Queuing System II

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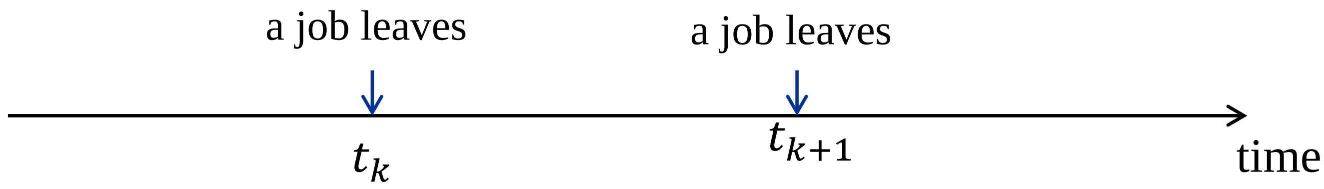
M/G/1 (1)

- Poisson Arrival Process
- General service time distribution

- Embedded Markov chain (Semi-Markov chain)
  - We observe the system at an instant that the served job departs the system
  - Then, since the service time does not need to be considered, the system has Markovian property



- Let X_k be a random variable representing the number of jobs in the system at the epoch t_k
- Embedded Markov chain is described as $\{X_k, k = 1, 2, 3, \cdots\}$

M/G/1 (2)

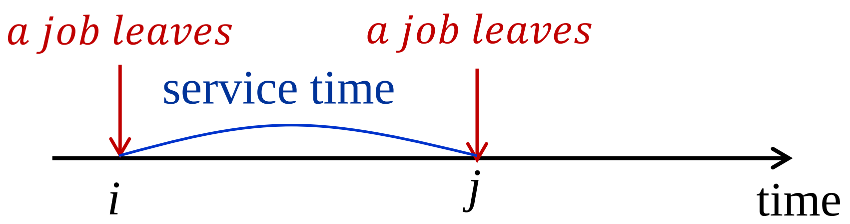
- State probability distribution in EMC
  - Let π_i be the probability of *i* jobs in system at a departure epoch

    $$\pi_i = \sum_j \pi_j P_{ji} \qquad \ldots (1)$$

    - where P_{ji} is the one-step transition probability from state *j* to state *i*
  - We need P_{ji}
  - Consider two cases in calculating P_{ji}
    - Case I: $X_k = i > 0$
    - Case II: $X_k = 0$

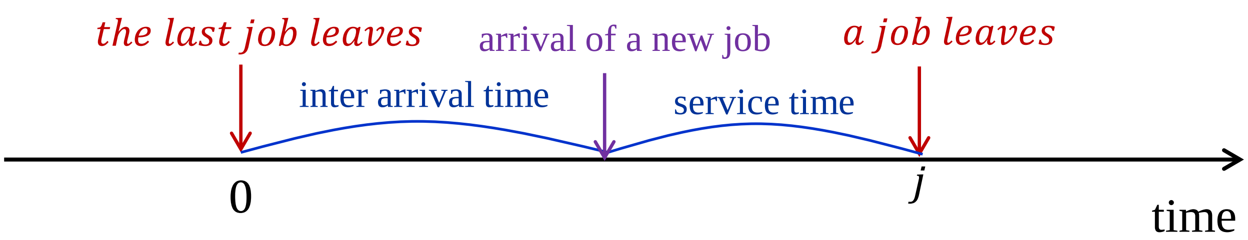
M/G/1(3)

- Case I



- If $j < i - 1, \ \ P_{ij} = 0$ … (2)
- If $j \geq i - 1, \ \ P_{ij} = q_{j-i+1}$ … (3)
  - ✓ where q_m is the probability of m arrivals in a service time

- Case II



- $P_{0j} = q_j$, j ≥ 0 … (4)

M/G/1 (4)

- Let $A(z) \coloneqq E[z^{\mathrm{x}}]$
  - If a $r.v.X$ represents the number of jobs within a system in stsady state, the mean number of jobs in the system, $\bar{N}$, is $A'(1)$.

- $$A(z) = \sum_{j=0}^{\infty} \pi_j z^j = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \pi_i P_{ij} z^j \qquad \text{from (1)}$$

$$= \pi_0 \sum_{j=0}^{\infty} P_{0j} z^j + \sum_{i=1}^{\infty} \pi_i \sum_{j=0}^{\infty} P_{ij} z^j$$

$$= \pi_0 \sum_{j=0}^{\infty} q_j z^j + \sum_{i=1}^{\infty} \pi_i \sum_{j=i-1}^{\infty} q_{j-i+1} z^j \qquad \text{from (2) – (4)}$$

$$= \pi_0 \sum_{j=0}^{\infty} q_j z^j + \sum_{i=1}^{\infty} \pi_i \, z^{i-1} \sum_{j=i-1}^{\infty} q_{j-i+1} z^{j-i+1}$$

$$= \pi_0 \sum_{j=0}^{\infty} q_j z^j + \sum_{i=1}^{\infty} \pi_i \, z^{i-1} \sum_{j=0}^{\infty} q_j z^j$$

M/G/1 (5)

– Let $Y(z)$ be PGF of a random variable representing the number of arrivals during a service time, i.e.,

$$Y(z) := \sum_{j=0}^{\infty} q_j z^j$$

$$A(z) = \pi_0 Y(z) + \sum_{i=1}^{\infty} \pi_i\, z^{i-1} Y(z)$$

$$= \pi_0 Y(z) + \sum_{i=1}^{\infty} \frac{\pi_i z^i}{z} Y(z) = \pi_0 Y(z) + \frac{Y(z)}{z} \sum_{i=1}^{\infty} \pi_i z^i$$

$$= \pi_0 Y(z) + \frac{(A(z) - \pi_0)}{z} Y(z)$$

$$A(z) = \frac{(z-1)\pi_0 Y(z)}{z - Y(z)}$$

M/G/1 (6)

- We need π_0 and $Y(z)$ for calculating $A(z)$
  - Note that $A(1) = 1$ and $Y(1) = 1$.

    But, since A(1) = $\frac{0}{0}$, we apply L'Hospital's rule.

$$\lim_{z\to 1} A(z) = \lim_{z\to 1} \frac{\pi_0 Y(z) + (z-1)\pi_0 Y'(z)}{1 - Y'(z)}$$

That is, $A(1) = \dfrac{\pi_0}{1 - Y'(1)} = 1$

$$\Rightarrow \quad \pi_0 = 1 - Y'(1)$$

- Calculation of $Y(z)$: we need q_m
  - Probability of *m* Poisson arrivals during *t* : $f_m(t) = \frac{(\lambda t)^m e^{-\lambda t}}{m!}$
  - Let $b(t)$ be probability density function of service time distribution
  - $q_m = \displaystyle\int_0^\infty f_m(t) b(t) dt = \int_0^\infty \frac{(\lambda t)^m}{m!} e^{-\lambda t} b(t) dt$

M/G/1 (7)

$$Y(z) = \sum_{m=0}^{\infty} q_m z^m = \sum_{m=0}^{\infty} \int_0^{\infty} \frac{(\lambda t)^m}{m!} e^{-\lambda t} b(t) dt \cdot z^m$$

$$= \int_0^{\infty} \underbrace{\sum_{m=0}^{\infty} \frac{(\lambda t z)^m}{m!}}_{e^{\lambda z t}} e^{-\lambda t} b(t) dt \quad = \int_0^{\infty} e^{-\lambda(1-z)t} b(t) dt$$

- Laplace transform of service time : $B^*(s) = \int_0^{\infty} e^{-st} b(t) dt$
- $Y(z) = \int_0^{\infty} e^{-\lambda(1-z)t} b(t) dt = B^*(\lambda - \lambda z) \quad \Rightarrow Y'(z) = -\lambda B^{*\prime}(\lambda - \lambda z)$
- $Y'(z) = \int_0^{\infty} \lambda t e^{-\lambda(1-z)t} b(t) dt \quad \Rightarrow \quad Y'(1) = \int_0^{\infty} \lambda t b(t) dt = \lambda E[s]$
- $\pi_0 = 1 - Y'(1) \quad \Rightarrow \quad \pi_0 = 1 - \lambda E[S]$
- $A(z) = \frac{(1-z)\pi_0 Y(z)}{Y(z)-z} = \frac{(1-z)(1-\lambda E[S]) B^*(\lambda(1-z))}{B^*(\lambda(1-z))-z}$

M/G/1 (8)

- Performance Measures
  - $\bar{N}$: Mean number of jobs in the system
    - Calculate $A'(1)$ by applying the L'Hospital's rule twice,

$$\bar{N} = A'(1) = Y'(1) + \frac{Y''(1)}{2\left(1 - Y'(1)\right)}$$

$$= \lambda E[S] + \frac{\lambda^2 E[S^2]}{2(1 - \lambda E[S]\,)}$$

  - $\bar{T}$: Mean sojourn time in the system
    - By the Little's law, $\bar{T} = \bar{N} / \lambda$

$$\bar{T} = E[S] + \frac{\lambda E[S^2]}{2(1 - \lambda E[S]\,)}$$

  - ρ : Utilization (server busy probability)

$$\rho = \lambda E[S]$$

M/G/1 (9)

- M/E_k/1 : one case of M/G/1

  - M/G/1: $\bar{N} = \lambda E[S] + \frac{\lambda^2 E[S^2]}{2(1-\lambda E[S])}$

  - M/E_k/1: $E[S]= \frac{1}{\mu}$, $E[S^2] = \frac{k+1}{k}\frac{1}{\mu^2}$

    $\bar{N} = \frac{\lambda}{\mu} + \frac{\lambda^2}{2}\frac{k+1}{k\mu(\mu-\lambda)}$

❖ M/E_k/1

$$\bar{N} = \frac{\lambda}{\mu} + \frac{\lambda(k+1)}{2k(\mu-\lambda)} - \frac{\lambda}{\mu} \cdot \frac{(k+1)}{2k} = \frac{\lambda}{\mu} + \frac{k+1}{2k}\frac{\lambda^2}{\mu(\mu-\lambda)}$$

M/G/1 (10)

- Note that $\bar{N}$ is not the mean number of jobs in the system as seen by any outside observer at any time, but $\bar{N}$ is the mean number of jobs left in the system as seen by the departing jobs

- Equality of state distribution at arrival epoch, departure epoch, any time

(Question) Are the probability distributions of the followings in steady state equal?

  - The number of jobs seen by departing jobs
  - The number of jobs seen by arrival jobs
  - The number of jobs seen by outside observers

Burke theorem

Wolff theorem

(Answer) Yes

M/G/1 (11)

- Wolff theorem (Poisson arrivals see time average)
  - If the arrival process is Poisson, the steady state distribution just prior to arrival epochs is the same as that for the number of jobs seen by an outside observer at any time

- Burke theorem
  - In any queuing system for which the state process is the step function with unit jump, the steady state distribution just prior to arrival epochs is the same as that just after departure epochs

M/G/1 (12)

- Proof of Burke theorem
  - Let $\{N(t): t \geq 0\}$ be a stochastic process of which any sample path is a step function with unit jump
  - Let $A_k(T)$ be the number of arrivals to the system having k jobs during T
  - Let $D_k(T)$ be the number of departures leaving k jobs in the system during T
    - $|A_k(T) - D_k(T)| \leq 1$ (from a step function with unit jump)
  - Let $A(T)$ be the total number of arriving jobs in the interval $[0, T]$
  - Let $D(T)$ be the total number of departing jobs in the interval $[0, T]$
  - Let $N(T)$ be the number of jobs in the system at time T
    - $N(T) = N(0) + A(T) - D(T)$

M/G/1 (12)

– Let $\pi_k{}^{\mathrm{d}}$ and $\pi_k{}^{\mathrm{a}}$ be the state probabilities seen by a departing job and an arriving job at time limit, respectively

– $$\pi_k{}^{\mathrm{d}} = \lim_{T\to\infty} \frac{D_k(T)}{D(T)} = \lim_{T\to\infty} \frac{D_k(T) - A_k(T) + A_k(T)}{N(0) + A(T) - N(T)}$$

$$= \lim_{T\to\infty} \frac{A_k(T)}{A(T)} \qquad (\text{since } |D_k(T) - A_k(T)| \le 1)$$

$$= \pi_k{}^{\mathrm{a}}$$

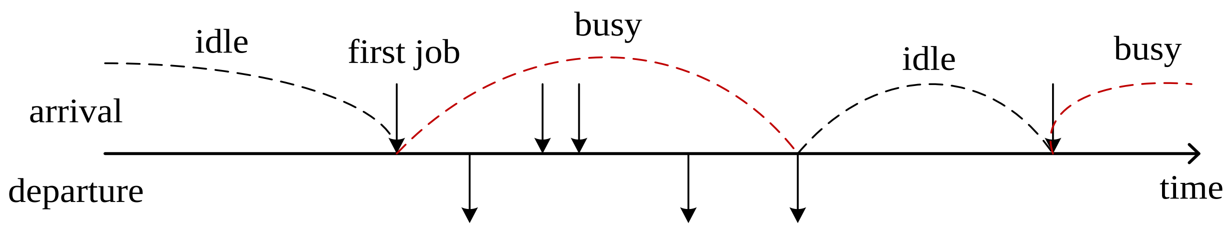
$$\therefore \; \underline{\underline{\pi_k{}^{\mathrm{d}}}} = \underline{\underline{\pi_k{}^{\mathrm{a}}}}$$

State probability
at departure epochs

State probability
at arrival epochs

M/G/1: Busy Period (1)

- Busy period
  - The busy period starts when a job arrives at the system in idle state and is continued until there remains no job to be served in the system



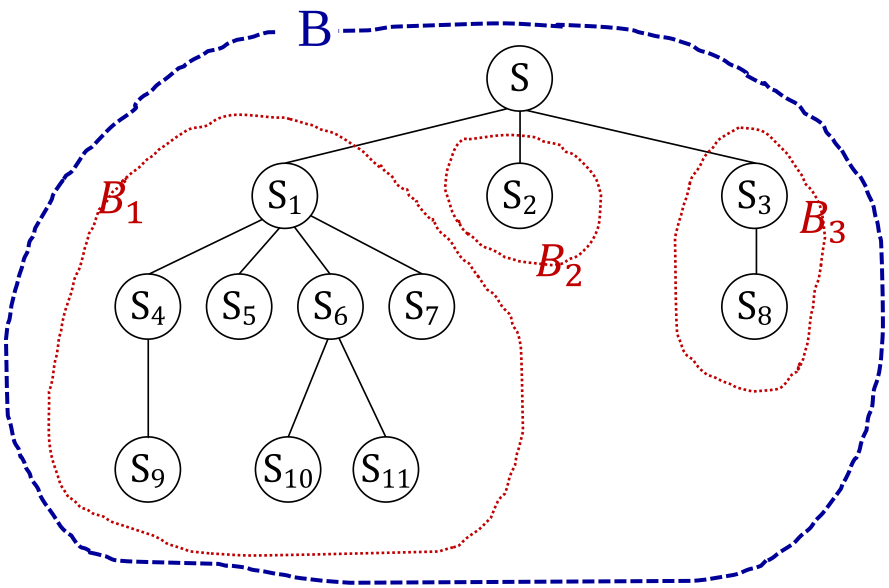
  - Let B be a random variable representing the duration of busy period
  - Let S and S_i be random variables representing the service times of the first job and the i-th arrived job after the first job, respectively
  - Let $A(S)$ be the number of arrivals while the first job is served

M/G/1: Busy Period (2)

– Let B_i be the total sum of service times of the i-th arrived job and its descendants

– Example

- $A(S) = 3$
- $B = S + B_1 + B_2 + B_3$
- $B_1 = S_1 + S_4 + S_5 + S_6 + S_7 + S_9 + S_{10} + S_{11}$
- $B_2 = S_2$
- $B_3 = S_3 + S_8$



M/G/1: Busy Period (3)

- Calculating the average of busy period, E[B]
  - $B^*(\theta) \coloneqq \mathrm{E}\left[e^{-\theta B}\right] = \int_0^\infty e^{-\theta x} f(x) dx$ where $f(x)$ is the pdf of B
  - $\lim_{\theta \to 0} \frac{dB^*(\theta)}{d\theta} = \lim_{\theta \to 0} \int_0^\infty (-x) e^{-\theta x} f(x) dx = -E[B]$
  - $B^*(\theta) = \mathrm{E}\left[e^{-\theta(S + B_1 + B_2 + \cdots + B_{A(S)})}\right]$

$$= \sum_{k=0}^\infty \int_0^\infty \mathrm{E}\left[e^{-\theta(S + B_1 + B_2 + \cdots + B_{A(S)})} \middle| S = x, A(S) = k\right] \times \Pr\{A(S) = k | S = x\} \times \Pr\{S = x\}$$

$$= \sum_{k=0}^\infty \int_0^\infty \mathrm{E}\left[e^{-\theta(x + B_1 + B_2 + \cdots + B_k)}\right] \frac{(\lambda x)^k}{k!} e^{-\lambda x} \underbrace{b(x)}_{\text{pdf of S}} dx$$

$$= \sum_{k=0}^\infty \int_0^\infty \mathrm{E}\left[e^{-\theta x}\right] \mathrm{E}\left[e^{-\theta B_1}\right] \cdots \mathrm{E}[e^{-\theta B_k}] \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x) dx$$

$$= \sum_{k=0}^\infty \int_0^\infty e^{-\theta x} \left(\mathrm{E}[e^{-\theta B}]\right)^k \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x) dx$$

since E[$e^{-\theta B_i}$] has the same value for all i

M/G/1: Busy Period (4)

$$B^*(\theta) = \sum_{k=0}^{\infty} \int_0^{\infty} e^{-\theta x} (B^*(\theta))^k \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x) dx$$

$$= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{\left(\lambda x B^*(\theta)\right)^k}{k!} e^{-(\lambda+\theta)x} b(x) dx$$

$$= \int_0^{\infty} e^{-\left(\lambda+\theta-\lambda B^*(\theta)\right)x} b(x) dx$$

$$= S^*\left(\lambda + \theta - \lambda B^*(\theta)\right) \leftarrow$$ $S^*(\varphi)$: Laplace transform of service time

$$-\mathrm{E}[B] = \lim_{\theta \to 0} \frac{dB^*(\theta)}{d\theta} = \lim_{\theta \to 0} \frac{dS^*\left(\lambda+\theta-\lambda B^*(\theta)\right)}{d\theta}$$

$$= \lim_{\theta \to 0} \int_0^{\infty} \left(-x + \lambda x \frac{dB^*(\theta)}{d\theta}\right) e^{-\left(\lambda+\theta-\lambda B^*(\theta)\right)x} b(x) dx$$

$$= \int_0^{\infty} (-x - \lambda x \mathrm{E}[B]) b(x) dx$$ since $B^*(0)=1$

$$= -\int_0^{\infty} x b(x) dx - \lambda \mathrm{E}[B] \int_0^{\infty} x b(x) dx$$

$$= -\mathrm{E}[S] - \lambda \mathrm{E}[B]\mathrm{E}[S]$$

$$\Rightarrow \qquad \mathrm{E}[B] = \frac{\mathrm{E}[S]}{1-\lambda \mathrm{E}[S]}$$

M/G/1: Busy Period (5)

- Another Approach
  - N: the number of jobs served during busy period
  - $\mathrm{E}[B] = \mathrm{E}[N] \times \mathrm{E}[S]$
  - $\rho = \lambda \mathrm{E}[S]$: server utilization, i.e., server busy probability
  - $\Pr\{N = n\} = \rho^{n-1}(1-\rho)$
  - $\mathrm{E}[N] = \sum_{n=1}^{\infty} n\rho^{n-1}(1-\rho)$

$$= (1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1}$$

$$= \frac{1}{1-\rho} = \frac{1}{1-\lambda \mathrm{E}[S]}$$

$$\underline{\underline{\mathrm{E}[B] = \frac{\mathrm{E}[S]}{1-\lambda \mathrm{E}[S]}}}$$

M/G/1: Busy Period (6)

- Another Approach for calculating E$[N]$
  - $N = 1 + N_1 + N_2 + \cdots + N_{A(S)}$
    - N_i is the number of arrivals while the i–th job is served
  - $\mathrm{G}(z) = \mathrm{E}[z^N] = \mathrm{E}\left[z^{(1+N_1+N_2+\cdots+N_{A(S)})}\right]$

$$= \sum_{k=0}^{\infty} \mathrm{E}\left[z^{(1+N_1+N_2+\cdots+N_{A(S)})} | A(S) = k\right] \times \Pr\{A(S) = k\}$$

$$= \sum_{k=0}^{\infty} \mathrm{E}\left[z^{(1+N_1+N_2+\cdots+N_k)}\right] \times \int_0^{\infty} \Pr\{A(S) = k | S = x\} \Pr\{S = x\}$$

$$= \sum_{k=0}^{\infty} \mathrm{z}(\mathrm{E}[z^N])^k \int_0^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x) dx$$ since $\mathrm{E}[z^{N_i}]$ has the same value for all i

$$= \mathrm{z}\int_0^{\infty} \sum_{k=0}^{\infty} \frac{(\lambda x G(z))^k}{k!} e^{-\lambda x}\, b(x) dx$$

$$= \mathrm{z}\int_0^{\infty} e^{-(\lambda-\lambda G(z))x}\, b(x) dx = \mathrm{z}\, S^*(\lambda - \lambda G(z))$$

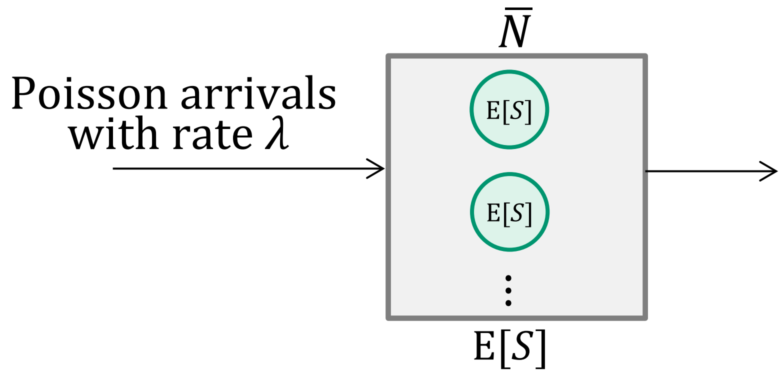
  - $\mathrm{E}[N] = G'(1)$

$$= \int_0^{\infty} e^{-(\lambda-\lambda G(1))x}\, b(x) dx + \lambda\, G'(1) \int_0^{\infty} x e^{-(\lambda-\lambda G(1))x}\, b(x) dx$$

$$= 1 + \lambda \mathrm{E}[N]\mathrm{E}[S]$$ since $G(1)=1$

$$\Rightarrow \mathrm{E}[N] = \frac{1}{1-\lambda \mathrm{E}[S]}$$

M/G/∞ (1)



- Mean number of jobs in the system: $\bar{N}$
  1. By Little's law, $\bar{N} = \lambda \mathrm{E}[S]$
  2. By Probability theory, E[N] = $\sum_{k=0}^{\infty} k\, P_k$
  - Since the jobs arrive according to Poisson process, the probability distribution on the number of jobs in system is also Poisson.
  - Probability of k jobs in the system equals the probability of k Poisson arrivals during E[S], $P_k = \frac{(\lambda \mathrm{E}[S])^k}{k!} \mathrm{e}^{-\lambda \mathrm{E}[S]}$

$$\bar{N} = \sum_{k=1}^{\infty} k \frac{(\lambda \mathrm{E}[S])^k}{k!} e^{-\lambda \mathrm{E}[S]} = \lambda \mathrm{E}[S] e^{-\lambda \mathrm{E}[S]} \sum_{k=0}^{\infty} \frac{(\lambda \mathrm{E}[\mathrm{S}])^{\mathrm{k}}}{\mathrm{k}!}$$
$$= \lambda \mathrm{E}[S]$$

M/G/∞ (2)

- Derivation of P_k
  - $P_k = \lim_{t\to\infty} \Pr\{N(t) = k\}$
    - $N(t)$: r.v. representing the number of jobs in the system at time *t*
  - Let $P_k(t) := \Pr\{N(t) = k\}$.
  - We first derive $P_k(t)$. Then, $P_k = \lim_{t\to\infty} P_k(t)$

< Derivation of $P_k(t)$ >

- Let $A(t)$ be the total number of arrivals for $[0, t]$
- $P_k(t) = \sum_{n=k}^{\infty} \Pr\{A(t) = n\} \times \Pr\{N(t) = k | A(t) = n\}$

$$= \sum_{n=k}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \times \Pr\{N(t) = k | A(t) = n\}$$

M/G/∞ (3)

– We need $\Pr\{N(t) = k | A(t) = n\}$, which is the probability that *k* jobs among *n* arrivals are still being served at time *t*.

- Let $q(t) \coloneqq \Pr\{\text{a job arrived at } [0, \mathrm{t}] \text{ is still in the server at time } t\}$

  Then, $\Pr\{N(t) = k | A(t) = n\} = \binom{n}{k} q(t)^k (1 - q(t))^{n-k}$

- We focus on *any one job* tagged by J.

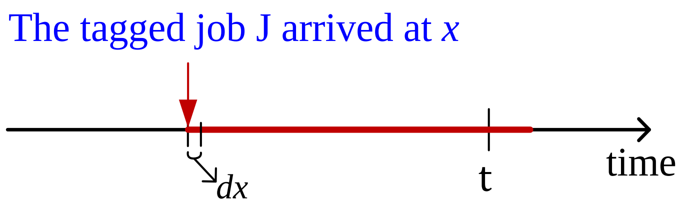
$$q(t) = \Pr\{\text{ J is still in the server at time } t | \text{ J arrived at} [0, t]\}$$

$$= \int_0^t \Pr\{\text{ J arrived at } x | \text{ J arrived at} [0, t]\}$$

$$\times \Pr\{\text{J is in the server at time } t \mid \text{J arrived at time } x\}$$

$$= \int_0^t \frac{dx}{t} \times \Pr\{S > t - x\}$$

where *S* is a r.v. representing the service time



M/G/∞ (4)

– $q(t) = \frac{1}{t}\int_0^t \Pr\{S > t - x\}\,dx = \frac{1}{t}\int_0^t \Pr\{S > y\}\,dy$

– $\Pr\{N(t) = k | A(t) = n\}$

$$= \binom{n}{k} q(t)^k (1 - q(t))^{n-k}$$

$$= \binom{n}{k}\left(\frac{1}{t}\int_0^t \Pr\{s > y\}\,dy\right)^k \left(1 - \frac{1}{t}\int_0^t \Pr\{s > y\}\,dy\right)^{n-k}$$

– $P_k(t) = \sum_{n=k}^{\infty} \Pr\{N(t) = k | A(t) = n\} \times \Pr\{A(t) = n\}$

$$= \sum_{n=k}^{\infty} \binom{n}{k}\left(\frac{1}{t}\int_0^t \Pr\{S > y\}\,dy\right)^k \left(1 - \frac{1}{t}\int_0^t \Pr\{S > y\}\,dy\right)^{n-k} \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= \left(\lambda \int_0^t \Pr\{S > y\}\,dy\right)^k \frac{e^{-\lambda t}}{k!} \times \sum_{n=k}^{\infty} \frac{\left(\lambda t - \lambda \int_0^t \Pr\{S > y\} dy\right)^{n-k}}{(n-k)!}$$

$$= \frac{\left(\lambda \int_0^t \Pr\{S > y\} dy\right)^k}{k!} e^{-\lambda \int_0^t \Pr\{S > y\} dy}$$

M/G/∞ (5)

- $P_k = \lim_{t\to\infty} P_k(t)$

$$= \frac{\left(\lambda \int_0^\infty \Pr\{S>y\}dy\right)^k}{k!} e^{-\lambda \int_0^\infty \Pr\{S>y\}dy}$$

- Since $\int_0^\infty \Pr\{S > y\}\, dy = \int_0^\infty \int_y^\infty f_S(x)dx\, dy$

$$= \int_0^\infty x f_S(x)dx$$

$$= \mathrm{E}[S]$$

- $$P_k = \frac{(\lambda \mathrm{E}[S])^k}{k!} e^{-\lambda \mathrm{E}[S]}$$

Poisson distribution