Linear Systems Analysis

in the Time Domain

Seoul National Univ.
School of Mechanical
and Aerospace Engineering

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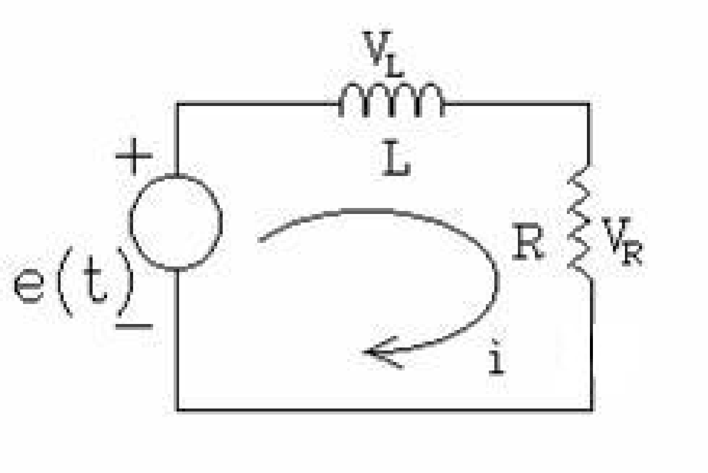
First Order Systems



$$v_L = L\frac{di}{dt}, \qquad v_R = Ri,$$

$$L\frac{di}{dt} + Ri = e(t)$$

$$x = i, \;\; \dot{x} = -\frac{R}{L}x + \frac{1}{L}e(t)$$

$$\frac{X(s)}{U(s)} = \frac{I(s)}{E(s)} = \frac{1}{Ls+R} = \frac{1}{R}\cdot\frac{1}{\frac{L}{R}s+1}$$

$$u(t) = e(t) = 1, \qquad i(0) = 0$$

$$i(t) = \frac{1}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

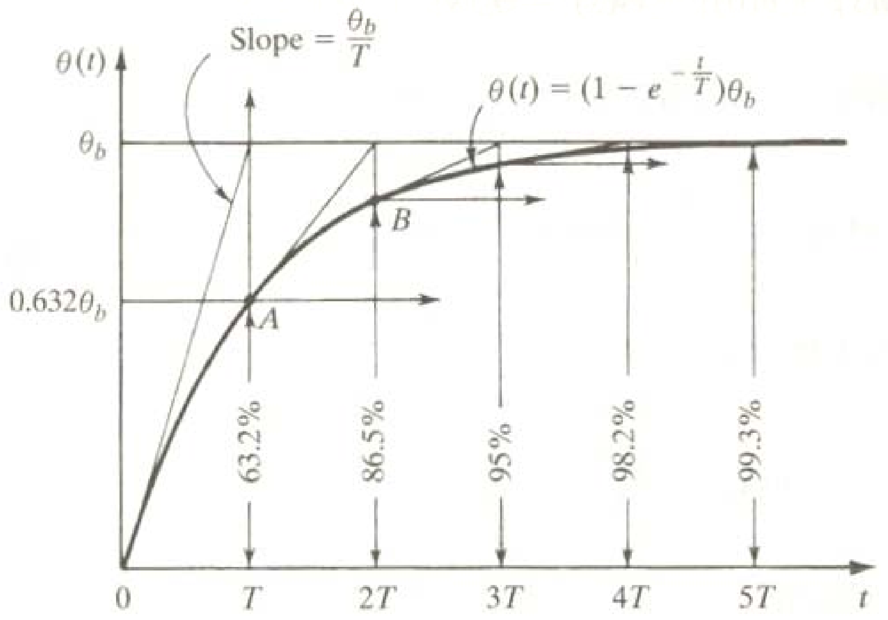
First Order Systems

$$\frac{Y(s)}{U(s)} = \frac{1}{Ts+1}, \qquad U(s) = \frac{1}{s}, \qquad u(t) = 1$$

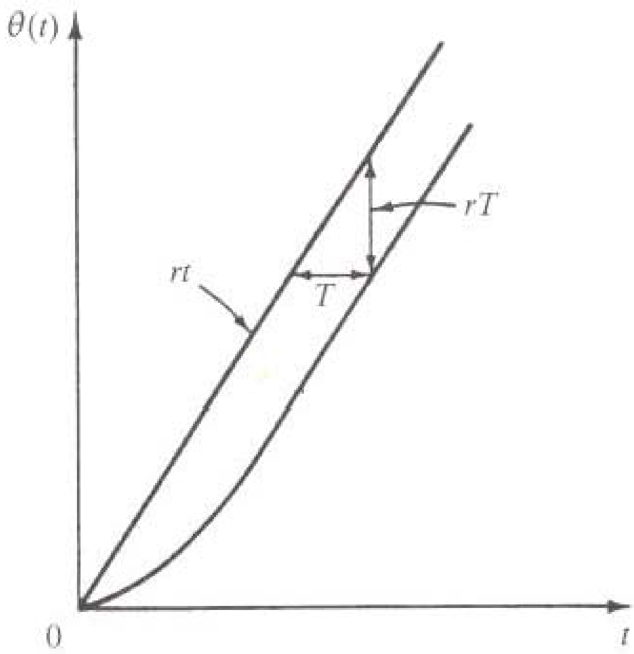
T : *time constant*

$$Y(s) = \frac{1}{Ts+1}\frac{1}{s} = \frac{1}{s} - T \cdot \frac{1}{Ts+1}$$

$$\therefore \quad y(t) = 1 - e^{-\frac{1}{T}t}, \qquad \dot{y}(t) = \frac{1}{T}e^{-\frac{1}{T}t}$$



First Order Systems



$$R(t) = rt$$

$$R(s) = r \cdot \frac{1}{s^2}$$

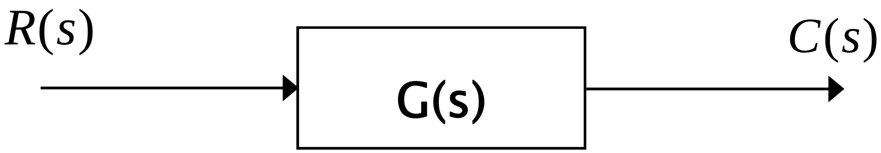
$$Y(s) = \frac{1}{Ts+1} \cdot r \cdot \frac{1}{s^2} = r\left(\frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+(1/T)}\right)$$

$$y(t) = r(t - T + Te^{-\frac{t}{T}})$$

$$e(t) = R(t) - y(t) = rT(1 - e^{-\frac{t}{T}})$$

$$e(\infty) = rT$$

Second Order Systems



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$R(s) = \frac{1}{s} \quad (step\ input), \qquad C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0, \qquad s = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

Second Order Systems

Underdamped case $s = -\zeta\omega_n \pm \sqrt{1-\zeta^2}\,\omega_n i\,, \qquad (\omega_d = \omega_n\sqrt{1-\zeta^2})$

$0 < \zeta < 1$

$$C(s) = \frac{\omega_n^2}{(s+\zeta\varpi_n)^2 + \omega_n^2(1-\zeta^2)} \cdot \frac{1}{s} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d{}^2}$$

$$\therefore C(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \eta)$$

$$\eta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Critically damped case

$\zeta = 1$

$$R(s) = \frac{1}{s}, \qquad C(s) = \frac{\omega_n{}^2}{(s+\zeta\omega_n)^2 s}$$

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t)$$

Second Order Systems

Overdamped case $\zeta > 1$

$$C(s) = \frac{\omega_n^2}{(s+\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s+\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})}e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t} - \frac{1}{2\sqrt{\zeta^2-1}(\zeta-\sqrt{\zeta^2-1})}e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

$$= 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}}{\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n} - \frac{e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}}{\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n}\right)$$

Approximation (After the faster term disappeared)

$$\frac{C(s)}{R(s)} = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}{s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}$$

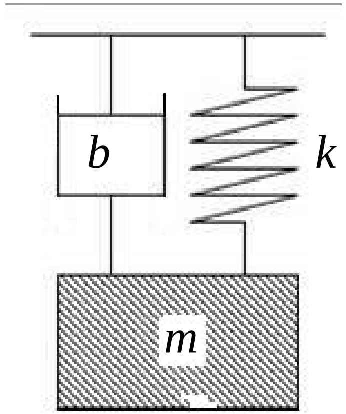
$$\therefore c(t) = 1 - e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

Experimental Determination of Damping Ratio

$$m\ddot{x} + b\dot{x} + kx = 0, \qquad \dot{x}(0) = 0$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n{}^2 x = 0$$

$$\zeta = \frac{1}{2\omega_n}\frac{b}{m} = \frac{b}{2\sqrt{mk}}$$

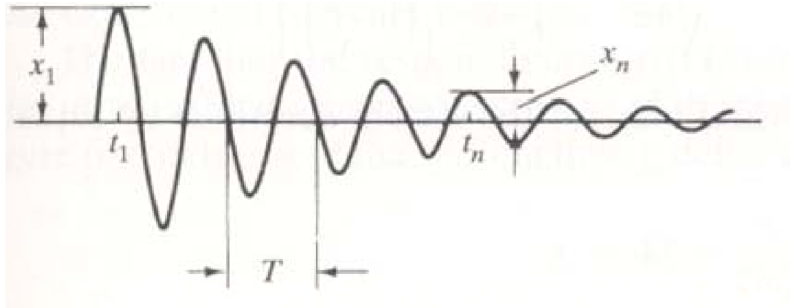


$$\left[s^2 X(s) - sx(0) - \dot{x}(0)\right] + 2\zeta\omega_n\left[sX(s) - x(0)\right] + \omega_n{}^2 X(s) = 0$$

$$X(s) = \frac{\left(s + 2\zeta\omega_n\right)x(0)}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{\frac{\zeta}{\sqrt{1-\zeta^2}}x(0)\sin\omega_d t + x(0)\cos\omega_d t\right\} = \frac{x(0)}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_d t - \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$\frac{x_1}{x_n} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1+(n-1)T)}} = e^{(n-1)\zeta\omega_n T}$$



Experimental Determination of Damping Ratio

Logarithmic decrement

$$\ln\frac{x_1}{x_2} = \zeta\omega_n T = \zeta\omega_n \cdot \frac{2\pi}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{1}{n-1}\left(\ln\frac{x_1}{x_n}\right)$$

$$\ln\frac{x_1}{x_n} = (n-1)\zeta\omega_n T$$

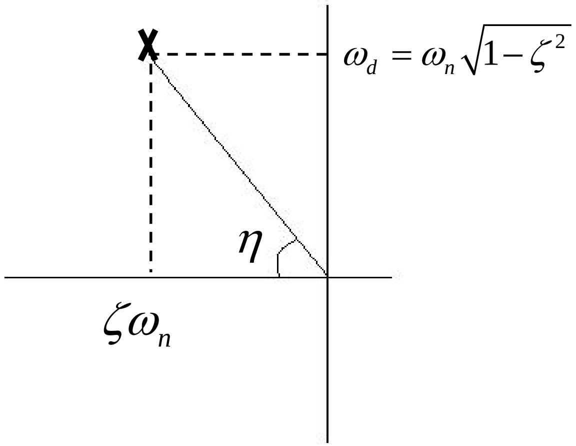
$$\Rightarrow \ \zeta = \frac{\frac{1}{n-1}\left(\ln\frac{x_1}{x_n}\right)}{\sqrt{4\pi^2 + \left\{\frac{1}{n-1}\left(\ln\frac{x_1}{x_n}\right)\right\}^2}}$$

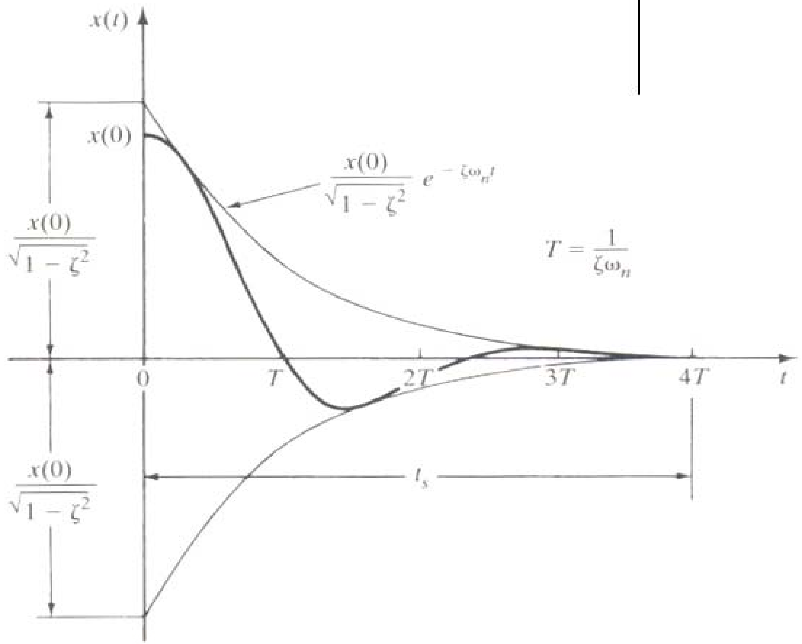
Estimate of Response Time

$$x(t) = \frac{x(0)}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \cos\left(\omega_d t - \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}} \right)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}, \qquad t = \frac{1}{\zeta\omega_n}, \qquad \omega_d t = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\pi}{2} - \eta$$



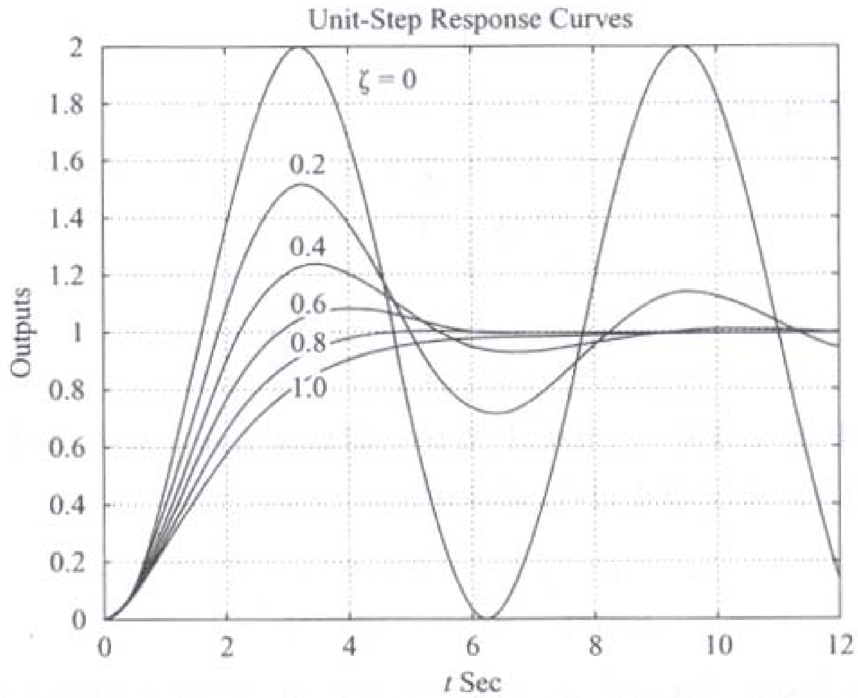


Second Order Transients

Step input response

$$C(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \eta)$$

$$\eta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$



Second Order Transients

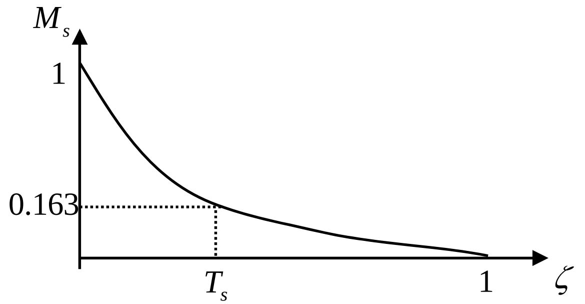
1) Peak overshoot M_p

$$\frac{dx}{dt} = \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n e^{-\zeta\omega_n t}\sin(\omega_d t+\phi) - \omega_n e^{-\zeta\omega_n t}\cos(\omega_d t+\phi) = 0, \qquad (\omega_d = \omega_n\sqrt{1-\zeta^2})$$

$$\Rightarrow \quad \tan(\omega_d t+\phi) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$t=0, \quad \omega_d t = \pi, 2\pi, \cdots$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$



$$M_p = y(t_p)$$

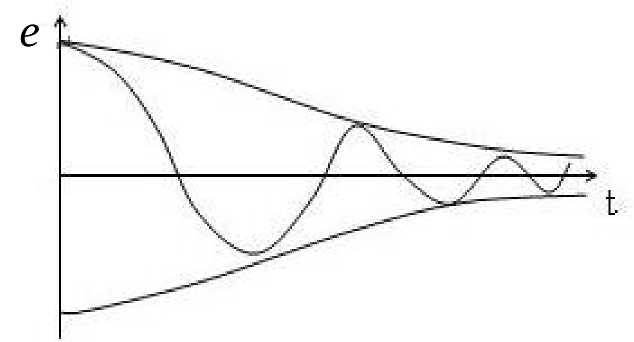
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \cdot \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \sin\left(\omega_n\sqrt{1-\zeta^2}\cdot\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} + \phi\right) = 1 + \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

$$per\ \ unit\ \ overshoot \qquad M_o = \frac{M_p - y_s}{y_s} = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

Second Order Transients

2) Settling time : The time required for the oscillations to decrease to a specified absolute percentage error. T_s

$$e = y - r = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$$

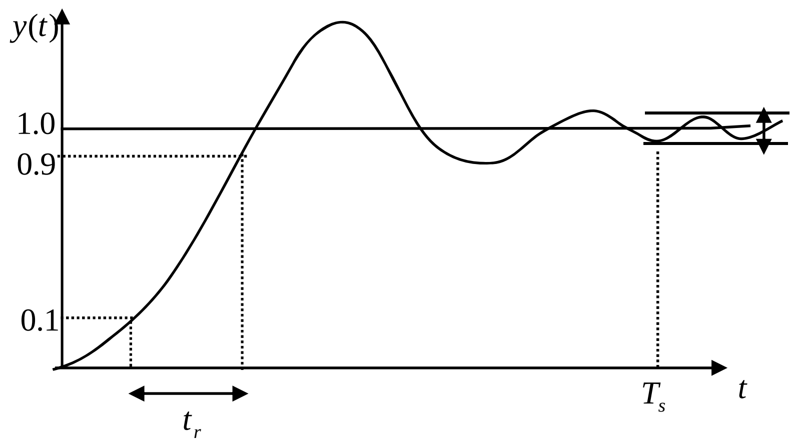


ex) $e = 2\%$ or 5%

2% case, $T_s \cong 4 \cdot T = 4 \cdot \dfrac{1}{\zeta\omega_n}$

5% case, $T_s \cong 3 \cdot T$

3) Rise time t_r



Second Order Transients

4) Frequency of oscillation of the transient

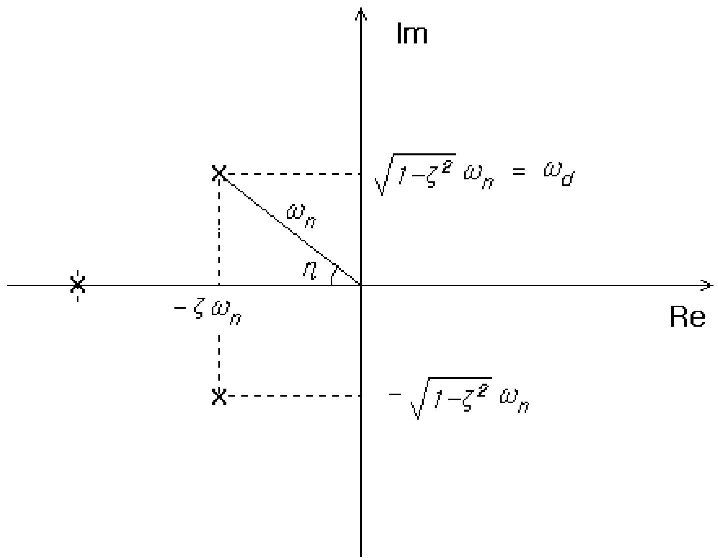
$$\omega_d = \omega_n\sqrt{1-\zeta^2}, \quad \omega_n = \sqrt{\frac{k}{m}},$$

$$\Rightarrow \qquad \omega_d = \sqrt{\frac{k}{m}} \cdot \sqrt{1-\zeta^2}$$

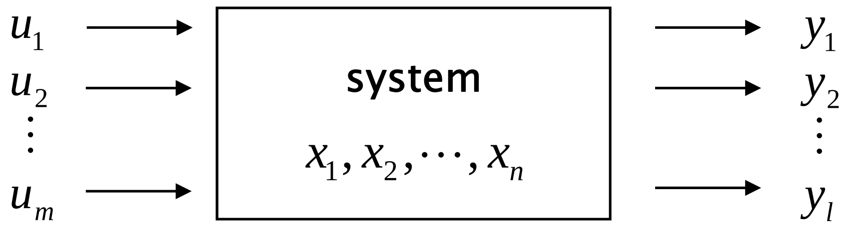
$$m^2 + 2\zeta\omega_n m + \omega_n^{\ 2} = 0$$

$$m = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= \sigma \pm j\omega_d$$



Solution of Linear (Time Invariant) State Equation



$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$$x(t) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$$

Basic Matrix Linear Algebra
-Homogeneous Solution

Scalar Function $u(t) = 0$

i) $\dot{x} = ax$

$$x(t) = Ce^{at} \qquad t = 0, \quad x(0) = C$$

$$x(t) = x(0)e^{at}$$

$$x(t) = e^{a(t-t_0)}x(t_0), \qquad t = t_0, \ x(t_0)$$

ii) $e^{at} = \exp(at) = 1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \cdots + \frac{(at)^k}{k!} + \cdots$

Homogeneous Solution

i) $\dot{\mathrm{x}} = \mathrm{A}\,\mathrm{x} \qquad \mathrm{A} : n \times n, \quad \mathrm{x} : n \times 1$

$$\mathrm{x}(t) = \exp\left[\mathrm{A}(t - t_0)\right]\mathrm{x}(t_0), \qquad \frac{d}{dt}(e^{\mathrm{A}t}) = \mathrm{A}\,e^{\mathrm{A}t}$$

ii) *How to evaluate* $e^{\mathrm{A}t}$

State Transition Matrix

$$\mathrm{x}(t) = e^{\mathrm{A}(t-t_0)}\,\mathrm{x}(t_0)$$
$$= \Phi(t-t_0)\,\mathrm{x}(t_0)$$

$$\Phi(t-t_0) = e^{A(t-t_0)} = \exp\left[\mathrm{A}(t-t_0)\right]$$: State transition matrix (STM)

: fundamental matrix of the system

Properties of STM

1. $\Phi(t_2-t_1)\Phi(t_1-t_0) = \Phi(t_2-t_0) \quad \textit{for any } t_0, t_1, t_2$
2. $\Phi(0) = I$
3. $\Phi(t)\Phi(t) = \Phi^2(t) = \Phi(2t) \quad \left(\because \left(e^{\mathrm{A}t}\right)^2 = e^{\mathrm{A}\cdot 2t} = \Phi(2t)\right)$

   $\Phi^g(t) = \Phi(gt)$
4. $\Phi^{-1}(t) = \Phi(-t)$
5. $\Phi(t)$ is nonsingular for all finite values of t (inverse exists)

Complete Solution of the State Equation

$$\dot{\mathrm{x}} = \mathrm{A}\,\mathrm{x} + \mathrm{B}u, \qquad \dot{\mathrm{x}} - \mathrm{A}\,\mathrm{x} = \mathrm{B}u,$$

integrating factor e^{-At}

$$e^{-At}\,\dot{\mathrm{x}} - \mathrm{A}\,e^{-At}\,\mathrm{x} = e^{-At}\,\mathrm{B}u, \qquad \frac{d}{dt}\left[e^{-At}\,\mathrm{x}\right] = e^{-At}\,\mathrm{B}u(t)$$

$$\Rightarrow e^{-At}\left(\mathrm{x}(t) - \mathrm{x}(0)\right) = \int_0^t e^{-A\tau}\,\mathrm{B}u(\tau)d\tau$$

$$\Rightarrow \mathrm{x}(t) = e^{At}\,\mathrm{x}(0) + \int_0^t e^{A(t-\tau)}\,\mathrm{B}u(\tau)d\tau$$

$$= \Phi(t)\,\mathrm{x}(0) + \int_0^t \Phi(t-\tau)\,\mathrm{B}u(\tau)d\tau$$

Complete Solution of the State Equation

$$for\ [t\ ,\ t_0],\qquad e^{-At}\,\mathrm{x}(t)-e^{-At_0}\,\mathrm{x}(t_0)=\int_{t_0}^{t}e^{-A\tau}\,\mathrm{B}u(\tau)d\tau$$

$$\Rightarrow \mathrm{x}(t)=\Phi(t-t_0)\,\mathrm{x}(t_0)+\int_{t_0}^{t}\Phi(t-\tau)\,\mathrm{B}u(\tau)d\tau$$

$$\Rightarrow \mathrm{x}(t)=\underline{\Phi(t)\ \mathrm{x}(0)}\quad+\quad\underline{\int_{0}^{t}\Phi(t-\tau)\,\mathrm{B}u(\tau)d\tau}$$

- Zero-input response
- free response

- Zero-state response
- forced response

$$change\ of\ variable\quad \tau,\qquad let\ \ \beta=t-\tau,\qquad \tau=0\rightarrow\beta=t$$

$$\tau=t\rightarrow\beta=0$$

$$d\beta=-d\tau$$

$$\Rightarrow \mathrm{x}(t)=\Phi(t)\,\mathrm{x}(0)+\int_{0}^{t}\Phi(\beta)\,\mathrm{B}u(\beta)d\beta$$

Matrix Exponential

$$\mathrm{x}(t) = e^{At}\,\mathrm{x}(0) \quad \cdots(\alpha)\,, \qquad \dot{\mathrm{x}} = \mathrm{A}\,\mathrm{x} \quad \cdots(\beta)$$

Matrix exponential e^{At}

$$e^{At} = \mathrm{I} + \mathrm{A}\,t + \frac{1}{2!}\mathrm{A}^2 t^2 + \frac{1}{3!}\mathrm{A}^3 t^3 + \cdots \qquad (*1)$$

$$\left(\because e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots \right)$$

$$\frac{d}{dt}(e^{At}) = \mathrm{A} + \mathrm{A}^2 t + \frac{1}{2}\mathrm{A}^3 t^2 + \cdots = \mathrm{A}\left\{ \mathrm{I} + \mathrm{A}\,t + \frac{1}{2}\mathrm{A}^2 t^2 + \cdots \right\}$$

$$= \mathrm{A}\,e^{At}$$

$$\dot{\mathrm{x}}(t) = \frac{d}{dt}\left(e^{At}\right)\mathrm{x}(0) = \mathrm{A}\,e^{At}\,\mathrm{x}(0) = \mathrm{A}\,\mathrm{x}(t)$$

$\Rightarrow (\alpha)$ is the solution of the matrix differential equation (β)

Matrix Exponential

- $e^{At} = \exp(\mathrm{A}\,t) = \sum_{k=0}^{\infty} \frac{\mathrm{A}^k t^k}{k!}$
- $\frac{d\left(e^{At}\right)}{dt} = \mathrm{A}\,e^{At}$
- $e^{(A+B)t} = e^{At}e^{Bt} \qquad if \quad \mathrm{AB} = \mathrm{BA}$

  $e^{(A+B)t} \neq e^{At}e^{Bt} \qquad if \quad \mathrm{AB} \neq \mathrm{BA}$

$$\Rightarrow e^{(A+B)t} = \mathrm{I} + (\mathrm{A}+\mathrm{B})t + \frac{(\mathrm{A}+\mathrm{B})^2}{2!}t^2 + \frac{(\mathrm{A}+\mathrm{B})^3}{3!}t^3 + \cdots$$

$$e^{At}e^{Bt} = \left(\mathrm{I} + \mathrm{A}\,t + \frac{\mathrm{A}^2 t^2}{2!} + \frac{\mathrm{A}^3 t^3}{3!} + \cdots\right)\left(\mathrm{I} + \mathrm{B}\,t + \frac{\mathrm{B}^2 t^2}{2!} + \frac{\mathrm{B}^3 t^3}{3!} + \cdots\right)$$

$$= \mathrm{I} + (\mathrm{A}+\mathrm{B})t + \frac{\mathrm{A}^2 t^2}{2!} + \mathrm{AB}t^2 + \frac{\mathrm{B}^2 t^2}{2!} + \frac{\mathrm{B}^3 t^3}{3!} + \frac{\mathrm{A}^2\mathrm{B}t^3}{2!} + \frac{\mathrm{AB}^2 t^3}{2!} + \frac{\mathrm{B}^3 t^3}{3!}\cdots$$

How to Evaluate e^{At}

Hence,

$$e^{(A+B)t} - e^{At}e^{Bt} = \frac{BA - AB}{2!}t^2 + \frac{BA^2 + ABA + B^2A + BAB - 2A^2B - 2AB^2}{3!}t^3 + \cdots$$

The difference between $e^{(A+B)t}$ and $e^{At}e^{Bt}$ vanishes,
if A and B commute.

How to Evaluate e^{At}

(*1) $\Rightarrow$ a) Diagonalized Form $\quad A = \begin{bmatrix} \lambda_1 & & \underline{0} \\ & \lambda_2 & \\ \underline{0} & & \lambda_3 \end{bmatrix}$

$$e^{At} = I + At + \frac{1}{2}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}^2 t^2 + \frac{1}{3!}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}^3 t^3 + \cdots$$

$$= \begin{bmatrix} 1 + \lambda_1 t + \frac{1}{2}\lambda_1^2 t^2 + \frac{1}{3!}\lambda_1^3 t^3 + \cdots & & \\ & \ddots & \\ & & \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_3 t} \end{bmatrix}$$

How to Evaluate e^{At}

b) Jordan Form

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad \rightarrow \quad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \quad \rightarrow \quad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2 e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

How to Evaluate e^{At}

c) General A : diagonalize !!

$\dot{x} = Ax + Bu$ i) $Q(\lambda) = \det[\lambda I - A] = 0$: Characteristic equation.

$A : n \times n$

$n \;\; sol \; = \lambda_i \qquad i = 1, \cdots, n$

ii) λ_i : eigenvalue of A

n distinct eigenvalues.

iii) $(\lambda_i I - A) p_i = 0 \qquad p_i : eigen\ vector$

$A p_i = \lambda_i p_i$

$$A\begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \qquad AP = P\Lambda$$

$$\Rightarrow \; P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \Lambda$$

Diagonalizable if $\lambda_i : n - distint\ eigenvalues$

How to Evaluate e^{At}

$$\mathrm{A}p_i = \lambda_i p_i$$

$if \quad p_i^T p_i = 1, \quad p_i$ *is a normalized eigenvector.*

Orthogonality of eigenvector

$$if \;\; \mathrm{A} = \mathrm{A}^T$$

$$\begin{cases} \mathrm{A}\, p_i = \lambda_i p_i \\ \mathrm{A}\, p_j = \lambda_j p_j \end{cases} \rightarrow \begin{cases} p_j^T\, \mathrm{A}\, p_i = \lambda_i p_j^T p_i = 0 \\ p_i^T\, \mathrm{A}\, p_j = \lambda_j p_i^T p_j = 0 \end{cases}$$

$$\mathrm{A}\begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\mathrm{P}^{\mathrm{T}}\mathrm{AP} = \Lambda$$

How to Evaluate e^{At}

$$let \quad P\hat{x} = x, \qquad \dot{x} = P\dot{\hat{x}}$$

$$\dot{x} = Ax + Bu$$

$$P\dot{\hat{x}} = AP\hat{x} + Bu$$

$$\dot{\hat{x}} = P^{-1}AP\hat{x} + P^{-1}Bu = \Lambda\hat{x} + P^{-1}Bu$$

$$\hat{x}(t) = e^{\Lambda t}\hat{x}(0) + \int_0^t e^{\Lambda(t-\tau)}P^{-1}Bu(\tau)d\tau$$

$$P\hat{x}(t) = Pe^{\Lambda t}P^{-1}x(0) + \int_0^t Pe^{\Lambda(t-\tau)}P^{-1}Bu(\tau)d\tau$$

$$\therefore x(t) = Pe^{\Lambda t}P^{-1}x(0) + \int_0^t Pe^{\Lambda(t-\tau)}P^{-1}Bu(\tau)d\tau$$

How to Evaluate e^{At}

d) General A (-> Jordan form)

repeated λ_i : multiple eigenvalue

$$\det\left[\lambda_i \mathrm{I} - \mathrm{A}\right] = (\lambda - \lambda_1)(\lambda - \lambda_2)^2 \qquad \mathrm{A}: 3\times 3$$

$$(\lambda_1 \mathrm{I} - \mathrm{A})\, p_1 = 0 \quad \Rightarrow \quad p_1$$

$$(\lambda_2 \mathrm{I} - \mathrm{A})\, p_2 = 0 \quad \Rightarrow \quad p_2$$

$$if \;\; rank(\lambda_2 \mathrm{I} - \mathrm{A}) = 2 \quad then \quad p_3\, ?$$

Find p $\;such\; that$

$$\mathrm{AP} = \mathrm{P}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & 1 \\ & & \lambda_2 \end{bmatrix} \quad \Rightarrow \quad \mathrm{A}\, p_1 = \lambda_1 p_1,$$

$$\mathrm{A}\, p_2 = \lambda_2 p_2$$

$$\mathrm{A}\, p_3 = p_2 + \lambda_2 p_3, \qquad (\mathrm{A} - \lambda_2 \mathrm{I})\, p_3 = p_2$$

Multiple Eigenvalue - Diagonal Form

ex1)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad Q(\lambda) = \det|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & 1 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix}$$

$$= (\lambda - 1)^2 (\lambda - 2)$$

$$eigenvalue \ : \ \lambda_1 = 1, \ \lambda_2 = 1, \ \lambda_3 = 2$$

$$eigenvector : \ Ap_i = \lambda_i p_i \qquad (\lambda_i I - A) p_i = 0$$

$$\lambda_{1,2} = 1, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} p_{1,2} = 0 \quad \Rightarrow \quad choose \ \ p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \ p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 2, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} p_3 = 0 \quad \Rightarrow \quad p_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Multiple Eigenvalue - Diagonal Form

$$\mathrm{P} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathrm{P}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \mathrm{P}^{-1}\mathrm{AP} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \Lambda$$

$$\dot{\mathrm{x}} = \mathrm{A\,x}$$

$$\dot{\hat{\mathrm{x}}} = \mathrm{P}^{-1}\,\mathrm{AP}\hat{\mathrm{x}} = \Lambda\,\hat{\mathrm{x}}$$

$$\hat{\mathrm{x}}(t) = e^{\Lambda t}\,\hat{\mathrm{x}}(0), \qquad \mathrm{P}\,\hat{\mathrm{x}} = \mathrm{x} \qquad \Rightarrow \qquad \mathrm{P}^{-1}\mathrm{x}(t) = e^{\Lambda t}\,\mathrm{P}^{-1}\,\mathrm{x}(0)$$

$$\therefore\ \mathrm{x}(t) = \mathrm{P}\,e^{\Lambda t}\,\mathrm{P}^{-1}\,\mathrm{x}(0) = e^{\mathrm{A}t}\,\mathrm{x}(0)$$

$$e^{\mathrm{A}t} = \mathrm{P}\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}\mathrm{P}^{-1} = \begin{bmatrix} e^t & 0 & e^t \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

Multiple Eigenvalue - Jordan Form

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \qquad Q(\lambda) = \det|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 & -2 \\ 0 & \lambda - 1 & -3 \\ 0 & 0 & \lambda - 2 \end{vmatrix}$$

$$= (\lambda - 1)^2 (\lambda - 2)$$

$$eigenvalue: \ \lambda_{1,2} = 1, \quad \lambda_3 = 2$$

$$eigenvector: \ A\, p_i = \lambda_i\, p_i \qquad (\lambda_i I - A)\, p_i = 0$$

$$\lambda_{1,2} = 1, \quad \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix} p_{1,2} = 0 \quad \Rightarrow \quad p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We can find only 1 eigenvector for $\lambda = 1$

$$\lambda_3 = 2, \quad \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} p_3 = 0 \quad \Rightarrow \quad p_3 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

Multiple Eigenvalue - Jordan Form

If not able to find 3 independent vectors, find p_2 which transforms A as Jordan form.

$$\mathrm{AP} = \mathrm{PJ} = \mathrm{P}\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\mathrm{A}\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \mathrm{A}\,p_1 & \mathrm{A}\,p_2 & \mathrm{A}\,p_3 \end{bmatrix} = \begin{bmatrix} p_1\lambda_1 & p_1 + \lambda_1 p_2 & \lambda_3 p_3 \end{bmatrix}$$

So that, $\mathrm{A}\,p_1 = \lambda_1 p_1 \qquad [\lambda_1 \mathrm{I} - \mathrm{A}]\,p_1 = 0 \quad \Rightarrow \quad p_1$

$\mathrm{A}\,p_1 = p_1 + \lambda_1 p_2 \qquad [\mathrm{A} - \lambda_1 \mathrm{I}]\,p_2 = p_1 \quad \Rightarrow \quad p_2$

$\mathrm{A}\,p_3 = \lambda_3 p_3 \qquad [\mathrm{A} - \lambda_3 \mathrm{I}]\,p_3 = 0 \quad \Rightarrow \quad p_3$

Multiple Eigenvalue - Jordan Form

$$\left[\mathrm{A}-\lambda_1\mathrm{I}\right]p_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} p_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad let, \quad p_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$b+2c=1, \quad 3c=0, \quad c=0 \qquad \rightarrow \qquad b=1, \qquad choose \quad p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathrm{P} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathrm{P}^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad \mathrm{P}^{-1}\mathrm{AP} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Jordan form

Multiple Eigenvalue - Jordan Form

$$if \quad \dot{x} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} x = Ax , \qquad x(t) = e^{At}\, x(0) \;\; ?$$

$$let \quad P\hat{x} = x, \qquad \dot{x} = P\dot{\hat{x}}$$

$$\dot{\hat{x}}(t) = P^{-1} AP\hat{x} = J\,\hat{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \hat{x}$$

$$e^{Jt} = \begin{bmatrix} e^{t} & te^{t} & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}, \qquad \hat{x}(t) = e^{Jt}\, \hat{x}(0)$$

$$x(t) = P\, e^{Jt}\, P^{-1} x(0) = e^{At}\, x(0)$$

Multiple Eigenvalue - Jordan Form

$$\therefore e^{\mathrm{A}t} = \mathrm{P}\, e^{\mathrm{J}t}\, \mathrm{P}^{-1} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & te^t & 5e^{2t} \\ 0 & e^t & 3e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & te^t & \left(-5e^t - 3te^t + 5e^{2t}\right) \\ 0 & e^t & \left(-3e^t + 3e^{2t}\right) \\ 0 & 0 & e^{2t} \end{bmatrix}$$

Summary

1. Diagonal Form

$$A = \begin{bmatrix} \lambda_1 & & \underline{0} \\ & \lambda_2 & \\ \underline{0} & & \lambda_3 \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_3 t} \end{bmatrix}$$

2. Jordan Form

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} \end{bmatrix},$$

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2 e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

Summary

3. General A

$$\dot{x} = A\,x + B\,u \qquad A\,p_i = \lambda_i p_i, \qquad P^{-1}AP = \Lambda$$

$$let \quad x = P\hat{x}$$

$$\dot{\hat{x}} = P^{-1}AP\hat{x} + P^{-1}B\,u = \Lambda\,\hat{x} + P^{-1}B\,u$$

$$\hat{x}(t) = e^{\Lambda t}\,\hat{x}(0) + \int e^{\Lambda(t-\tau)}\,P^{-1}B\,u(\tau)d\tau$$

$$P\hat{x}(t) = x(t) = P\,e^{\Lambda t}\,P^{-1}\,x(0) + \int P\,e^{\Lambda(t-\tau)}\,P^{-1}B\,u(\tau)d\tau$$

$$P\,e^{\Lambda t}\,P^{-1} = e^{At}$$

Laplace Transformation Method

$$\dot{x} = A x + B u\,, \qquad x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

Laplace Transformation

$$s X(s) - x_0 = A X(s) + B u(s)\,, \qquad X(s) = (sI - A)^{-1} x_0 + (sI - A)^{-1} B u(s)$$

$$\therefore e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

$$(sI - A)^{-1} = \frac{1}{s}\left(I - \frac{1}{s}A\right)^{-1}$$

$$= \frac{1}{s}\left(I + \frac{1}{s}A + \frac{1}{s^2}A^2 + \frac{1}{s^3}A^3 \cdots\right)$$

$$= \frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \frac{1}{s^4}A^3 \cdots$$

Laplace Transformation Method

Laplace transformation table

$$\mathcal{L}[t]=\frac{1}{s^2},\qquad \mathcal{L}\left[\frac{1}{(n-1)!}t^{n-1}\right]=\frac{1}{s^n},\qquad \mathcal{L}\left[t^n f(t)\right]=(-1)^n\frac{d^n}{ds^n}F(s)$$

$$\Rightarrow \mathcal{L}^{-1}\left[(s\mathrm{I}-\mathrm{A})^{-1}\right]=\mathcal{L}^{-1}\left[\frac{1}{s}\mathrm{I}+\frac{1}{s^2}\mathrm{A}+\frac{1}{s^3}\mathrm{A}^2+\frac{1}{s^4}\mathrm{A}^3\cdots\right]$$

$$=\mathrm{I}+\mathrm{A}t+\frac{1}{2!}\mathrm{A}^2t^2+\frac{1}{3!}\mathrm{A}^3t^3\cdots$$

$$\therefore e^{\mathrm{A}t}=\mathrm{I}+\mathrm{A}t+\frac{1}{2!}\mathrm{A}^2t^2+\frac{1}{3!}\mathrm{A}^3t^3+\cdots$$

Solution of Linear (Time Invariant) State Equation

Method 1. Diagonalization

Method 2. Laplace Transformation

Method 3. Sylvester's Interpolation Formula