12.5. Diffusion in Fully Ionized Plasmas

⊛ Fluid Description:

Ignoring inertia (Diffusion is a slow process),

$$\vec{j} \times \vec{B} = \vec{\nabla} p \qquad (12,28)$$

and generalized Ohm's law

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j} + \frac{\vec{j} \times \vec{B} - \vec{\nabla} P_e}{ne} \qquad (12,27)$$

are satisfied.

⊛ Take $\vec{B} = B\hat{z}$ and $P = P(x)$, then from (12,28),

$$j_x = 0, \quad j_y = \frac{1}{B}\frac{dp}{dx} \qquad (12,29).$$

From (12,27), (and writing $\vec{j} \times \vec{B} - \vec{\nabla} P_e = \vec{\nabla} P_i$), and (12,29),

$$u_y = -\frac{E_x}{B} + \frac{1}{neB}\frac{dP_i}{dx} \quad \text{and} \quad \underline{u_x = -\frac{\eta}{B^2}\frac{dp}{dx}} \qquad (12,30)$$

i.e., Particles flow from high pressure region to low pressure region.

$$\Rightarrow \boxed{\vec{u}_\perp = -\frac{\eta}{B^2}\vec{\nabla}_\perp p} \qquad (12,31)$$

Classical Diffusion of Plasmas

⊛ Using mass continuity eqn and assuming isothermal plasmas,

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot(\rho \vec{u}_\perp) = \vec{\nabla}_\perp \cdot \left(\frac{\rho \eta}{B^2} \vec{\nabla}_\perp p\right) = \vec{\nabla}_\perp \cdot \left(\frac{\eta P}{B^2} \vec{\nabla}_\perp \rho\right) \quad (12.32)$$

$$\therefore \quad \boxed{D_\perp = \frac{\eta P}{B^2}}$$

Classical Diffusion Coefficient (12.33)

⊛ Using $\eta \simeq \frac{m\nu_{ei}}{ne^2}$ and $P = n(T_e + T_i)$, we obtain

$$D_\perp \sim \frac{\nu_{ei} m (T_e + T_i)}{e^2 B^2} \sim \underline{\nu_{ei} \langle r_{Le}^2 \rangle} \left(1 + \frac{T_i}{T_e}\right) \quad (12.34)$$

$\sim$ "estimation from random walk argument."

⊛ Recall: $D_\parallel \sim \frac{v_{Th}^2}{\nu} \sim \frac{T}{m\underline{\underline{\nu}}} \sim \underline{\underline{T^{5/2}}}$, while

$* \; D_\perp \sim \underline{\underline{\nu}} \cdot r_L^2 \sim \underline{\underline{T^{-1/2}}}$ (→ higher temperature plasmas diffuse slower!?)

Ambipolarity Constraint

⊛ Since $\vec{\nabla} \cdot \vec{j} = 0$ is satisfied, both electrons and ions diffuse out of plasma at the same rate.

- Underlying reason for this intrinsic ambipolarity is momentum conservation in electron-ion collisions.

⊛ Recall that a rough estimation based on random walk could have led to

$$D_{\perp \, \text{ion}} \sim \nu_{ii} \, r_{Li}^2 \sim \left(\frac{M_i}{m_e}\right)^{1/2} \nu_{ei} \, r_{Le}^2$$

Eq.(12.34)

But this does Not occur in plasmas, because collisions between like ptls (ion-ion collisions or electron-electron collisions) don't lead to any diffusion!

- This is also a consequence of conservation of momentum in collisions! -

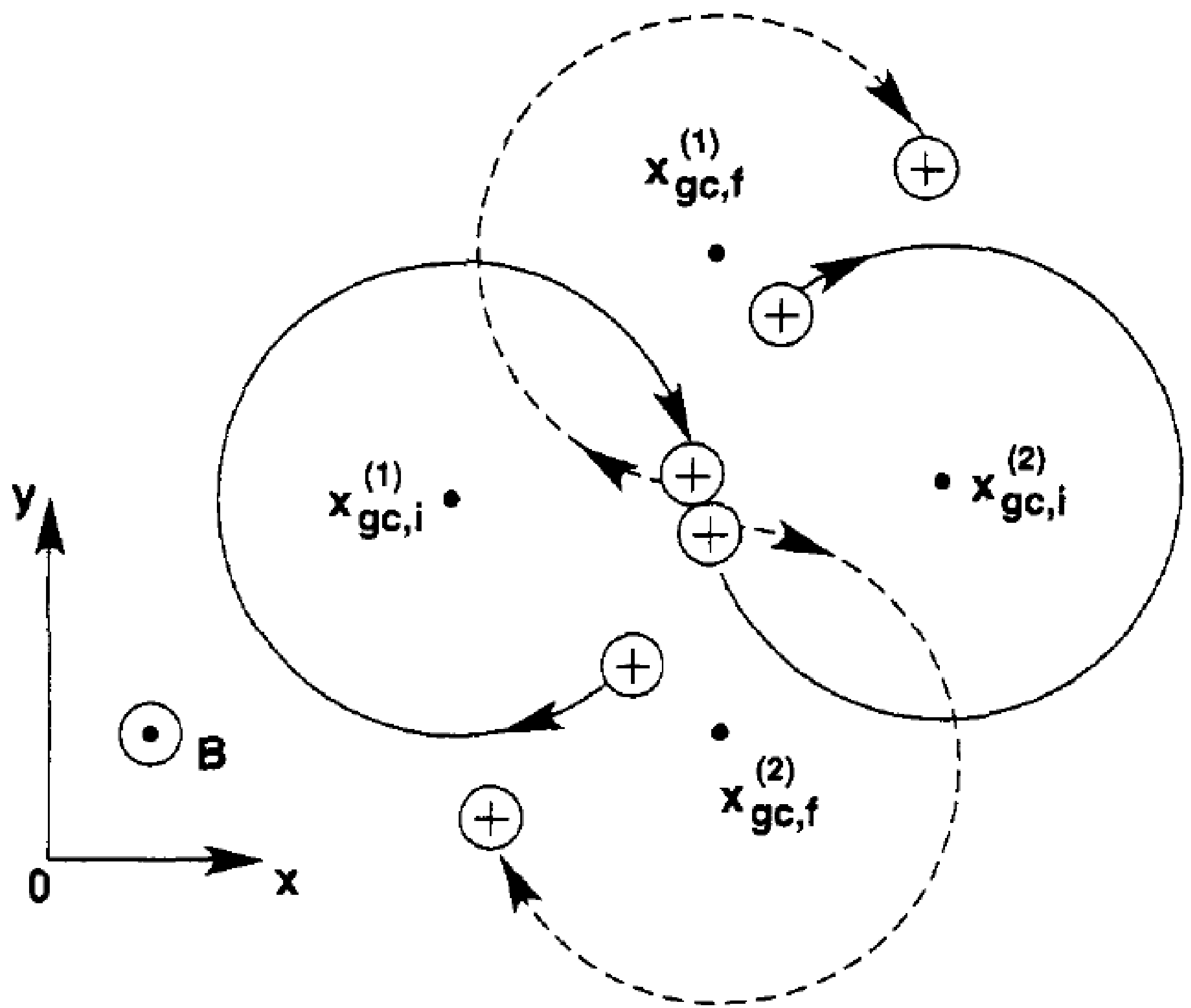


Figure 12.2. Initial (full lines) and final (broken lines) Larmor orbits of two ions making a 90° collision. The initial positions of the two guiding centers are $x^{(1)}_{gc,i}$ and $x^{(2)}_{gc,i}$; the final positions are $x^{(1)}_{gc,f}$ and $x^{(2)}_{gc,f}$.

12.8. Diffusion of Energy (Heat Conduction)

⊛ Conduction : Heat Flux driven by a Temperature Gradient.

When conduction dominates (over convection), the energy transfer equation can be written as :

$$\boxed{\frac{3}{2} n \frac{\partial T}{\partial t} = -\vec{\nabla} \cdot \vec{Q} = \vec{\nabla}_\perp \cdot (K_\perp \vec{\nabla}_\perp T) + \vec{\nabla}_\parallel \cdot K_\parallel (\vec{\nabla}_\parallel T)} \quad (12.58).$$

" $K_\perp$ " and " $K_\parallel$ " are the thermal conductivities perpendicular and parallel to $\vec{B}$, respectively. " K/n " is called the thermal diffusivities.

⊛ Thermal conduction // to $\vec{B}$ is dominated by electrons since

$$\frac{K_\parallel}{n} \sim \frac{v_{Te}^2}{\nu_e} \quad \left(\gg \frac{v_{Ti}^2}{\nu_{ie}} \right) \qquad (12.59)$$

⊛ Thermal Conduction $\perp$ to $\vec{B}$ is, on the other hand, dominated by ions.

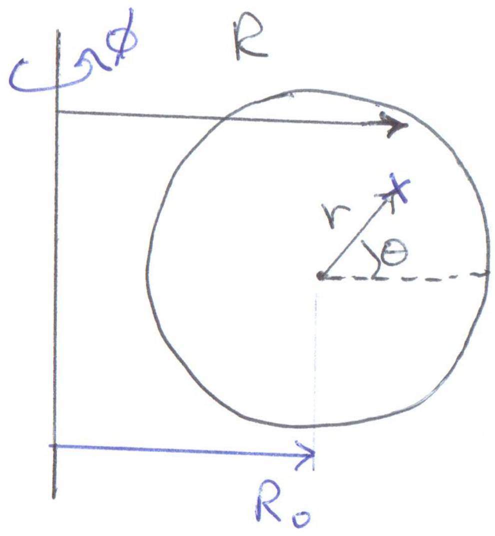
$$\frac{K_\perp}{n} \sim \nu_{ii} r_{Li}^2 \quad (\gg \nu_e r_{Le}^2) \qquad (12.60)$$

∴ No constraint on spatial transfer of energy !

Magnetically Trapped Particles in Tokamak Plasmas

⊛ Equilibrium : $\vec{B} = \dfrac{B_0}{1+\frac{r}{R_0}\cos\theta}\,\hat{\phi} + \dfrac{\epsilon}{q(r)}\,\dfrac{B_0}{1+\frac{r}{R_0}\cos\theta}\,\hat{\theta}$

"toroidal" "poloidal"



- $\epsilon = r/R_0$: inverse aspect ratio
- $q(r) \simeq rB_\phi / RB_\theta$: safety factor.
- $B \equiv |\vec{B}| \simeq B_\phi = \dfrac{B_0 R_0}{R} = \dfrac{R_0}{1+(\frac{r}{R_0})\cos\theta}$

⊛ (Axisymmetric)-Equilibrium :

no dependence on "ϕ" : $\dfrac{\partial}{\partial\phi}$(equilibrium quantity) $= 0$,

"ϕ" is a cyclic (ignorable) coordinate.

⊛ ϕ is a cyclic variable $\Rightarrow$ Canonical Angular momentum P_ϕ is conserved.

$$P_\phi = R\left(mv_\phi + \frac{Z|e|}{c}A_\phi\right) \simeq R\left(mv_\parallel + \frac{Z|e|}{c}A_\phi\right) = Rmv_\parallel - \frac{Z|e|}{c}\psi.$$

for ion

where ψ is the poloidal flux function $(d\psi = RB_\theta\, dr)$

⊛ Unperturbed particle orbits in a tokamak stay on the surface of constan $\underline{P_\phi}$, constant $\underline{\mu}$, and constant $\underline{E}$.

⊛ Let's consider a guiding-center motion along $\vec{B}$.

Let "l" be a distance along $\vec{B}$:

$$dl^2 = R^2 d\phi^2 + r^2 d\theta^2 = (qR_0)^2 d\theta^2 + r^2 d\theta^2 \simeq (qR_0)^2 d\theta^2,$$

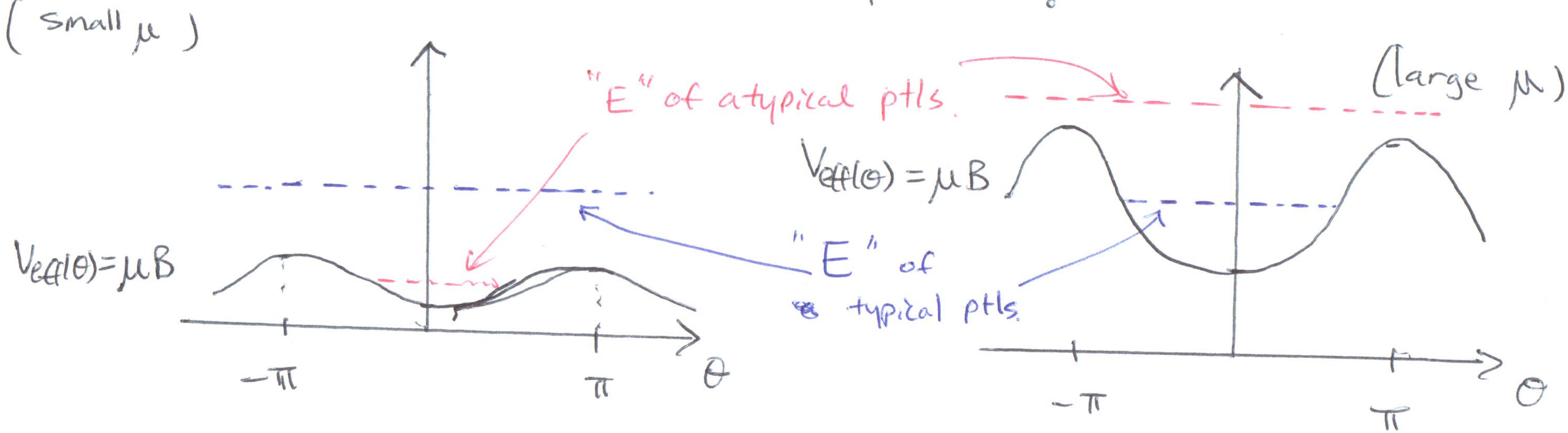
where $q = d\phi/d\theta = rB_\phi/RB_\theta$, $\Rightarrow v_\parallel = \frac{dl}{dt} = qR_0\frac{d\theta}{dt}$

⊛ For $\frac{r}{R_0} \ll 1$, $\Phi_0 = 0$, $\vec{E}_0 = 0$

$$E = \frac{1}{2}mv_{\parallel}^2 + \frac{1}{2}mv_{\perp}^2 = \frac{1}{2}mq^2R_0^2\left(\frac{d\theta}{dt}\right)^2 + \underline{\mu B_0\left(1 - \frac{r}{R_0}\cos\theta\right)} \; ;$$

- An effective potential for motion along $\vec{B}$ projected to θ originated from $\perp$ kinetic energy.

⊛ Note that the value of μ is a constant in time, but different for different particles.



⊛ For trapped particles, there is a θ_T such that

$$E = \mu B(\theta = \theta_T) = \frac{1}{2} m v_\parallel^2(\theta=0) + \mu B(\theta=0).$$

θ_T : is the turning point where $v_\parallel = 0$.

⊛ For passing particles, $v_\parallel \neq 0$ even at $\theta = \pm\pi$ where B is maximum, (and $|v_\parallel|$ is minimum) as a function of θ.

⇒ trapped-passing boundary is determined from

$$\frac{1}{2} m v_\parallel^2(\theta=\pm\pi) + \mu B_0\left(1+\frac{r}{R_0}\right) = \frac{m}{2} v_\parallel^2(\theta=0) + \mu B_0\left(1-\frac{r}{R_0}\right).$$

By requiring $v_\parallel^2(\theta=\pm\pi) > 0$ for passing ptls, we obtain

$$v_\parallel^2(\theta=0) \geq 4\frac{r}{R_0}\frac{\mu B_0}{m},$$

(*) Fraction of trapped ptls $= \dfrac{\#\text{ of trapped ptls}}{\#\text{ of total ptls}} \propto \sqrt{\epsilon}$

for isotropic distribution in $\vec{v}$.

(*) Radial Width of Orbits :

- Calculate the radial deviation from the reference flux surface ψ_0 (as a function of θ) :

Since $P_\phi = -\frac{Z|e|}{c}\psi + mRv_\parallel(\theta) = -\frac{Z|e|}{c}\psi_0$ for each trapped ptl,

$$\Delta_b = \frac{2\Delta\psi}{RB_\theta} = \frac{2}{RB_\theta}\frac{mc}{Z|e|}R\,v_\parallel(\theta=0) = \frac{2mc}{Z|e|B_\theta}v_\parallel(\theta=0);$$

for a typical trapped ptl, $v_\parallel(\theta=0) \simeq \sqrt{\epsilon}\,v$

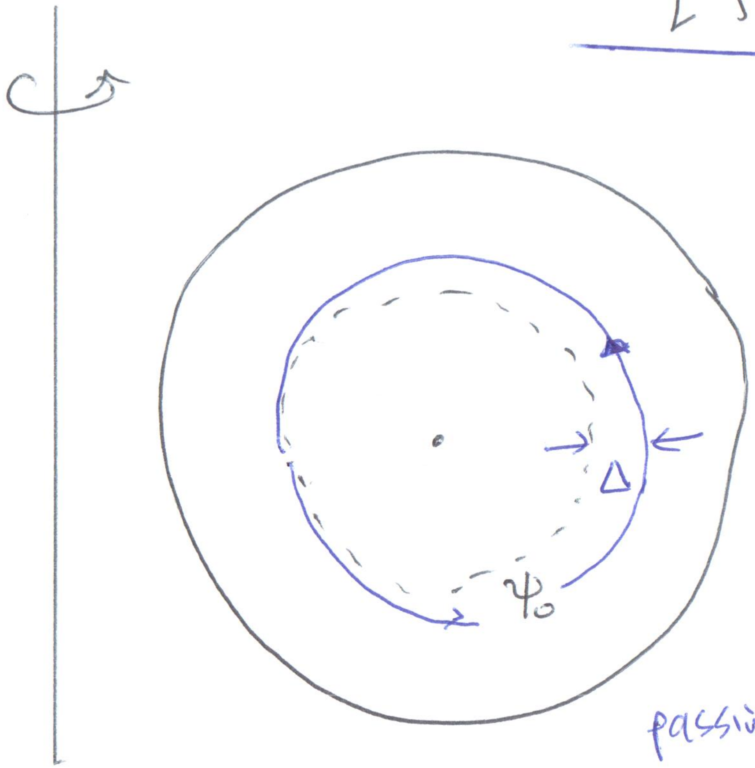
$$* \quad \underline{\Delta_b \simeq \sqrt{\epsilon}\,\rho_i\frac{q}{\epsilon} \simeq \frac{q}{\sqrt{\epsilon}}\rho_i}$$

: banana width.

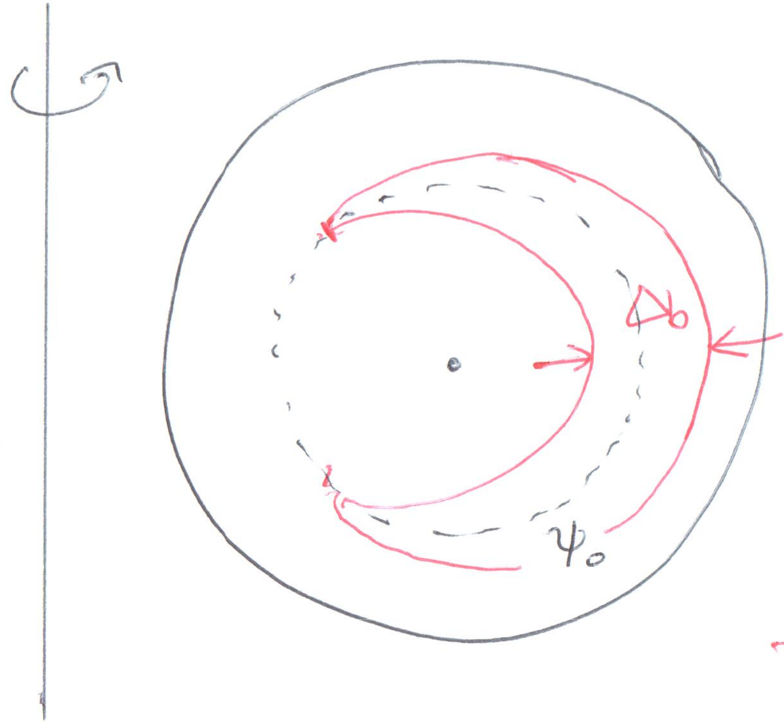
⊛ Meanwhile, for strongly passing ptls, variations of $v_{\parallel}$ along $\vec{B}$ is not significant.

$$\rightarrow \underline{\Delta} \simeq \frac{2\Delta\psi}{RB_\theta} \simeq \frac{2}{RB_\theta}\frac{mc}{Z|e|} v_{\parallel}(\theta=0)\,\Delta R \simeq \frac{2mc}{Z|e|B_\theta}\frac{r}{R} v_{\parallel}$$

$$\simeq \underline{2q\rho_i} \qquad (v_{\parallel} \gg v_{\perp} \rightarrow v_{\parallel} \simeq v).$$



passing ptl



trapped ptl

* NeoClassical Transport

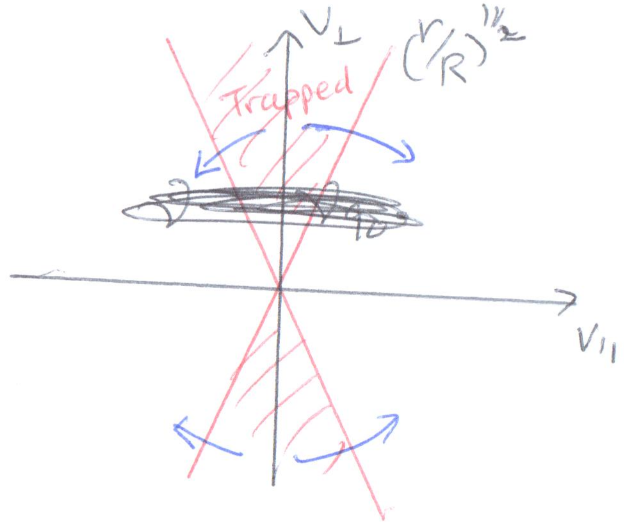
~ Coulomb collisional transport of charged particles influenced by toroidal geometry.

* Banana Orbits:

- Radial Width: $\Delta_b \simeq \left(\frac{r}{R}\right)^{1/2} \rho_\theta$, $\rho_\theta = \frac{V_{Th}}{\Omega}\frac{B_\phi}{B_\theta} = \frac{B_\phi}{B_\theta}\rho$,

- Effective Collisional Frequency for scattering through pitch-angle $\Delta(v_{\parallel}/v)$:

$$\nu_{eff} \cong \nu / \left[\Delta\left(\frac{v_{\parallel}}{v}\right)\right]^2 \cong \left(\frac{R}{r}\right)\nu ,$$



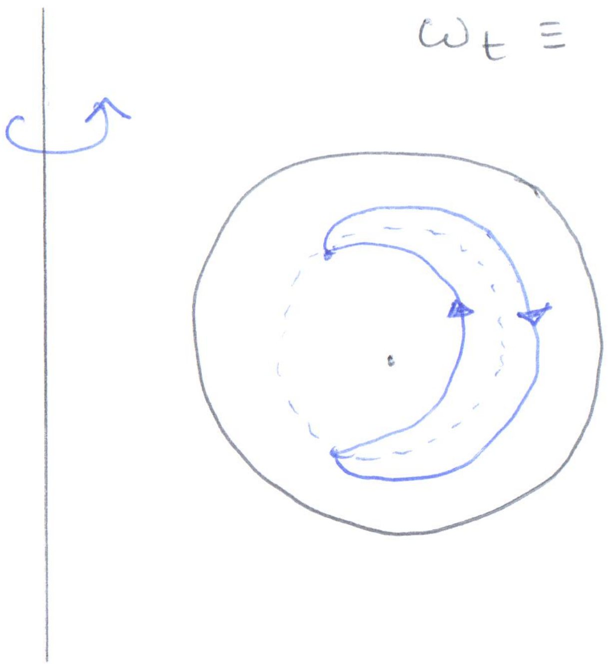
⊛ Collisionality Regimes

I. Banana (Collisionality) Regime ;

" Charged ptl can execute several banana orbits ~~without~~ before being scattered into loss cone and become passing ptl."

→ " $\nu_{eff} < \omega_b$ " ; bounce frequency $\approx \left(\frac{r}{R}\right)^{1/2} \frac{v_{Th}}{qR}$.

$\omega_t \equiv \frac{v_{Th}}{qR}$: transit frequency.



$\omega_b \sim \frac{1}{\tau_{bounce}}$ of banana orbit execution.

II. "Plateau" Regime ;

for

$$\left(\frac{r}{R}\right)^{3/2} < \nu \Big/ \left(\frac{v_{Th}}{qR}\right) < 1$$

- somewhere between "banana" and "P-S" regime
- Diffusion rate is almost independent of collisionality "ν"

→ "plateau".

III. Pfirsch-Schlüter Regime ;

for $\nu \cdot (qR / v_{Th}) > 1$, i.e., mean free path of untrapped ptls is shorter than the "connection length" $\sim qR$ between top and bottom of torus.

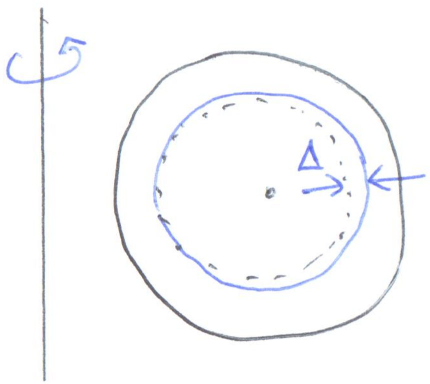
"Banana orbits cannot maintain their identities even during one execution."

⊛ Radial Diffusion Coefficient

⊛ $D \approx$ (fraction of participating ptls) $\times$ (effective collision freq.) $\times$ (radial stepsize)^2.

$\sim \Delta x^2/\Delta t$ from random walk.

III. Pfirsch-Schlüter Regime:



Δ: radial deviation from reference flux surface of passing ptls. $\simeq q\rho$

$$D_{ps} \approx \nu (q\rho)^2 \simeq q^2 (\nu \rho^2)$$

"classical" diffusion coeff. (in cylinder).

$(\quad " \quad) \simeq (1-(\frac{r}{R})^{1/2}) \simeq 1.$

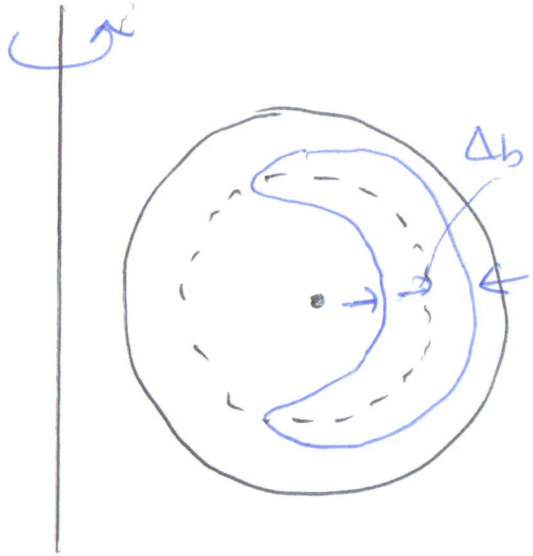
I. Banana Regime:

$$D_{Banana} \approx \left(\frac{r}{R}\right)^{1/2} \cdot \nu_{eff} \cdot \Delta_b^{\,2}$$

$$\approx \left(\frac{r}{R}\right)^{1/2} \cdot \frac{\nu}{(r/R)} \cdot \left(\frac{q}{\left(\frac{r}{R}\right)^{1/2}} \rho \right)^2$$

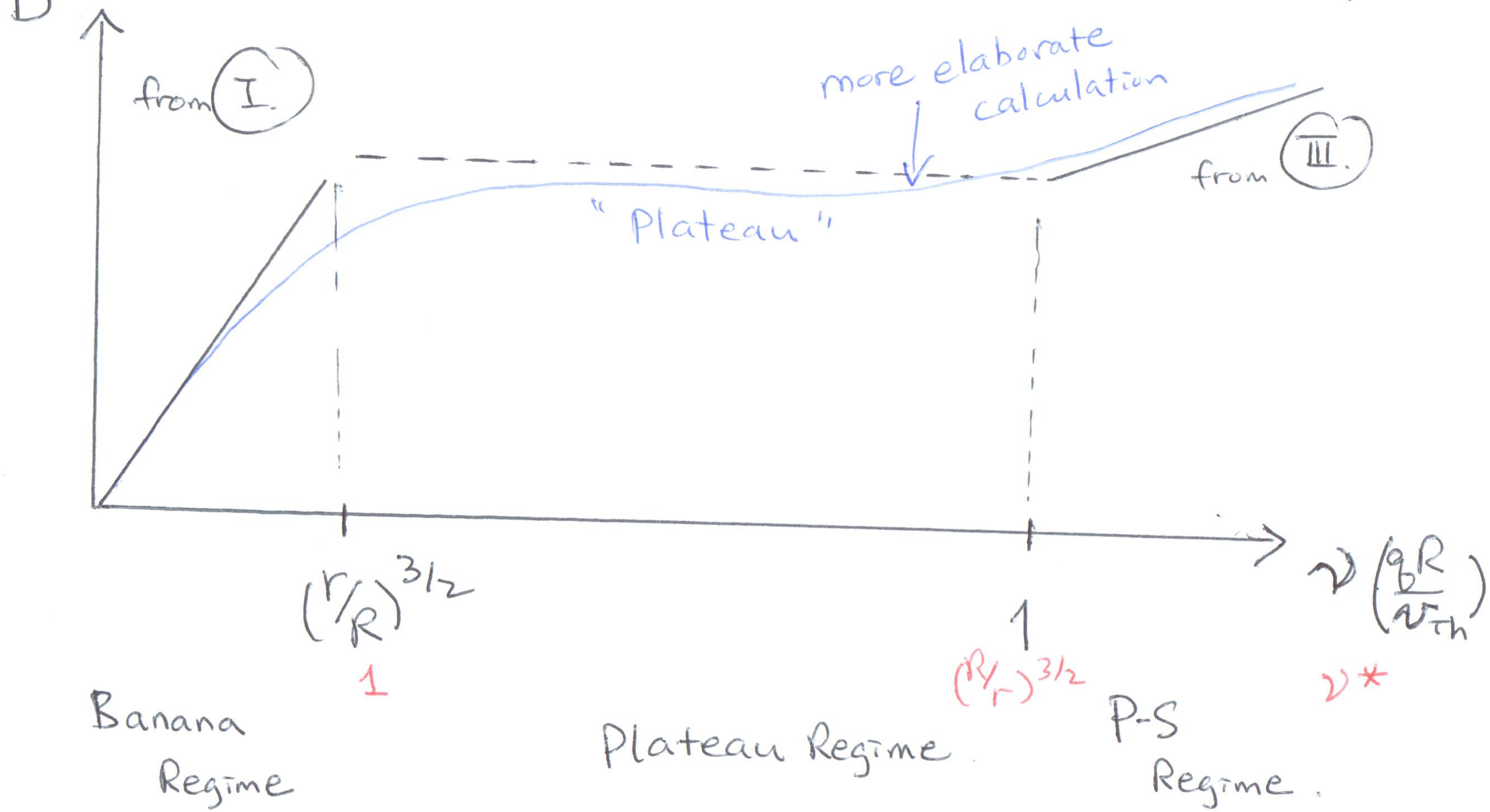
$$\approx \underline{\frac{q^2}{(r/R)^{3/2}}} (\nu \rho^2) \approx \left(\frac{r}{R}\right)^{1/2} \nu \rho_\theta^{\,2}.$$

banana enhancement factor over classical diffusion



⊛ Neo Classical Diffusion

as a function of collisionality



⊛ $\boxed{\nu_* \equiv \nu_{eff}/\omega_b = \nu\left(\frac{qR}{v_{Th}}\right)\left(\frac{R}{r}\right)^{3/2}}$ Standard Definition!

⊛ Remarks on Neo Classical Transport :

* Disparities among $\nu_{ee}, \nu_{ei} \leftrightarrow \nu_{ii} \leftrightarrow \nu_{ie}$ still exist.

* Ambipolar constraints for particle diffusion still exist for axi-symmetric tokamaks,

* Predicted rate of transport is much lower than experimental estimations unless turbulence is sufficiently suppressed.

* It can be relevant for heavy metal impurities due to $Z_I \gg 1$.