11.3. Plasma Resistivity

- Ohm's Law:

$$\vec{E} = \eta \vec{j} \tag{11.26}$$

* Its origin is the electron momentum equation

$$m_e n_e \frac{d\vec{u}_e}{dt} = -e n_e \vec{E} + \vec{R}_{ei}, \tag{11.27}$$

where

$$\vec{R}_{ei} = -m n_e \langle \nu_{ei} \rangle (\vec{u}_e - \vec{u}_i) \tag{11.28}$$

* Since $\vec{j} = -n_e e (\vec{u}_e - \vec{u}_i)$,

$$\boxed{\eta = \frac{m_e \langle \nu_{ei} \rangle}{n_e e^2} = \frac{2^{1/2} m_e^{1/2} Z e^2 \ln\Lambda}{12 \pi^{3/2} \epsilon_0^2 T_e^{3/2}}} \tag{11.30}$$

- This expression based on the shifted Maxwellian of f_e is an overestimation by a factor of "2".
- In fact, more accurate calculation taking into account of the distortions of f_e from shifted Maxwellian has been performed.

⊛ $\eta \propto T_e^{-3/2}$

→ The "freezing" of plasma to $\vec{B}$ field lines

(a property of ideal MHD), works better

for high-temperature plasma .

→ Ohmic heating : $P = \vec{j} \cdot \vec{E} = \eta j^2$

∴ efficiency drops sharply as $T_e \uparrow$ at fixed j.

not sufficient for fusion purpose .

→ Auxiliary heating methods are necessary.

Neutral Beam Heating, Ion Cyclotron Resonance Heating,

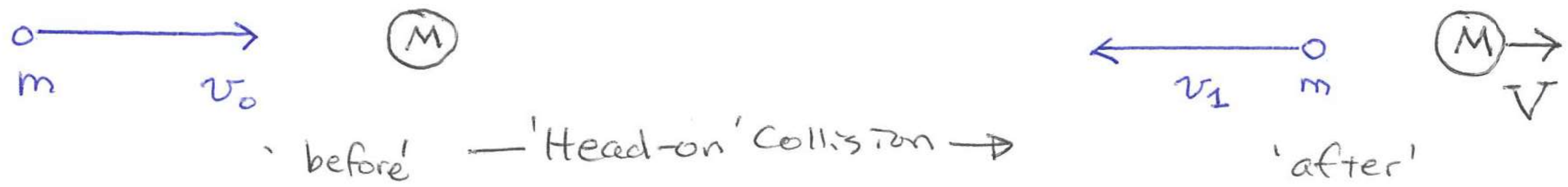
Electron " ,

etc.,

11.4. Energy Transfer

(*) Consider energy transfer between electrons with T_e and ions with T_i via collisional process, for $T_e \gg T_i$.

(*) This process will take the "equilibration time" τ_{eq} which is much longer than times for electrons and ions separately to come to thermodynamic equilibrium among themselves, which are ν_{ee}^{-1} and ν_{ii}^{-1} respectively.



* Momentum Conservation : $\quad m v_0 + m v_1 = M V \qquad (11.32)$

* Energy Conservation : $\quad \frac{1}{2} m v_0^2 - \frac{1}{2} m v_1^2 = \frac{1}{2} M V^2 \qquad (11.33)$

$$\Longrightarrow \quad \frac{1}{2} M V^2 \approx \underline{\left(\frac{4m}{M}\right)} \frac{m v_0^2}{2} \quad \text{for } v_1 \approx v_0 . \qquad (11.34)$$

Energy Transfer

(*) Consider Coulomb collisions btwn electrons and ions.

From $m \Delta \vec{v} = - M \Delta \vec{V}$, we obtain

$$\frac{1}{2} M |\Delta \vec{V}|^2 = \frac{m^2}{2M} |\Delta \vec{v}|^2. \qquad (11.36).$$

Recall $\Delta \vec{v}$ is mostly in $\perp$ direction on avg (Fig 11.1).

$$\therefore \quad \frac{1}{2} M |\Delta \vec{V}|^2 = \frac{m^2}{2M} (\Delta v_\perp)^2 \qquad (11.37).$$

(*) For the case where many electrons colliding with many ions, from (11.11), we have

$$\frac{d}{dt} \langle (\Delta v_\perp)^2 \rangle = \frac{n_i Z^2 e^4 \ln\Lambda}{2\pi \epsilon_0^2 m^2 v} \qquad (11.38).$$

(*) Total rate of energy loss from electrons by collisional transfer to ions,

$$\frac{d}{dt} W_e = -\frac{m^2}{2M} \int \frac{d}{dt} \langle (\Delta v_\perp)^2 \rangle f_e(v) \, d^3v \qquad (11.40).$$

⊛ From $W_i = \frac{3}{2} n_i T_i$ and $\frac{d}{dt} W_i = -\frac{d}{dt} W_e$, (11.41)

we get

$$\frac{dT_i}{dt} = \frac{m^2}{3 n_i M} \int \frac{d}{dt} \langle (\Delta v_\perp)^2 \rangle f_e(v)\, d^3 v \quad (11.42)$$

$$= \frac{Z^2 e^4 \ln \Lambda}{6\pi \epsilon_0^2 M} \int \frac{f_e(v)}{v} d^3 v \quad (11.43)$$

For a Maxwellian f_e, $\int \frac{f_e(v)}{v} d^3 v = \left(\frac{2}{\pi}\right)^{1/2} \frac{n_e m_e^{1/2}}{T_e^{1/2}}$ (11.44)

$$\Rightarrow \quad \frac{d}{dt} T_i = \frac{T_e}{\tau_{eq}}. \quad (11.45)$$

where

$$\boxed{\tau_{eq}^{-1} = \frac{n_e Z^2 e^4 m_e^{1/2} \ln \Lambda}{3\pi (2\pi)^{1/2} \epsilon_0^2 M T_e^{3/2}}} \quad (11.46)$$

'Temperature equilibration rate'

Note that $\tau_{eq}^{-1} \approx 2 \frac{m}{M} \langle \nu_{ei} \rangle$ (11.47)

⊛ This can be generalized to the case with finite T_e,

$$\frac{d}{dt} T_i = \frac{T_e - T_i}{\tau_{eq}} \qquad (11.48).$$

$$\left(\rightarrow \frac{d}{dt} T_e = \frac{n_i}{n_e} \frac{T_i - T_e}{\tau_{eq}} \right)$$

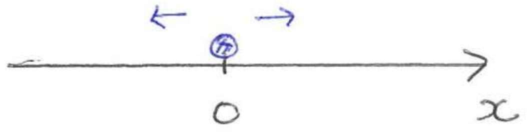
from energy conservation

$\boxed{\text{Homework}}$

⊛ Problem 11.4. on page 180

Ch. 12: Diffusion in Plasmas

12.1. Diffusion as a Random Walk:



- ptls take one step at every Δt. (^ time interval)
- Probability of a step to Right $= 1/2$
  " $= 1/2$.

After a sufficient length of time, if we take an average over all ptls,

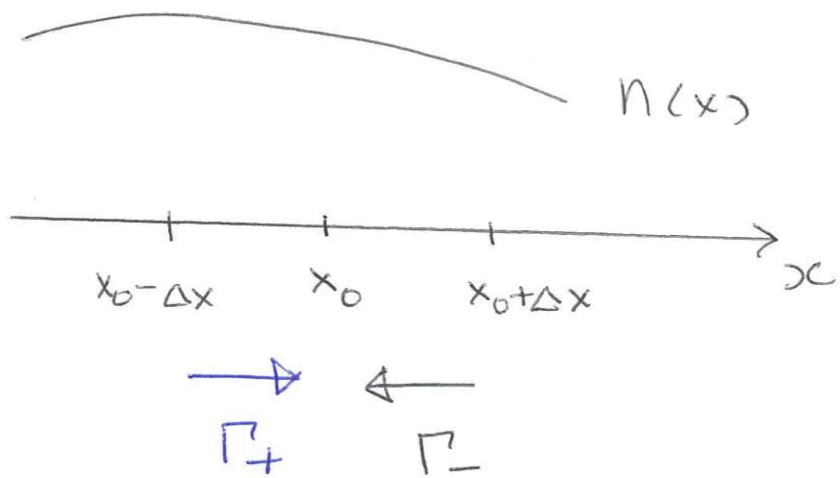
$$\langle x \rangle = 0 \qquad (12.1)$$

but

$$\boxed{\frac{d}{dt}\langle x^2 \rangle = \frac{\Delta x^2}{\Delta t}} \qquad (12.2)$$

(*) Reading Assignment: 12.2. on page 186-187.

12.3. The Diffusion Equation



⊛ Evaluate the ptl flux at $x = x_0$ in a time interval Δt from random walk process; assuming $\Delta x \to 0$, $\Delta t \to 0$;

$$\dashrightarrow \quad \Gamma = \Gamma_+ + \Gamma_- = -\frac{(\Delta x)^2}{2\Delta t}\frac{dn(x)}{dx} \equiv -D\frac{dn(x)}{dx} \qquad (12.4\text{-}5).$$

⊛ From continuity equation (ptl number conservation), we obtain the Diffusion Equation :

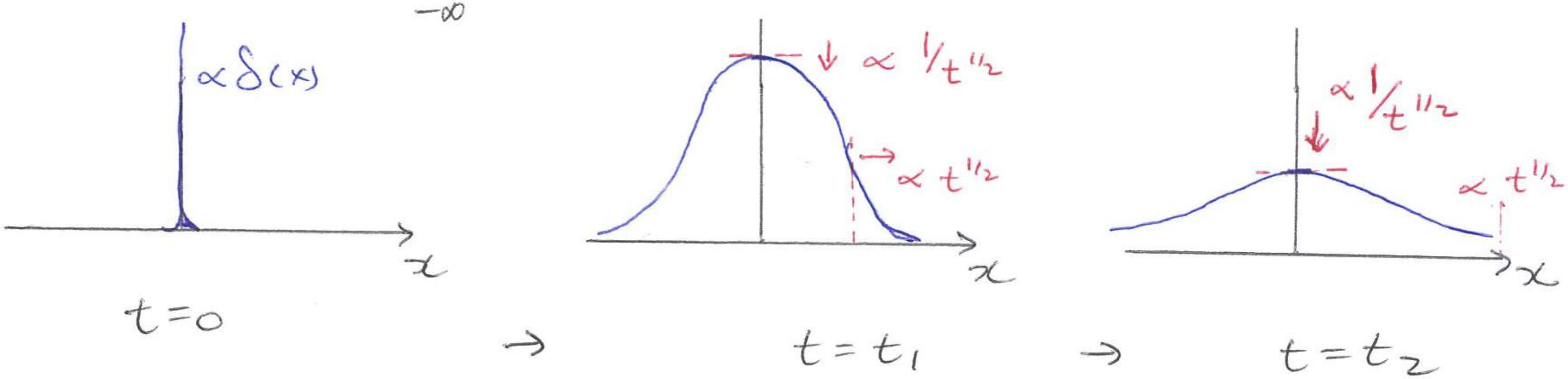
$$\frac{\partial}{\partial t} n = -\frac{\partial \Gamma}{\partial x} = \frac{\partial}{\partial x} \underline{\underline{D}} \frac{\partial}{\partial x} n \qquad (12.3).$$

particle diffusion coefficient

⊛ If all the ptls begin at $t=0$ at $x=0$, the exact solution of Eq (12.3) is:

$$* \quad n(x,t) = \frac{N}{(4\pi D t)^{1/2}} \exp\left[-\frac{x^2}{4Dt}\right] \qquad (12.6)$$

where $N = \int_{-\infty}^{\infty} n(x,t)\,dx$ is the total number of ptls.



$$\langle x^2 \rangle = N^{-1} \int_{-\infty}^{\infty} x^2\, n(x,t)\,dx = 2Dt = \left(\frac{\Delta x^2}{\Delta t}\right) t,$$ in agreement with Eq. (12.2).

Diffusion in Plasmas

* With no $\vec{B}$ or $\vec{E}$ field;

Adopting the result from random walk model, we estimate

$$D \sim \frac{\Delta x^2}{\Delta t} \sim \nu \lambda_{mfp}^2 \qquad (12.8)$$

- $\nu \sim \frac{1}{\Delta t}$: collisional frequency
- $\lambda_{mfp} \sim \Delta x$: mean free path $\sim \frac{v_{Th}}{\nu}$

$$\therefore \boxed{D \sim \frac{v_{Th}^2}{\nu} \sim \frac{T}{m\nu}} \qquad (12.9)$$

* This scaling is also applicable to the diffusion // to $\vec{B}$ when $|\vec{B}| \neq 0$. i.e., for "$D_{\parallel}$" : parallel diffusion coefficient.

~~Ch.21. Diffusion in Plasmas~~

~~TK 23~~

III-26.

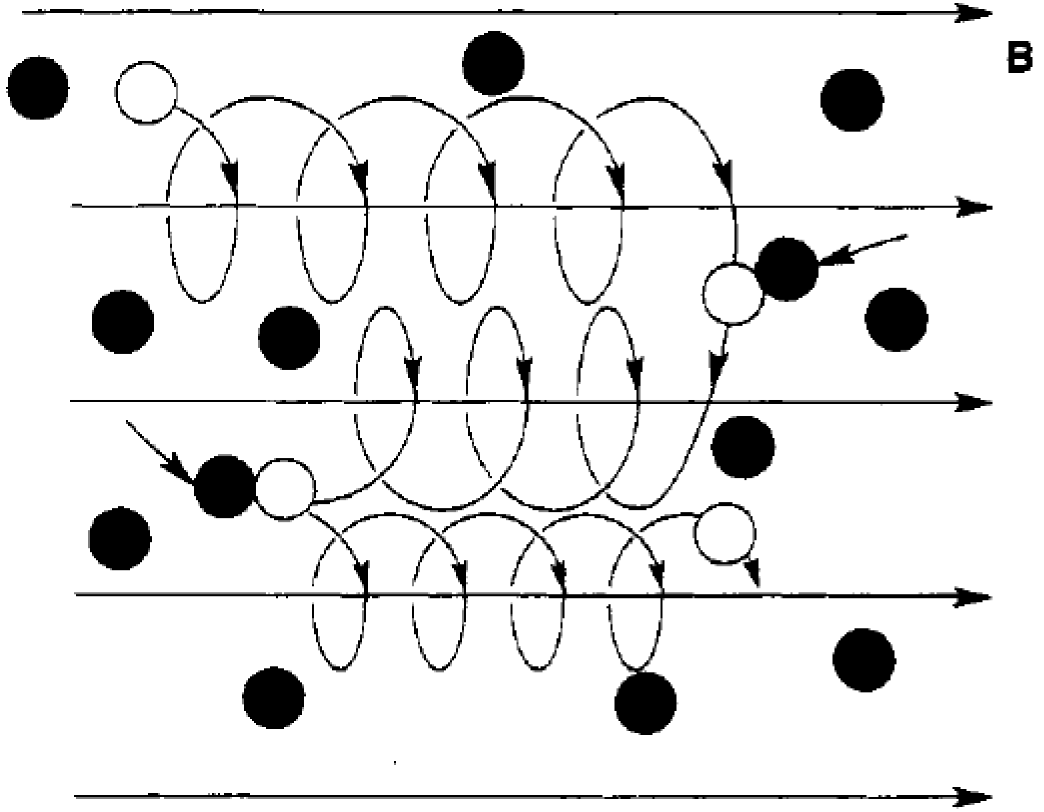
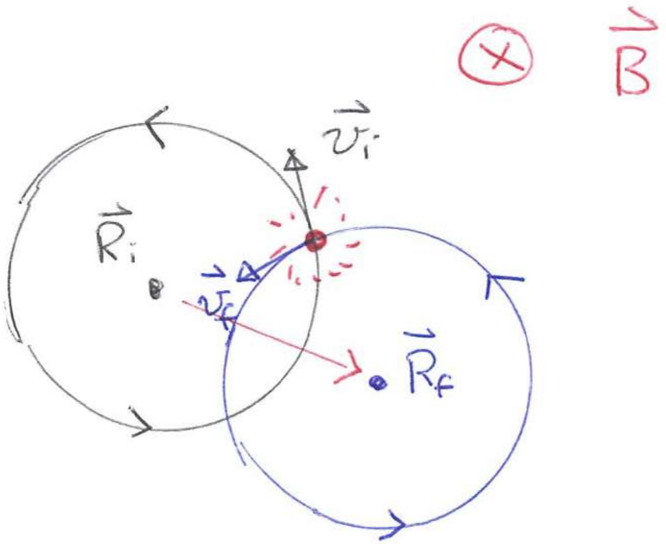


Figure 12.1. Diffusion of a charged particle (open circle) in a magnetic field due to collisions with other particles, either neutral or charged (full circles). Two collisions are shown, each of which contributes to diffusion along the field and, by changing the phase angle of the Larmor gyration, to diffusion across the field.



(Simplified picture of)

"Coulomb collision in magnetized plasmas!"

⊛ Even in uniform $\vec{B}$ field, the charged ptls can diffuse across $\vec{B}$ due to successive coulomb collisions.

⊛ When a charged ptl collides with another ptl at ⊙,
its velocity vector changes its direction from $\vec{v}_i$ to $\vec{v}_f$.
While it gyrates with the same r_L, its gyro-phase changes abruptly
$\Rightarrow$ its location of guiding-center gets shifted!

Perpendicular Diffusion

⊛ $\therefore \quad \Delta x \sim r_L$ (not λ_{mfp})

and

$\Delta t \sim \nu^{-1}$

$$\rightarrow \quad \boxed{D_\perp \sim \nu r_L^2} \qquad (12.10)$$

⊛ This heuristic derivation considered collisions as "abrupt large angle scatterings". But we learned that the cumulative effects of small-angle scatterings are more important.

⊛ Recognizing that collisional scattering is itself a diffusive process in velocity space, the angle of scattering $\Delta\theta$ increases in time according to $(\Delta\theta)^2 \sim \nu \Delta t$. Since $\underline{\Delta x \sim r_L \Delta\theta}$,

$D_\perp \sim \Delta x^2/\Delta t \sim \nu r_L^2$, recovering Eq. (12.10).