8.3 The 'Small Larmor Radius' Approximation

* Compare the last terms of the generalized Ohm's law, to $\vec{u}\times\vec{B}$ term

$$\frac{\vec{\nabla} P_e / ne}{\vec{u}\times\vec{B}} \sim \frac{T}{euBL} \sim \frac{M v_{Ti}^2}{euBL} \sim \frac{M v_{Ti}}{eBL} \sim \frac{v_{Ti}}{\omega_{ci} L} \sim \frac{r_{Li}}{L} \qquad (8.22)$$

where $u \sim v_{Ti}$ has been used.

Also, $\dfrac{\vec{j}\times\vec{B}/ne}{\vec{u}\times\vec{B}} \sim \cdots \sim \dfrac{r_{Li}}{L}$, since $\vec{\nabla} p \sim \vec{j}\times\vec{B}$.

* For $\dfrac{r_{Li}}{L} \ll 1$, the Ohm's can be simplified.

$$* \quad \underline{\vec{E} + \vec{u}\times\vec{B}} = \eta \vec{j} \qquad (8.23)$$

the effective electric field seen by a fluid element moving with velocity $\vec{u}$ across $\vec{B}$

(for $|\vec{u}| \ll c$).

"MHD model" consists of the set of following equations.

- $$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$
- $$\vec{\nabla}\cdot\vec{j} = 0 \qquad (8.24)$$
- $$\rho \frac{d}{dt}\vec{u} = -\vec{\nabla}p + \vec{j}\times\vec{B}$$
- $$\vec{E} + \vec{u}\times\vec{B} = \eta\vec{j} \qquad (8.23)$$

together with the required versions of the Maxwell eqns.

- $$\vec{\nabla}\times\vec{B} = \mu_0\vec{j}$$
- $$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B} \qquad (8.25)$$
- $$\vec{\nabla}\cdot\vec{B} = 0$$

8.4. The Approximation of "Infinite Conductivity"

η is very small in magnitude in a high-temperature plasma

→ Some dynamical phenomena at large scale can often be described with an approximation of $\eta = 0$
or
infinite conductivity

→ $\boxed{\text{ideal MHD.}}$

The relation

$\vec{E} + \vec{u} \times \vec{B} = 0$ leads to strong constraints on the evolution of $\vec{u}$ and $\vec{B}$

"Plasma is tied to the magnetic field lines."

"Any two elements of an ideal plasma that lie initially on a given field line will still lie on the same field line after an arbitrary motion of the plasma."

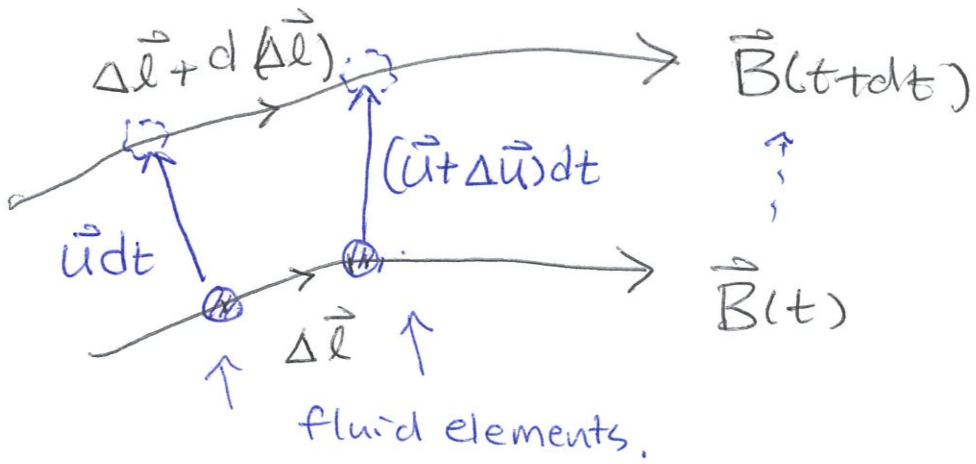
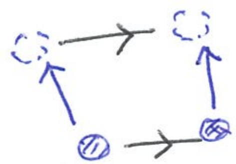


Fig 8.1,

* Let's consider two plasma fluid elements separated by $\Delta\vec{l}$ on the same magnetic field line $\vec{B}(t)$.

* After "dt", the two fluid elts move $\vec{u}\,dt$ and $(\vec{u}+\Delta\vec{u})dt$ respectively, while $\vec{B}(t)$ moves to $\vec{B}(t+dt)$.

* Let's calculate "$d(\Delta\vec{l})$." From Fig 8.1,



the vectors should add up.

$$\rightarrow \quad \Delta\vec{l} + d(\Delta\vec{l}) = \Delta\vec{l} + (\vec{u}+\Delta\vec{u})dt - \vec{u}\,dt = \Delta\vec{l} + \Delta\vec{u}\,dt \qquad (8.28)$$

on top

$$\text{So} \quad \frac{d\,\Delta\vec{l}}{dt} = \Delta\vec{u} = (\Delta\vec{l}\cdot\vec{\nabla})\,\vec{u}$$

from Taylor expansion (8.27)

* On the other hand, starting from the Faraday's law,

$$\frac{\partial}{\partial t}\vec{B} = -\vec{\nabla}\times\vec{E} \underset{\text{Ohm's law}}{=} \vec{\nabla}\times(\vec{u}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}\cdot\vec{u}) \qquad (8.30)$$

$$\because \vec{\nabla}\cdot\vec{B} = 0$$

* The total derivative of $\vec{B}$, following the motion of the plasma is

$$\frac{d}{dt}\vec{B} = \left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{u} - \vec{B}(\vec{\nabla}\cdot\vec{u}). \qquad (8.31)$$

* Let's evaluate

$$\frac{d}{dt}(\Delta\vec{l}\times\vec{B}) = \frac{d}{dt}\Delta\vec{l}\times\vec{B} + \Delta\vec{l}\times\frac{d}{dt}\vec{B} = \underbrace{[(\Delta\vec{l}\cdot\vec{\nabla})\vec{u}]\times\vec{B}}_{\sim} + \underbrace{\Delta\vec{l}\times[(\vec{B}\cdot\vec{\nabla})\vec{u}}_{\sim} - \underline{\vec{B}(\vec{\nabla}\cdot\vec{u})}] \qquad (8.32)$$

$= 0$ because $\Delta\vec{l}\times\vec{B} = 0$ initially

- Two terms underlined by ~ cancel, if $\Delta\vec{l} \parallel \vec{B}$ (the same expression if $\Delta\vec{l}$ and $\vec{B}$ are interchanged).

$\therefore$ $\boxed{\frac{d}{dt}(\Delta\vec{l}\times\vec{B}) = 0}$ and

$\Delta\vec{l}$ moves so as to remain parallel to $\vec{B}$.

8.5. Conservation of Magnetic Flux

⊛ "The magnetic flux through any closed contour that moves with the plasma is constant."

$$\Phi \equiv \int \vec{B} \cdot d\vec{S}$$

Magnetic Flux, closed contour differential element of area.

$$\frac{d}{dt}\Phi = \int \vec{\nabla}\times(\vec{u}\times\vec{B}) \cdot d\vec{S} + \int \vec{B} \cdot \frac{d}{dt}(\Delta\vec{S}) \qquad (8.36)$$

due to $\left(\frac{\partial \vec{B}}{\partial t}\right)$ due to movement of area.

$$\frac{d}{dt}(\Delta\vec{S}) = \vec{u}\times\Delta\vec{\ell} \quad (8.37) \leftarrow \quad d\Delta\vec{S} = \vec{u}\,dt\times\Delta\vec{\ell}$$

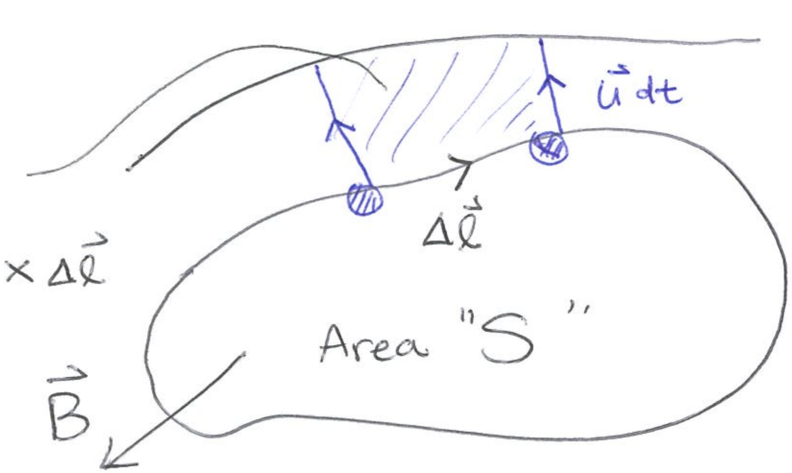


Figure 8.2.

From Eq. (8.36)

$$\frac{d}{dt}\Phi = \int \vec{\nabla}\times(\vec{u}\times\vec{B})\cdot d\vec{S} + \int \vec{B}\cdot\frac{d}{dt}\Delta\vec{S}$$

↓ Stokes' Theorem   Eq. (8.37) ↓

$$= \int(\vec{u}\times\vec{B})\cdot\Delta\vec{\ell} + \int\vec{B}\cdot(\vec{u}\times\Delta\vec{\ell}) = 0 \;!$$

from vector identity.

"The magnetic flux through an area bounded by any closed contour 'painted' on the plasma is unchanged in any motion of the plasma."

8.6. Conservation of Energy

⊛ The total energy of a system described by ideal MHD;

$$W = \int \left(\frac{\rho}{2} |\vec{u}|^2 + \frac{P}{\gamma - 1} + \frac{\epsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2 \right) d^3x \qquad (8.40)$$

Kinetic energy density ~ directed motion — thermal energy density ~ random motion. — electric field energy density — magnetic field energy density

" $\dfrac{dW}{dt} = 0$ "

Recall: $\rho \left(\frac{\partial}{\partial t} \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = \sigma \vec{E} + \vec{j} \times \vec{B} - \vec{\nabla} P \qquad (8.10).$

" Single-fluid equation of motion "

Contribution from LHS of Eq. (8.10); from taking $\vec{u}\cdot$ (8.10) and integrating over all space,

$$\circledast \quad \int \rho \vec{u}\cdot\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right)d^3x = \frac{1}{2}\int\left(\rho\frac{\partial}{\partial t}|\vec{u}|^2 + \rho\vec{u}\cdot\vec{\nabla}|\vec{u}|^2\right)d^3x$$

$$= \frac{1}{2}\int\left(\rho\frac{\partial}{\partial t}|\vec{u}|^2 - |\vec{u}|^2\,\vec{\nabla}\cdot(\rho\vec{u})\right)d^3x = \frac{1}{2}\int\left(\rho\frac{\partial}{\partial t}|\vec{u}|^2 + |\vec{u}|^2\frac{\partial\rho}{\partial t}\right)d^3x$$

→ from integration by part

→ Continuity equation

$$= \frac{\partial}{\partial t}\int\frac{\rho}{2}|\vec{u}|^2\,d^3x$$

Contribution from the 3rd term on RHS of Eq. (8.10);

$$\circledast \quad \int \vec{u}\cdot\vec{\nabla}p\,d^3x = -\int p\,\vec{\nabla}\cdot\vec{u}\,d^3x = \frac{1}{\gamma-1}\int\rho\,\frac{d}{dt}\left(\frac{P}{\rho}\right)d^3x$$

IBP

from Eq.(8.42) below:

$$0 = \frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = \frac{1}{\rho^{\gamma-1}}\frac{d}{dt}\left(\frac{P}{\rho}\right) - \frac{(\gamma-1)P}{\rho^{\gamma+1}}\frac{d\rho}{dt} = \frac{1}{\rho^{\gamma-1}}\frac{d}{dt}\left(\frac{P}{\rho}\right) + \frac{(\gamma-1)P}{\rho^\gamma}\vec{\nabla}\cdot\vec{u} \qquad (8.42)$$

adiabatic eqn of state

8.7. Magnetic Reynolds Number

⊛ Ideal MHD assumes infinite electrical conductivity (zero resistivity).
"What is the consequence of non-zero resistivity?"

⊛
$$\frac{\partial}{\partial t}\vec{B} = -\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(\vec{u}\times\vec{B}) - \vec{\nabla}\times(\eta\vec{j}) = \vec{\nabla}\times(\vec{u}\times\vec{B}) + \frac{\eta}{\mu_0}\nabla^2\vec{B} \qquad (8.44)$$

Ohm's law

Ampère's law.

$$= -(\vec{u}\cdot\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}\cdot\vec{u}) + (\vec{B}\cdot\vec{\nabla})\vec{u} + \underline{\frac{\eta}{\mu_0}\nabla^2\vec{B}} \qquad (8.45)$$

Convection of $\vec{B}$ with plasma

Plasma Compression

" " $\perp$ to $\vec{B}$ "

"Resistive Diffusion" of ~~Plasma~~ $\vec{B}$ across the plasma

⊛ For motion with characteristic scale length "L"
and " plasma speed "u",

$$\frac{\text{"Convection"}}{\text{"diffusion"}} \sim \frac{\mu_0 u L}{\eta} \equiv R_M.$$

- Magnetic Reynolds #
- Lundquist #

$> 10^8$! for fusion-relevant plasmas.

Homework

Problem 8.2 : on page 127

Ch. 9. MHD Equilibrium

9.1. Assuming $\eta = 0$, $\vec{u} = 0$, isotropic pressure,

a steady-state solution of MHD equations satisfies;

$$\boxed{\vec{\nabla} P = \vec{j} \times \vec{B}}, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \qquad (9.1)$$

$$\Rightarrow \quad \vec{j}_\perp = \frac{\vec{B} \times \vec{\nabla} P}{B^2} \quad ; \text{ diamagnetic current} \qquad (9.2)$$

From quasi-neutrality, $\vec{\nabla} \cdot \vec{j} = 0$; and with $\vec{\nabla} \cdot \vec{B} = 0$,

$$\Rightarrow \quad (\vec{B} \cdot \vec{\nabla}) \left(\frac{j_\parallel}{B} \right) + \vec{\nabla} \cdot \vec{j}_\perp = 0 \qquad (9.5)$$

9.2. Magnetic Pressure

$$\vec{\nabla} P = \vec{j} \times \vec{B} = \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) \times \vec{B} = \frac{1}{\mu_0}\left\{(\vec{B} \cdot \vec{\nabla})\vec{B} - \vec{\nabla}\left(\frac{B^2}{2}\right)\right\} \tag{9.6}$$

Ampère's law

$$\Rightarrow \quad \boxed{\vec{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}(\vec{B} \cdot \vec{\nabla})\vec{B}} \tag{9.7}$$

"Pressure-balance condition"!

where

$$(\vec{B} \cdot \vec{\nabla})\vec{B} = B^2(\hat{b} \cdot \vec{\nabla})\hat{b} + \hat{b}(\hat{b} \cdot \vec{\nabla})\frac{B^2}{2}$$

bending force ($\perp \vec{B}$) parallel compression of field lines.

In some cases, the field lines are approximately straight and parallel, $\Rightarrow$ RHS of (9.7) ≈ 0.

$$\Rightarrow \quad \text{"} \; \underbrace{P}_{\text{plasma pressure}} + \underbrace{\frac{B^2}{2\mu_0}}_{\text{magnetic-field pressure}} = \text{constant} \; \text{"}$$

$$\boxed{\beta = 2\mu_0 \frac{P}{B^2}} \qquad (9.9)$$

"a measure of the degree to which the magnetic field is holding a ~~a~~ non-uniform plasma in equilibrium."

⊛ To achieve magnetic fusion efficiently, we need plasmas with higher β value.

* Conventional tokamak plasmas : $\beta < 10\%$.
* Relatively innovative configurations such as spherical torus : $\beta < 1$.
* Astrophysical plasmas : $\beta \geq 1$

9.3. Cylindrical Pinch

(*) cylindrical plasma with B_θ only and j_z only.

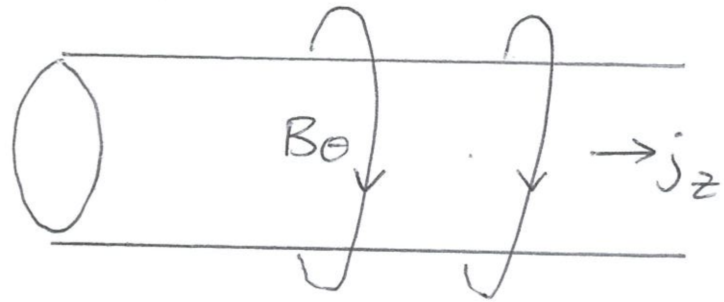


Fig 9.2.

Since $\dfrac{\partial}{\partial \theta}\hat{\theta} = -\hat{r}$, Eq. (9.7) becomes

$$\boxed{\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2}{2\mu_0}\right) = -\frac{B_\theta^2}{\mu_0 r}} \qquad (9.10)$$

From integration from 0 to r, we obtain

$$P(r) = P(0) - \frac{B_\theta^2(r)}{2\mu_0} - \frac{1}{\mu_0}\int_0^r \frac{B_\theta^2}{r}\,dr \qquad (9.11)$$

There exist infinitely many possible equilibria of this kind.

For illustration, let's consider

$$\begin{cases} j_z(r) = j_{z0}, & r \le a \\ j_z(r) = 0, & r > a \end{cases} \qquad (9.12).$$

For that current profile, we obtain $I = \pi a^2 j_{z0}$ and,

$$B_\theta(r) = \frac{B_{\theta a}\, r}{a}, \quad \text{and} \quad p(r) = p(0) - \frac{B_{\theta a}^2}{\mu_0}\frac{r^2}{a^2} \qquad (9.13)$$

i.e. Parabolic pressure profile.

Since $p(a) = 0$, we obtain " $p(0) = \dfrac{B_{\theta a}^2}{\mu_0} = \dfrac{\mu_0 I^2}{4\pi^2 a^2}$ " (9.14).

Pinch Condition

— In this configuration, j_z is entirely from the plasma diamagnetic current, and this provides the entire magnetic field.

— Unfortunately, this configuration is strongly unstable against MHD instability.

9.4. Force-free Equilibria: 'Cylindrical' Tokamak

Consider a low-β cylindrical plasma,

$$\Rightarrow \quad \boxed{0 = \vec{j} \times \vec{B}} \qquad (9.15)$$

Force-free Equilibria

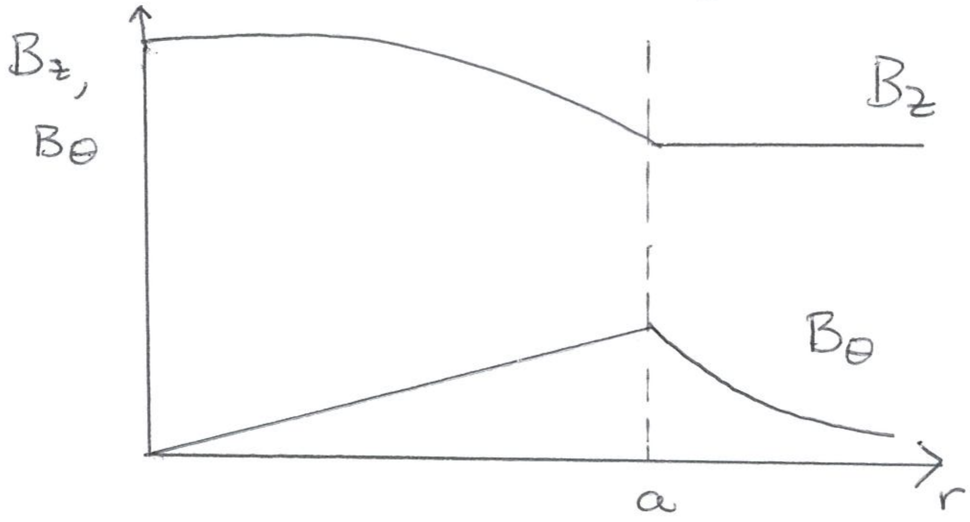
$$\rightarrow \quad \frac{\partial}{\partial r}\left(\frac{B_\theta^2}{2} + \frac{B_z^2}{2}\right) = -\frac{B_\theta^2}{r} \qquad (9.16)$$

Then consider the current density $j_z(r)$ which is uniformly distributed within the plasma so that $B_\theta(r) = B_{\theta a}\left(\frac{r}{a}\right)$.

Integrating Eq. (9.16) $\Rightarrow$

$$B_z(r)^2 = B_z(0)^2 - B_\theta(r)^2 - 2\int_0^r \frac{B_\theta^2}{r}\,dr = B_z(0)^2 - 2B_\theta(r)^2 \qquad (9.17)$$

Eq (9.17) is illustrated in Fig. 9.4.



← "Paramagnetic" plasma

$B_z(0) > B_z(a)$

But, when a moderate amount of plasma pressure $P > B_0^2/2\mu_0$ is added, the plasma becomes "diamagnetic".

i.e, $B_z(0) < B_z(a)$.

Homework

Problem 9.2 on page 135.