Point Process

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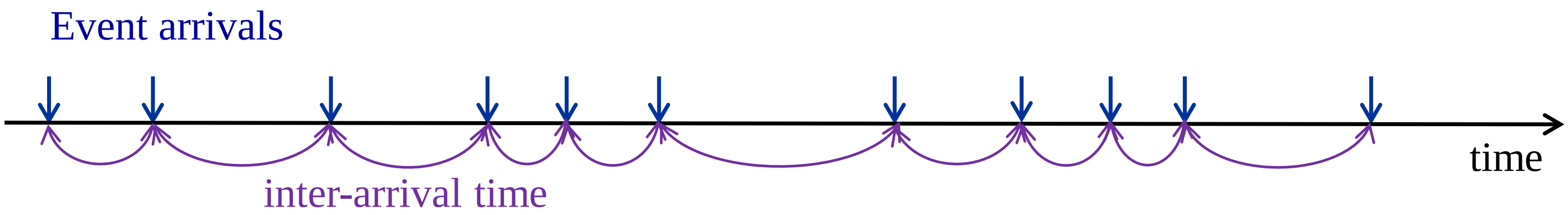
Temporal Point Process

- Temporal (1-dimension) Point Process
  - Example: Poisson process



- Homogeneous

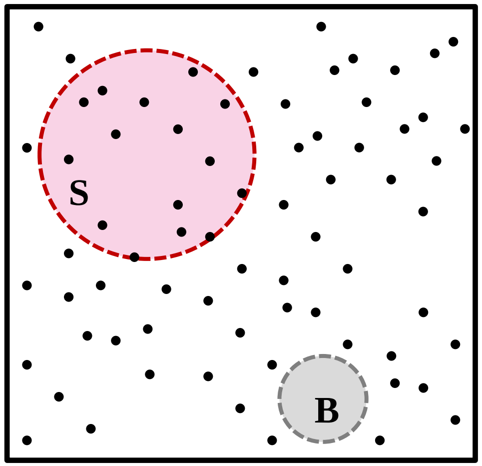
$$\Pr\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

- In-homogeneous

$$\Pr\{N(t) = n\} = \frac{\left(\int_0^t \lambda(u)du\right)^n}{n!} e^{-\int_0^t \lambda(u)du}$$

Spatial Point Process

- Spatial (2-dimensions) Point Process



✓ For given area S, how many points (N(S)) are there?

✓ For given area B, is there any point or not (V(B))?

- V(B)∈{0,1} is vacancy in area B

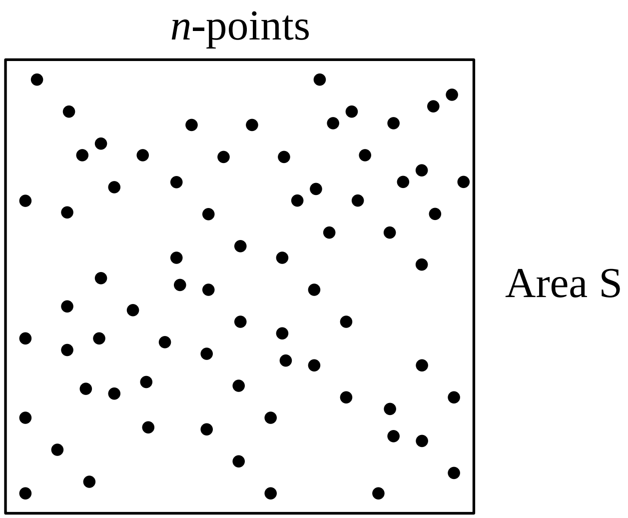
✓ Thinning

- For a point process with intensity λ, when either retaining each point with probability q or deleting it with probability 1-q, independently other points, the new process is a point process with intensity $q\lambda$

- Applications
  - Location modeling of some objects
    - Trees in a forest
    - Birds nests
    - Stars in galaxy
    - Galaxies in space

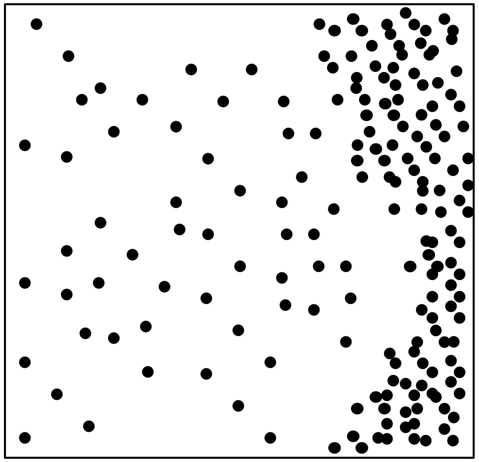
Spatial Poisson Process (1)

- Homogeneous
  - $\Pr\{N(S) = n\} = \frac{(\lambda|S|)^n}{n!} e^{-\lambda|S|}$
    - |S| represents the area size
    - Locations of *n*-points in S are independent and uniformly distributed random variable



Spatial Poisson Process (2)

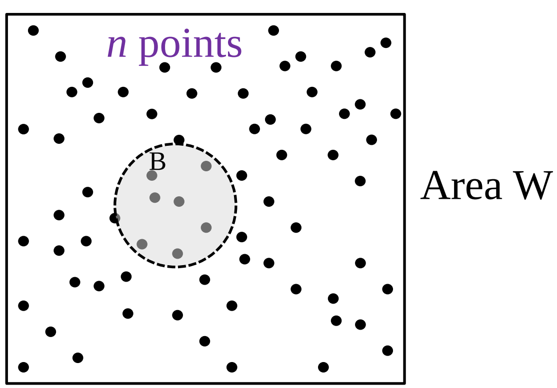
- In-homogeneous



  - $\Pr\{N(S) = n\} = \frac{\left(\int\int_S \lambda(x,y)dxdy\right)^n}{n!} e^{-\int\int_S \lambda(x,y)dxdy}$
    - Where $\lambda(x, y)dxdy$ is the probability that a point occurs in the region $\{(x, x + dx), (y, y + dy)\}$

Spatial Poisson Process (3)

- Consider a Poisson point process in $\mathbb{R}^2$ with uniform intensity (homogeneous) $\beta > 0$
- Given that $N(W) = n$, the condition distribution of $N(B)$ for $B \subseteq W$ is binomial



$$\Pr\{N(B) = k | N(W) = n\}$$

$$= \frac{\Pr\{N(B)=k\}\Pr\{N(W\setminus B)=n-k\}}{\Pr\{N(W)=n\}}$$

$$= \frac{\frac{(\cancel{\lambda}|B|)^k}{k!} \cancel{e^{-\lambda|B|}} \; \frac{(\cancel{\lambda}|W\setminus B|)^{n-k}}{(n-k)!} \cancel{e^{-\lambda|W\setminus B|}}}{\frac{(\cancel{\lambda}|W|)^n}{n!} \cancel{e^{-\lambda|W|}}}$$

$$= \frac{n!}{k!(n-k)!} \left(\frac{|B|}{|W|}\right)^k \left(\frac{|W|-|B|}{|W|}\right)^{n-k}$$

$$= \binom{n}{k} \left(\frac{|B|}{|W|}\right)^k \left(1 - \frac{|B|}{|W|}\right)^{n-k}$$

binomial distribution

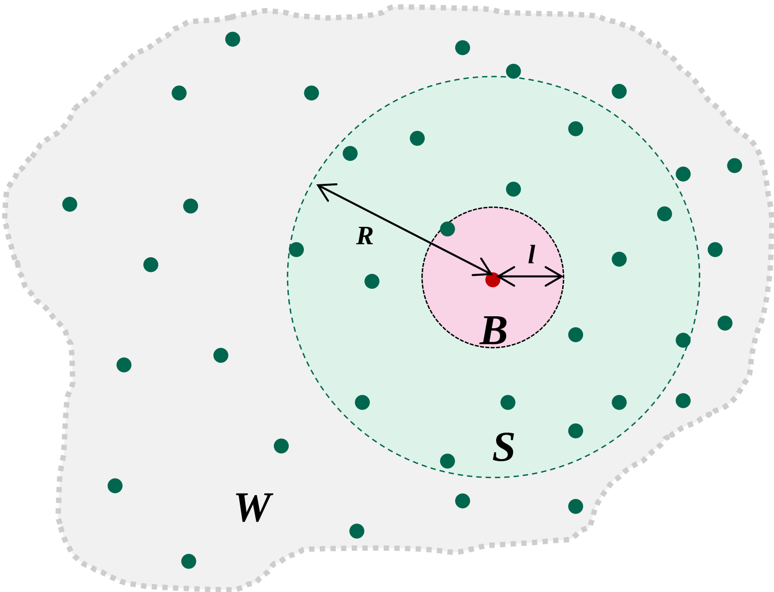
Spatial Poisson Process (4)

< Example >

Consider a city where police cars are distributed according to a Poisson process with an intensity of λ [cars per square meters]. Assume that an incident requiring police presence occurs somewhere in the city.

What is the probability density function of distance L between the location of incident and the nearest police car?

Spatial Poisson Process (5)



- $\Pr\{N(W) = n\} = \frac{(\lambda|W|)^n}{n!} e^{-\lambda|w|}$
- $\Pr\{N(S) = k\} = \frac{\left(\lambda\pi R^2\right)^k}{k!} e^{-\lambda\pi R^2}$

- $\Pr\{L \le l\} = 1 - \Pr\{L > l\}$
- $\Pr\{L > l\}$ is the probability that there is no point in area B with size πl^2
- $\Pr\{L > l\} = \Pr\{N(B) = 0\} = e^{-\lambda|B|} = e^{-\lambda\pi l^2}$
- $\Pr\{L \le l\} = 1 - e^{-\lambda\pi l^2}$
- PDF of L : $f_L(l) = 2\pi\lambda l e^{-\lambda\pi l^2}$