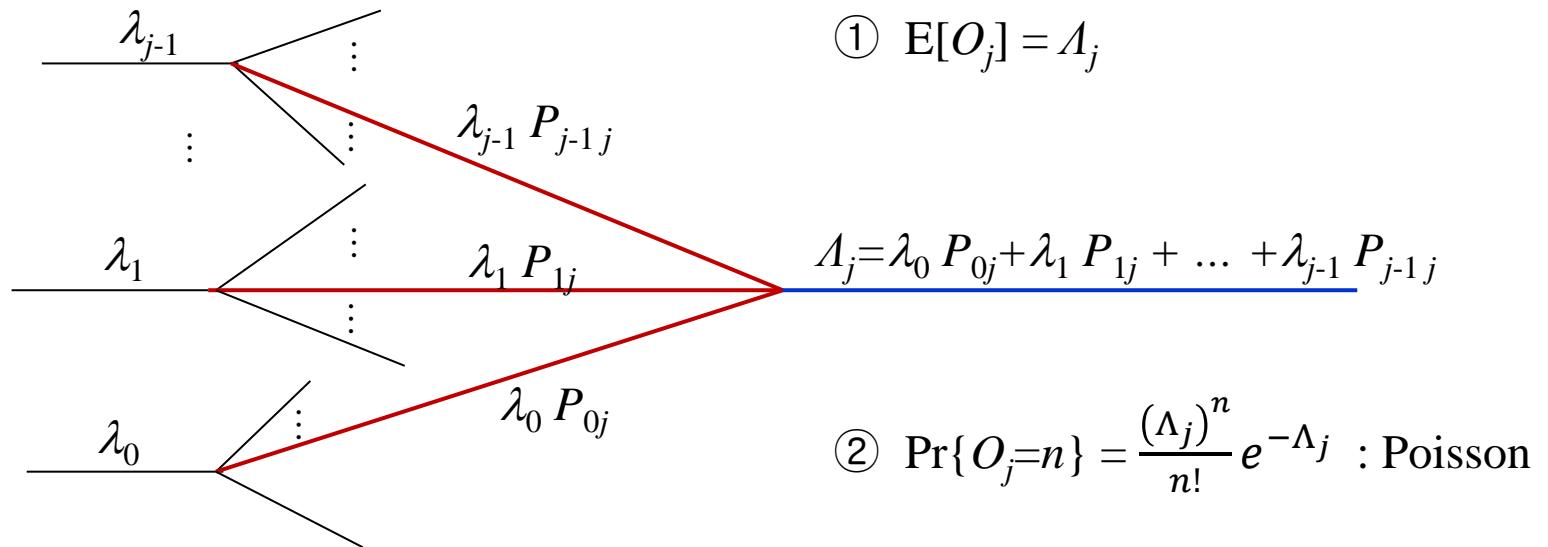
$$\begin{aligned}\sum_{n=0}^{\infty} P_n(t) z^n &= e^{-\int_0^t \lambda(u)du} \sum_{n=0}^{\infty} \frac{(z \int_0^t \lambda(u)du)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\left(\int_0^t \lambda(u)du\right)^n}{n!} e^{-\int_0^t \lambda(u)du} z^n\end{aligned}$$

- Therefore,

$$P_n(t) = \frac{\left(\int_0^t \lambda(u) du\right)^n}{n!} e^{-\int_0^t \lambda(u)du}$$

# Exercise 1: HPP

- Consider an elevator that starts in the basement (floor 0) and travels upward. Let  $N_i$  denote a random variable representing the number of people that get in the elevator at floor  $i$ . Assume that  $N_i$ 's are independent and that  $N_i$  is Poisson with mean  $\lambda_i$ . Each person entering at  $i$  will, independent of everything else, get off at  $j$  with probability  $P_{ij}$  ( $\sum_{j>i} P_{ij} = 1$ ). Let  $O_j$  be the number of people getting off the elevator at floor  $j$ .
- ① Compute  $E[O_j]$
  - ② What is the probability distribution of  $O_j$ ?



## Exercise2: NHPP

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- Suppose that each event of a Poisson process  $X(t)$  with rate  $\lambda$  is classified as being either of type 1, 2, ...,  $k$ . If the event occurs at time  $s$ , then, independently of all else, it is classified as type  $i$  with probability  $p_i(s)$ . Note that the probability  $p_i(s)$  depends on the event arrival time  $s$ . Let  $N_i(t)$  denote the number of type  $i$  arrivals in  $[0, t]$ . What is the probability distribution of  $N_i(t)$ ,  $\Pr\{N_i(t) = n\}$ ?
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- Arrival rate of type  $i$  events at time  $s$ :  $\lambda P_i(s)$
- Average number of type  $i$  events in  $[0, t]$ :  $\int_0^t \lambda P_i(s) ds$

$$\Pr\{N_i(t) = n\} = \frac{\left(\int_0^t \lambda P_i(s) ds\right)^n}{n!} e^{-\int_0^t \lambda P_i(s) ds}$$