UNIT 3 : Collisional Processes in Plasmas

Ch 11. Collisions in fully ionized Plasmas

11.1 Coulomb Collisions

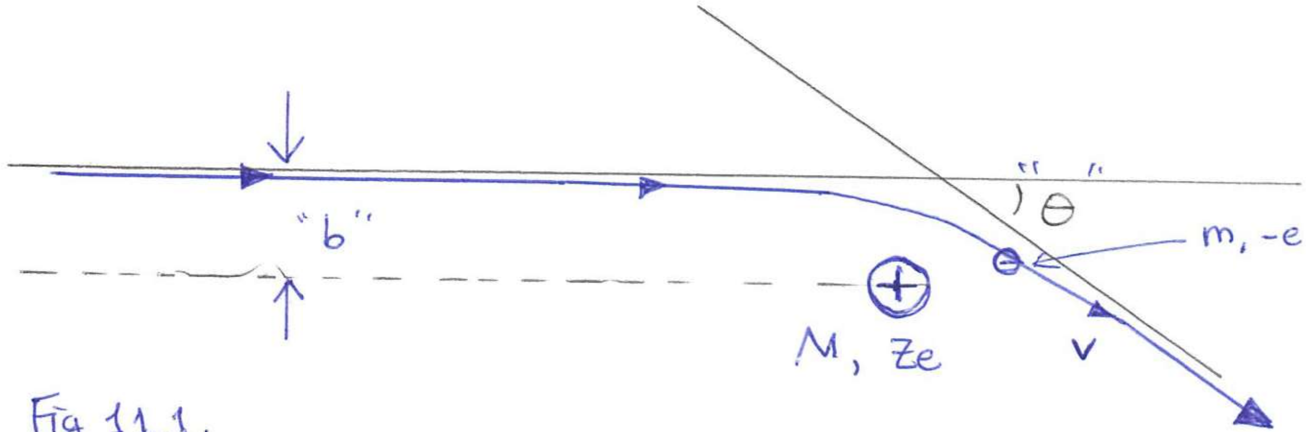


Fig 11.1.

⊛ Consider an electron with velocity v approaching a fixed ion of charge "Ze". The angle of deflection "θ" is related to "the impact parameter" $\underline{\underline{b}}$, and is due to Coulomb interaction with an inverse-square-law force.

$$\boxed{\tan\frac{\theta}{2} = \frac{Z e^2}{4\pi\epsilon_0 m v^2 b}} \qquad (11.2)$$

"Rutherford Scattering"

⊛ For scattering through 90° (i.e., $\theta/2 = 45°$, $\tan\frac{\theta}{2} = 1$), the impact parameter "b" is given by

$$b_0 = \frac{Z e^2}{4\pi\epsilon_0 m v^2} \qquad (11.3).$$

$$\rightarrow \quad \tan\left(\frac{\theta}{2}\right) = b_0/b\,.$$

⊛ "Cross section" of the ion for 90°-scattering:

$$\sigma_{i,90^\circ} = \pi b_0^2 = \frac{\pi Z^2 e^4}{(4\pi\epsilon_0)^2 m^2 v^4} \qquad (11.4).$$

⊛ But for most situations in plasmas, the cumulative effect of many small-angle deflections turns out to be larger than the effect of relatively fewer large-angle deflections.

⇒ "Effective cross section for Coulomb scattering is considerably larger than Eq. (11.4)."

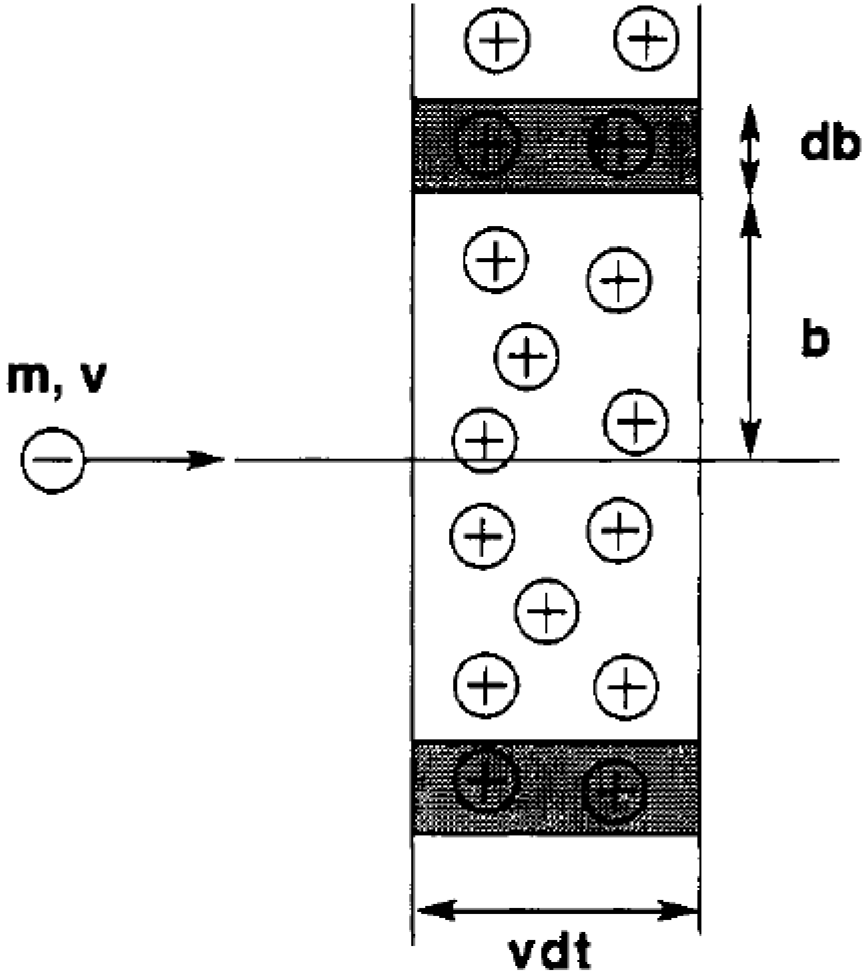
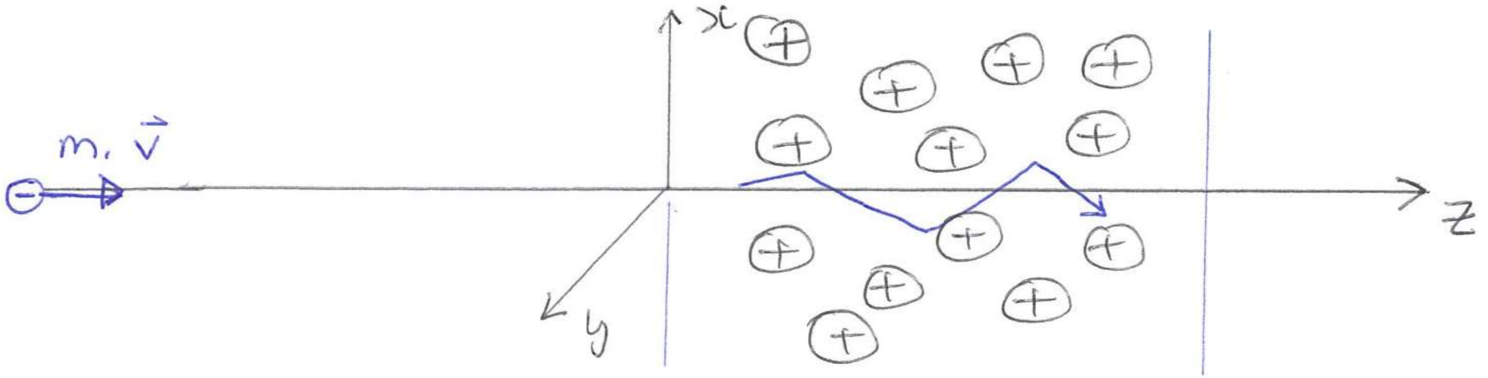


Figure 11.2. Electron Coulomb scattering by ions in an annular element of volume with impact parameters between b and $b + \mathrm{d}b$ as the electron moves a distance $v\mathrm{d}t$.



⊛ Taking averages over many small-angle scattering events,

$\langle \Delta v_x \rangle = \langle \Delta v_y \rangle = 0$, because there's no preferred direction for scattering.

However, $\langle \Delta v_x^2 \rangle = \langle \Delta v_y^2 \rangle = \frac{1}{2} \langle \Delta v_\perp^2 \rangle \neq 0$.

⊛ From $\tan\left(\frac{\theta}{2}\right) = b_0/b$, we obtain $\left(\because \sin\theta = \frac{2\tan\theta/2}{1+\tan^2(\theta/2)} \right)$

$$\boxed{\sin\theta = \frac{2\, b/b_0}{1+(b/b_0)^2}} \qquad (11.8)$$

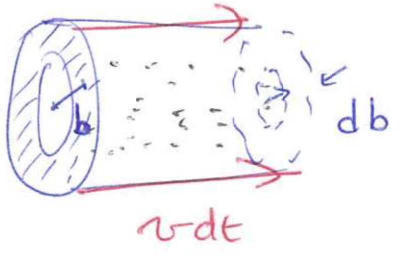
$\therefore$ For a single scattering event, we have

$$\boxed{(\Delta v_\perp)^2 = v^2 \sin^2\theta = \frac{4v^2 (b/b_0)^2}{[1+(b/b_0)^2]^2}} \qquad (11.9)$$

* According to Fig 11.2,

the number of ions in a scattering element between "b" and "b+db" is given by

" $\underbrace{n_i}_{\text{ion number density}}\ \underbrace{2\pi b\, db}_{\text{area}}\ \underbrace{v\, dt}_{\text{distance}}$ "



* Integrating over impact parameters and differentiating w.r.t. time,

$$\frac{d}{dt}\langle(\Delta v_\perp)^2\rangle = 4\pi n_i v \int (\Delta v_\perp)^2 b\, db = 8\pi n_i v^3 \int_0^{b_{max}} \frac{(b/b_0)^2\, b\, db}{[1+(b/b_0)^2]^2} \qquad (11.10)$$

The integral is well-defined at $b=0$, but diverges for $b_{max} \to \infty$. We will avoid this problem by physics consideration.

* $$\boxed{\frac{d}{dt}\langle(\Delta v_\perp)^2\rangle = 4\pi n_i v^3 b_0^2 \left[\ln\left[1+\left(\frac{b_{max}}{b_0}\right)^2\right] + \text{"smaller terms for } b_{max}/b_0 \gg 1\text{"} \right] = 8\pi n_i v^3 b_0^2 \cdot \ln\Lambda = \frac{n_i Z^2 e^4}{2\pi \epsilon_0^2 m^2 v} \ln\Lambda} \qquad (11.11)$$

⊛ Since the electron energy is conserved in the collision for $\frac{m_e}{M_i} \to 0$,

$$(v + \Delta v_{\parallel})^2 + \Delta v_{\perp}^2 = v_{\parallel}^2 \implies v \Delta v_{\parallel} + \frac{1}{2} (\Delta v_{\perp})^2 = 0.$$

$$\therefore \frac{d}{dt} \Delta v_{\parallel} = -4\pi n_i v^2 b_0^2 \ln \Lambda = -\frac{n_i Z^2 e^4}{4\pi \epsilon_0^2 m^2 v^2} \ln \Lambda$$

$$\equiv -\underline{\nu_{ei}}\, v$$

⊛ The collision frequency (for loss of electron momentum):

$$\boxed{\nu_{ei} = 4\pi n_i v b_0^2 \ln \Lambda = \frac{n_i Z^2 e^4}{4\pi \epsilon_0^2 m^2} \frac{\ln \Lambda}{v^3}} \qquad (11.16)$$

Here,

- $\Lambda \equiv b_{max.}/b_0 \gg 1$ and $\boxed{\ln \Lambda \text{ is called the Coulomb logarithm}}$.

⊛ In a quasi-neutral plasma, a charged ptl will interact weakly with ptls further removed from it than the "Debye length λ_D". much (weaker than "$1/r^2$-law" implies).

1.7 Debye shielding

⊛ Consider a plasma in which electrons and ions are in thermal equilibrium among themselves with T_e and T_i respectively.

Suppose T_e and T_i are uniform and $n_e = Zn_i \equiv n_{e\infty}$ at infinity.

$$n_e(x) = n_{e\infty} \exp[e\phi(x)/T_e]$$

$$Zn_i(x) = n_{e\infty} \exp[-eZ_i\phi(x)/T_i] \qquad (1.31).$$

④ The Poisson equation in 1-d is,

$$\epsilon_0 \frac{d^2}{dx^2}\phi(x) = e[n_e - Z n_i] = en_{e\infty}\left[e^{e\phi(x)/T_e} - e^{-eZ\phi(x)/T_i}\right]$$

$$\simeq en_{e\infty}\left[\frac{e\phi(x)}{T_e} + \frac{eZ\phi(x)}{T_i}\right], \qquad (1.32-33)$$

for $Z\frac{e\phi}{T_e} \ll 1$.

$$\Rightarrow \quad \phi \propto \exp[-x/\lambda_D] \qquad (1.35).$$

Debye Length.

$$\lambda_D \equiv \left[\frac{\epsilon_0 T_e}{n_e e^2 (1 + Z T_e / T_i)} \right]^{1/2} \qquad (1.36).$$

$$\left(\lambda_{De} \equiv \left(\epsilon_0 T_e / n_0 e^2 \right)^{1/2} \right)$$

For $T_e = 3\,\mathrm{eV}$, $n_e = 10^{19}\,\mathrm{m}^{-3}$, $\lambda_{De} \simeq 3 \times 10^{-6}\,\mathrm{m}$.

Characteristics of Plasmas:

1. System size $\gg \lambda_D$
2. Number of ptls in Debye sphere: $n \left(\frac{4\pi}{3} \lambda_D^3 \right) \gg 1$.

Both are easily satisfied for systems we consider.

In 3-d too, the Debye shielding results in an exponential decrease in the electric potential for $r > \lambda_D$.

" $\phi(r) \propto \exp(-r/\lambda_D)/r$ " [Problem 1.3.]

- Back to $\boxed{\text{Coulomb logarithm}}$

$\Lambda = b_{max.}/b_0 \sim \lambda_D/b_0$,

where $b_0 \sim Ze^2/12\pi\epsilon_0 T \sim \left(\frac{Z}{12\pi}\right)(n\lambda_D^2)^{-1}$

so $\Lambda \sim \left(\frac{12\pi}{Z}\right) n\lambda_D^3$ is typically a very large number.

(Recall: $n\lambda_D^3 \gg 1$ was one of definitions of plasma).

Coulomb Cross Section

From $\nu_{ei} = n_i \sigma_{ei} v$ (II.18)

we obtain,

$$\boxed{\sigma_{ei} = \frac{Z^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m^2 v^4}} \left(= \underbrace{4\ln\Lambda}_{\sim 70} \sigma_{i\,90°} \text{ of Eq(II.4)} \right). \quad \text{(II.19)}.$$

(a consequence of $1/r^2$ - law).

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Table 11.1. Values of $\ln\Lambda$ for naturally occurring and laboratory plasmas.

	$n(\mathrm{m}^{-3})$	$T(\mathrm{eV})$	$\ln\Lambda$
Solar wind	10^7	10	26
Van Allen belts	10^9	10^2	26
Earth's ionosphere	10^{11}	10^{-1}	14
Solar corona	10^{13}	10^2	21
Gas discharge	10^{16}	10^0	12
Process plasma	10^{18}	10^2	15
Fusion experiment	10^{19}	10^3	17
Fusion reactor	10^{20}	10^4	18

11.2. Electron and Ion Collision Frequencies

(*) So far, we discuss collision frequency of each ptl, i.e, $\nu_{ei} \propto v^{-3}$. Now, we derive collision frequencies averaged over distribution of particles.

(*) Let's evaluate the frictional force on a distribution of electrons drifting through stationary ions,

$$\vec{F} = -n_e m \underline{\langle \nu_{ei} \vec{v} \rangle} \qquad (11.20)$$

avg. over distribution of electrons.

Consider drifting Maxwellian:

$$f_e(\vec{v}) = \frac{n_e}{(2\pi)^{3/2} v_{Te}^3} \exp\left[-\frac{(\vec{v}-\vec{u})^2}{2v_{Te}^2}\right] \approx \left(1 + \frac{u_z v_z}{v_{Te}^2}\right) f_{eo}(\vec{v}),$$

where $f_{eo}(\vec{v})$ is the "unshifted" Maxwellian. $u_z \ll v_{Te}$ is assumed, and $\vec{u} = u_z \hat{z}$ is the non-zero mean velocity.

⊛ $$F_z = -m\int \nu_{ei} v_z f_e \, d^3v = -m u_z \int \frac{v_z^2}{v_{Te}^2} \nu_{ei} f_{eo} \, d^3v$$

$$= -\frac{m u_z}{3} \int \frac{v^2}{v_{Te}^2} \nu_{ei} f_{eo} \, d^3v .$$

Here, we used the spherical symmetry of $f_{eo}(\vec{v}^2)$.

The velocity space integral is (because $\nu_{ei} \propto 1/v^3$);

$$\int \frac{f_{eo}(v)}{v} \underline{d^3v} = 4\pi \int_0^\infty f_{eo}(v)\, v \, dv = \left(\frac{2}{\pi}\right)^{1/2} \frac{n_e}{v_{T.e}} . \qquad (11.21)$$

" $4\pi v^2 dv$ "

⊛ The frictional force becomes : $F_z = -n_e m \langle \nu_{ei} \rangle u_z$

where

$$\boxed{\langle \nu_{ei} \rangle = \frac{2^{1/2} n_i Z^2 e^4 \ln\Lambda}{12\pi^{3/2} \epsilon_0^2 m^{1/2} T_e^{3/2}}} \qquad (11.22)$$

- Note that this is independent of ion mass $\because$ $\frac{m_e}{M_i} \ll 1$ and ions are assumed to be stationary during scattering.

⊛ In a plasma containing many different ions,

"$n_i Z^2$" in Eq (11, 22) is replaced by "$n_e Z_{eff}$",

where $\boxed{Z_{eff} = \sum_i n_i Z_i^2 / n_e}$: measure of impureness of a plasma.

⊛ For electron-electron or ion-ion collisions, we should consider the frictional force on a population of electrons (ions) drifting through another population of " (").
the same species

In this situation, the Coulomb (Rutherford) scattering calculation with a fixed target particle can be repeated if we consider a scattering event at the center-of-mass frame.

For a consideration of relative motion of two ptls,

" Now, the incoming (effective) ptl suffering a collisional scattering has a reduced mass $\mu = \frac{M_1 M_2}{M_1 + M_2}$ ".

⊛ Reduced mass for electron-electron collision : $m_e/2$

" ion-ion " : $M_i/2$

(In fact "m" in Eq. (11.22) should have been $\mu \equiv \frac{m_e M_i}{m_e + M_i} \ (\approx m_e)$)

to be more precise

⊛ Noting that Frictional force "F_z" $\propto m^{1/2}$, and identifying the origins of various parametric dependences of Eq. (11.22),

we can deduce

$$\boxed{\langle \nu_{ee} \rangle = \frac{n_e e^4 \ln \Lambda}{12 \pi^{3/2} \epsilon_0^2 m^{1/2} T_e^{3/2}}} \qquad (11.23)$$

and

$$\boxed{\langle \nu_{ii} \rangle = \frac{n_i Z^4 e^4 \ln \Lambda}{12 \pi^{3/2} \epsilon_0^2 M^{1/2} T_i^{3/2}}} \qquad (11.24)$$

(*) Momentum exchange for ions through electrons is generally not very important, since the momentum gained or lost by the ion in such a collision is relatively small.

(*) Comparing electron and ion collision frequencies in a plasma with $T_e \sim T_i$, we have

$$\nu_{ei} \sim \nu_{ee} \gg \nu_{ii} \gg \nu_{ie}$$

and

$$\boxed{\frac{\nu_{ei}}{\nu_{ii}} \sim \left(\frac{M_i}{m_e}\right)^{1/2}, \quad \frac{\nu_{ii}}{\nu_{ie}} \sim \left(\frac{M_i}{m_e}\right)^{1/2}} \qquad (7).25)$$

(*) $\langle \nu_{ei} \rangle \sim \langle \nu_{ee} \rangle \sim 5 \times 10^{-11}\, n / T_e^{3/2}$ (s^{-1}) (temp in eV)

$\langle \nu_{ii} \rangle \sim 10^{-12}\, n / T_i^{3/2}$ (s^{-1}).

Note : $\boxed{\sigma_{Coulomb} \propto T^{-2}}$

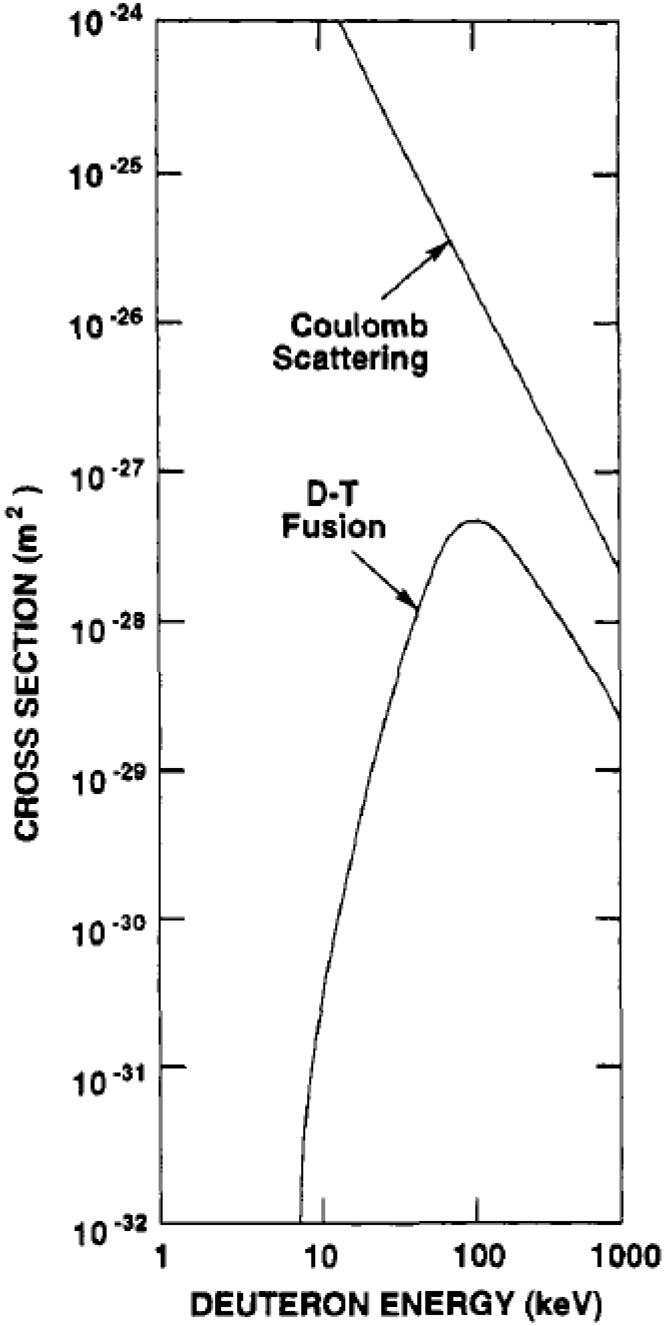

Figure 11.3. Coulomb and fusion cross sections compared for a deuterium ion (deuteron) in a deuterium–tritium plasma.