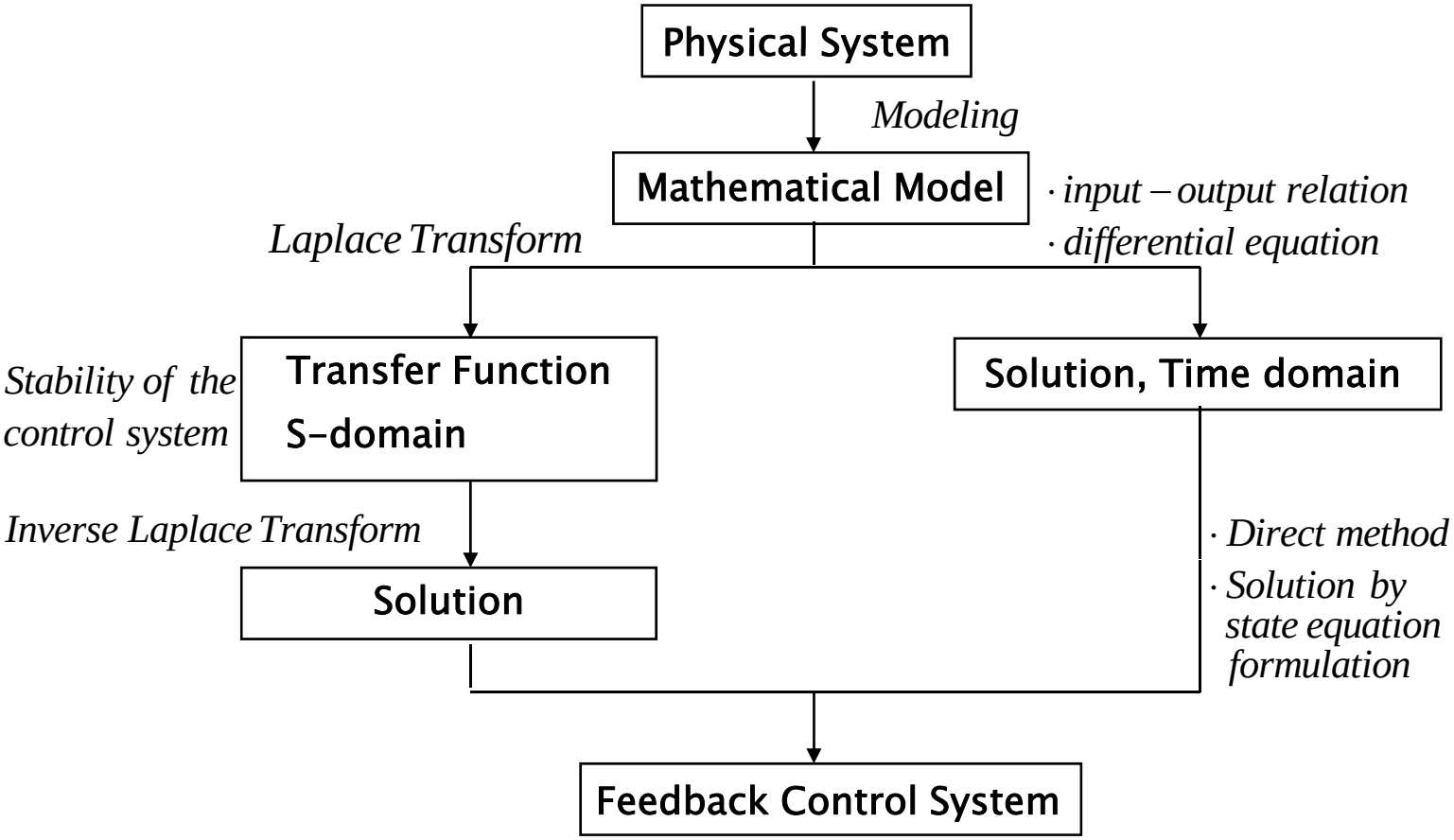
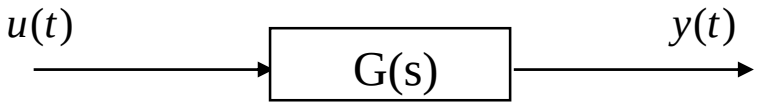
Frequency Response


Physical System
Modeling
Mathematical Model
· input – output relation
· differential equation
Laplace Transform
Stability of the control system
Transfer Function
S-domain
Solution, Time domain
Inverse Laplace Transform
Solution
· Direct method
· Solution by state equation formulation
Feedback Control System

Frequency Response



$$u(t) = P\sin\omega t$$

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+s_1)(s+s_2)\cdots(s+s_n)}$$

$$Y(s) = G(s)\cdot u(s)\,, \qquad u(s) = P\cdot\frac{\omega}{s^2+\omega^2}$$

$$Y(s) = G(s)\cdot\frac{P\omega}{s^2+\omega^2}$$

$$= \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \frac{b_1}{s+s_1} + \frac{b_2}{s+s_2} + \cdots + \frac{b_n}{s+s_n}$$

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + b_1e^{-s_1t} + b_2e^{-s_2t} + \cdots + b_ne^{-s_nt}$$

Frequency Response

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

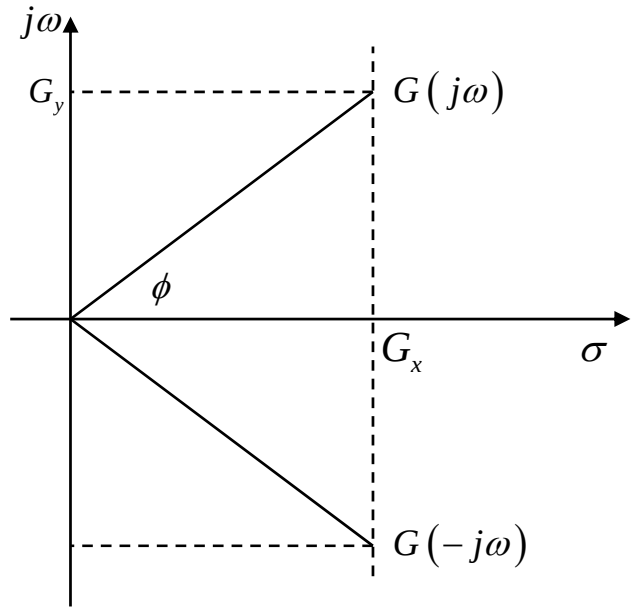
$$a = G(s)\cdot\frac{P\omega}{s^2+\omega^2}(s+j\omega)\Big|_{s=-j\omega} = -\frac{P}{2j}G(-j\omega)$$

$$\bar{a} = G(s)\cdot\frac{P\omega}{s^2+\omega^2}(s-j\omega)\Big|_{s=j\omega} = \frac{P}{2j}G(j\omega)$$

$$G(j\omega) = G_x + jG_y$$

$$= |G(j\omega)|\cos\phi + j|G(j\omega)|\sin\phi$$

$$= |G(j\omega)|(\cos\phi + j\sin\phi) = |G(j\omega)|e^{j\phi}$$



Similarly, $G(-j\omega) = |G(-j\omega)|e^{-j\phi}$

$$\Rightarrow \quad a = -\frac{P}{2j}G(-j\omega) = -\frac{P}{2j}|G(j\omega)|e^{-j\phi}$$

$$\bar{a} = \frac{P}{2j}G(j\omega) = \frac{P}{2j}|G(j\omega)|e^{j\phi}$$

Frequency Response

$$\therefore\ y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

$$= -\frac{P}{2j}\left|G(j\omega)\right|e^{-j\phi}e^{-j\omega t} + \frac{P}{2j}\left|G(j\omega)\right|e^{j\phi}e^{j\omega t}$$

$$= \left|G(j\omega)\right|\frac{P}{2j}\left(e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}\right)$$

$$= \left|G(j\omega)\right|P\sin(\omega t+\phi)$$

Frequency Response of First Order Systems

Consider, $\dfrac{Y(s)}{R(s)} = G(s) = \dfrac{1}{Ts+1}$

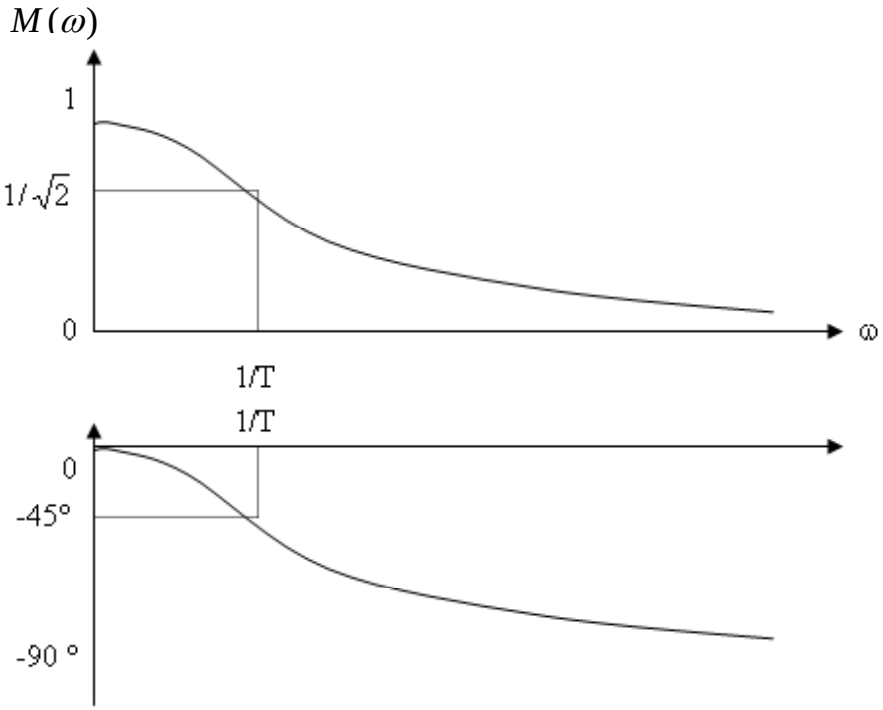
$$G(j\omega) = \frac{1}{Tj\omega + 1}$$

$$M(j\omega) = |G(j\omega)| = \frac{1}{\sqrt{T^2\omega^2 + 1}}$$

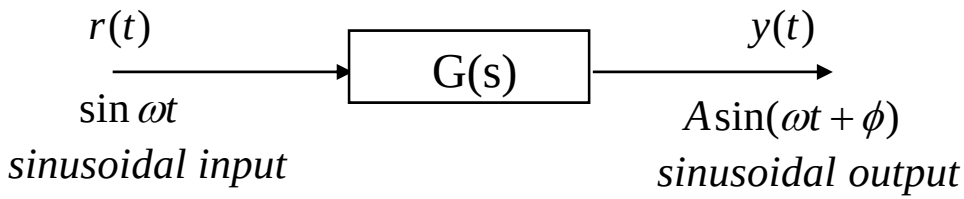
$$\phi(j\omega) = \angle \underline{G(j\omega)} = -\angle(Tj\omega + 1)$$

$$= -\tan^{-1} T\omega$$

$$y(t) = \frac{1}{\sqrt{T^2\omega^2 + 1}} \sin\left(\omega t - \tan^{-1} T\omega\right)$$



Frequency Response of Second Order Systems



$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}, \qquad 0<\zeta<1$$

$$R(s)=\frac{\omega}{s^2+\omega^2}, \qquad r(t)=\sin\omega t$$

$$Y(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\frac{\omega}{s^2+\omega^2}=\frac{as+b}{s^2+2\zeta\omega_n s+\omega_n^2}+\frac{cs+d}{s^2+\omega^2}$$

$$y(t)=ae^{-\zeta\omega_n t}\cos\omega_d t+\frac{b}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t+c\cos\omega_d t+d\sin\omega_d t$$

$$=\underbrace{Ae^{-\zeta\omega_n t}\sin(\omega_d t+\theta)}_{\text{transient response}}+\underbrace{B\sin(\omega t+\phi)}_{\text{Steady-state response}}$$

Steady State Frequency Response

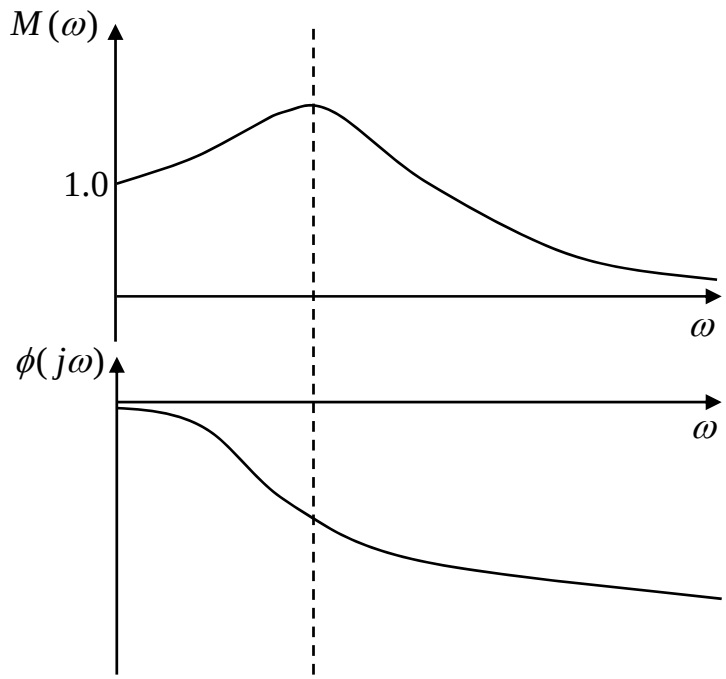
$$y(t) = A(j\omega)\sin(\omega t + \phi)$$

Magnitude ratio

$$M(\omega) = \left|\frac{y(t)}{r(t)}\right| = |G(j\omega)|$$

Phase $\quad \phi(j\omega) = \angle G(j\omega)$

$\Rightarrow$ Frequency response



Steady State Frequency Response

$$\frac{Y(s)}{R(s)} = G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

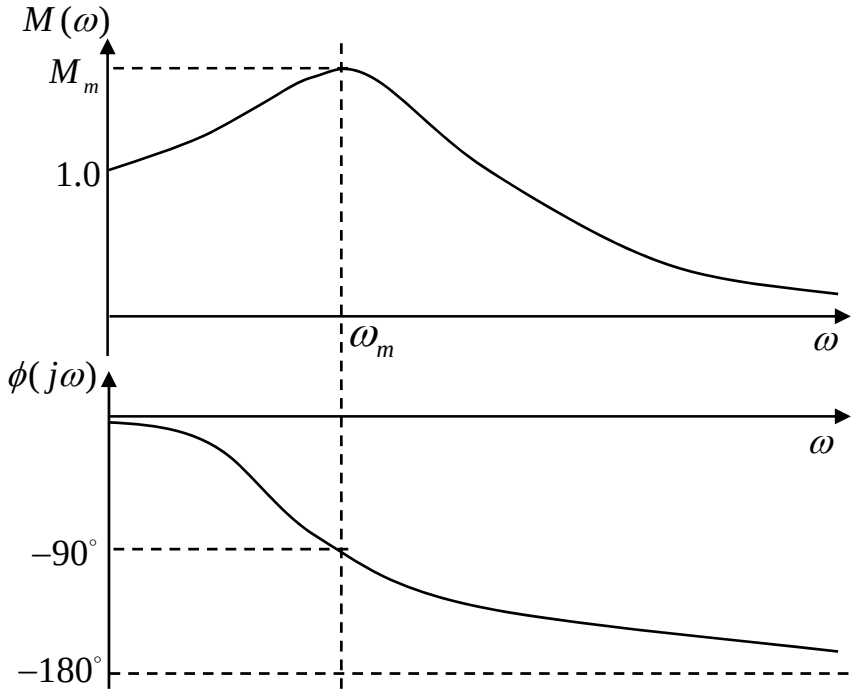
$$\frac{Y(j\omega)}{R(j\omega)} = G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n\omega j + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + 2\zeta\omega_n\omega j}$$

$$M(\omega) = \left|\frac{Y(j\omega)}{R(j\omega)}\right| = \left|\frac{\omega_n^2}{\left(\omega_n^2 - \omega^2\right)^2 + 4\zeta^2\omega_n{}^2\omega^2} \cdot \left(\omega_n^2 - \omega^2 - 2\zeta\omega_n\omega j\right)\right|$$

$$= \frac{\omega_n^2}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + 4\zeta^2\omega_n{}^2\omega^2}} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \cdot \frac{\omega^2}{\omega_n{}^2}}}$$

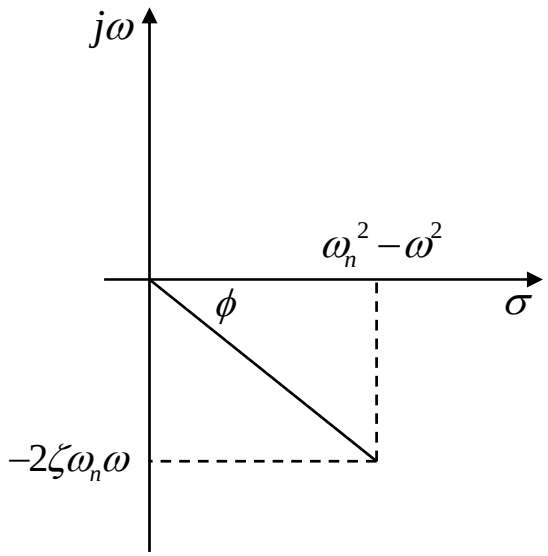
$$\frac{Y(j\omega)}{R(j\omega)} = M(j\omega)\angle G(j\omega)$$

Steady State Frequency Response



$$\phi(j\omega) = -\tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^{\ 2} - \omega^2}$$

$$\lim_{\omega\to\infty}\phi(j\omega) = -\tan^{-1}0 = -180^\circ$$



Steady State Frequency Response

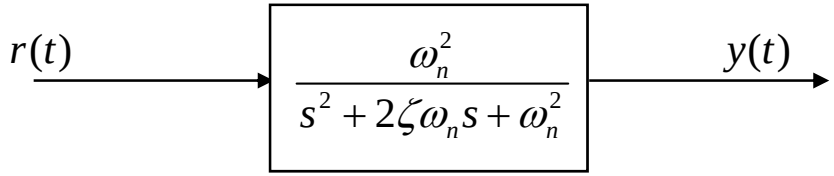
$$M^2(\omega) = \frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \cdot \frac{\omega^2}{\omega_n{}^2}}$$

$$\frac{dM^2(\omega)}{d\omega} = \frac{-\left[2\left(1-\frac{\omega^2}{\omega_n^2}\right)\left(-\frac{2\omega}{\omega_n^2}\right) + 8\zeta^2 \cdot \frac{\omega}{\omega_n{}^2}\right]}{\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \cdot \frac{\omega^2}{\omega_n{}^2}\right]^2} = 0$$

$$\Rightarrow\ 2\left(1-\frac{\omega^2}{\omega_n^2}\right)\left(-\frac{2\omega}{\omega_n^2}\right) + 8\zeta^2 \cdot \frac{\omega}{\omega_n{}^2} = 0\,, \qquad -\left(1-\frac{\omega^2}{\omega_n^2}\right) + 2\zeta^2 = 0$$

$$\therefore\ \omega = \omega_n\sqrt{1-2\zeta^2} = \omega_m\,, \qquad M_m = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

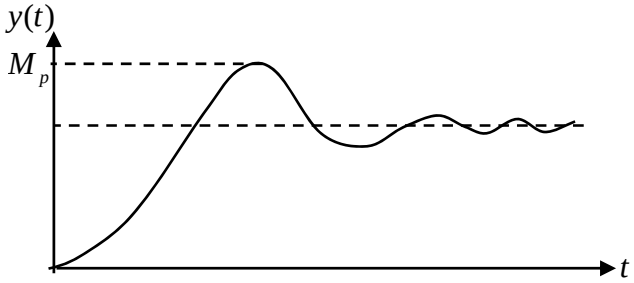
Unit Step Response VS Frequency Response



$$r(t) = u_{-1}(t)$$

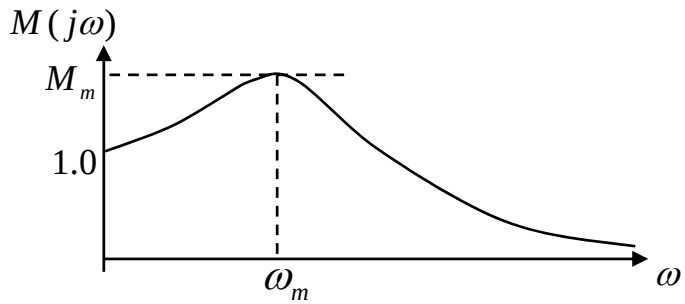
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}\zeta\right)$$

Unit step response



$$M_p = 1 + \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

Frequency response



$$\omega_m = \omega_n \sqrt{1-2\zeta^2} \quad , \qquad M_m = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

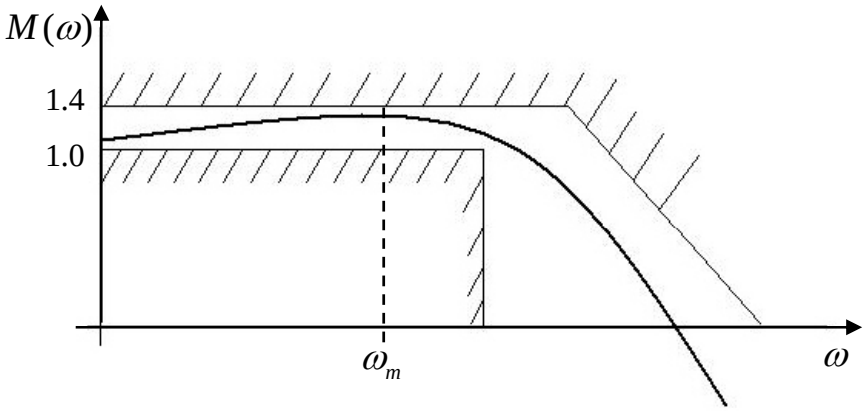
Control System Design in Frequency Response

1. large $\omega_m \rightarrow$ fast time response

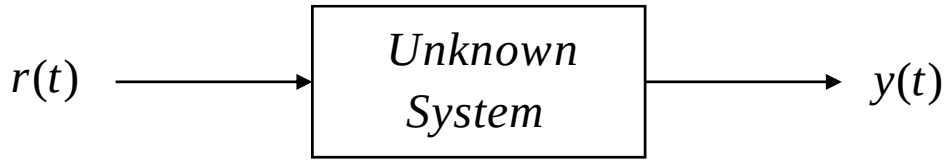
2. M_m, M_p : function of damping ratio ζ
   large $M_m \rightarrow$ large M_p

3. good damping characteristics $1 < M_m < 1.4$

4. minimum effect of any undesirable noise



Exponential Determination of Transfer Function



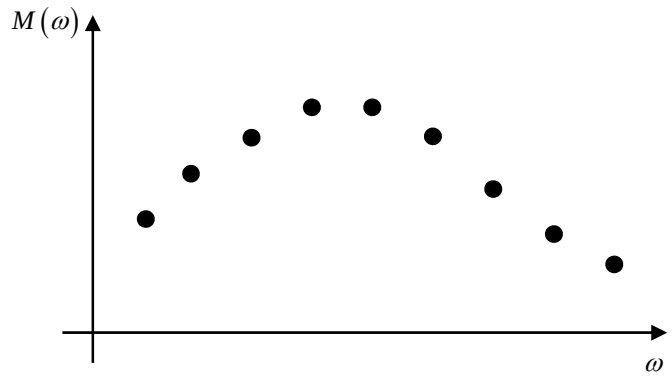
$$r(t) = \sin \omega_i t \quad ; \quad y(t) = M(\omega_i) \sin\left(\omega_i t - \phi\right)$$

some frequency ω_i $\rightarrow$ measure $M\left(\omega_i\right)$

steady state

$$G(s) = G\left(j\omega\right)$$

$$M(\omega) = \left|G\left(j\omega\right)\right|$$



Transfer Functions and Frequency Magnitude Plot

- Bode Plot (Logarithmic Plots)

i) Log Magnitude $L_m G(j\omega) = 20\log|G(j\omega)|\,dB$

ii) dB : decibel, Logarithm of the magnitude

iii) 1 decade : $1Hz \sim 10Hz$ frequency width

$2.5Hz \sim 25Hz$

- Drawing the Bode Plots

The reason of using logarithm – mathematical operation $\times,\ \div \rightarrow +,\ -$

– typical three types : simple straight line

asymtotic

approximations

Transfer Functions and Frequency Magnitude Plot

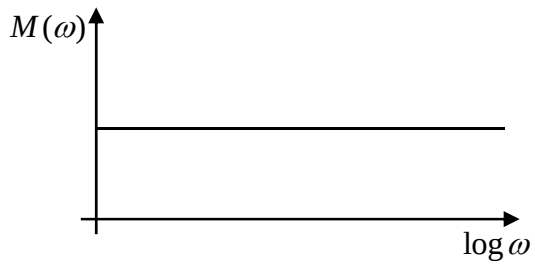
ex) $$G(s)=\frac{K(1+sT_1)(1+sT_2)}{s^2(1+sT_3)(1+as+bs^2)}$$

$$L_m G(j\omega)=L_m K+L_m(1+j\omega T_1)+L_m(1+j\omega T_2)$$

$$-2L_m(j\omega)-L_m(1+j\omega T_3)-L_m\left\{1+aj\omega+b(j\omega)^2\right\}$$

$i)\ L_m\cdot K=20\log K\ dB$

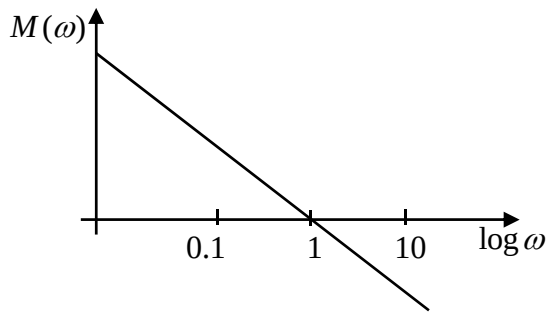
straight line



$ii)\ L_m\left(\frac{1}{j\omega}\right)=20\log\left|\frac{1}{j\omega}\right|=-20\log\omega$

straight line

With negative slope -20dB/decade



Transfer Functions and Frequency Magnitude Plot

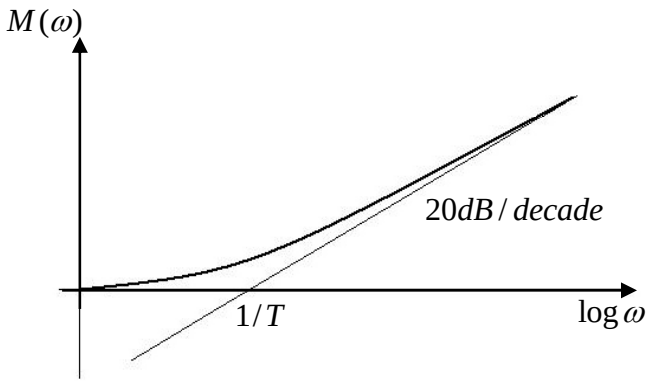
$iii)\ \ (1+j\omega T)$

$$L_m(1+j\omega T) = 20\log|1+j\omega T| = 20\log\sqrt{1+\omega^2T^2}$$

$$\omega T << 1 \quad \rightarrow \quad L_m(1+j\omega T) \cong 20\log 1 = 0dB$$

$$\omega T >> 1 \quad \rightarrow \quad L_m(1+j\omega T) \cong 20\log \omega T$$

$$= 20\log\omega + 20\log T$$



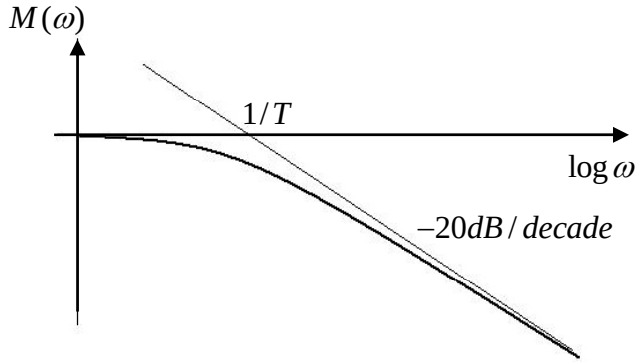
$iv)\ \ \dfrac{1}{1+j\omega T}$

$$L_m(1+j\omega T)^{-1} = -20\log|1+j\omega T| = -20\log\sqrt{1+\omega^2T^2}$$

$$\omega T << 1 \quad \rightarrow \quad L_m(1+j\omega T)^{-1} \cong -20\log 1 = 0dB$$

$$\omega T >> 1 \quad \rightarrow \quad L_m(1+j\omega T)^{-1} \cong -20\log \omega T$$

$$= -20\log\omega - 20\log T$$



Transfer Functions and Frequency Magnitude Plot

$$v)\quad \frac{1}{\left(1+\frac{2\zeta}{\omega_n}s+\frac{1}{\omega_n^{\,2}}s^2\right)} \quad\Rightarrow\quad \frac{1}{\left[1+\frac{2\zeta}{\omega_n}j\omega+\frac{1}{\omega_n^{\,2}}(j\omega)^2\right]}$$

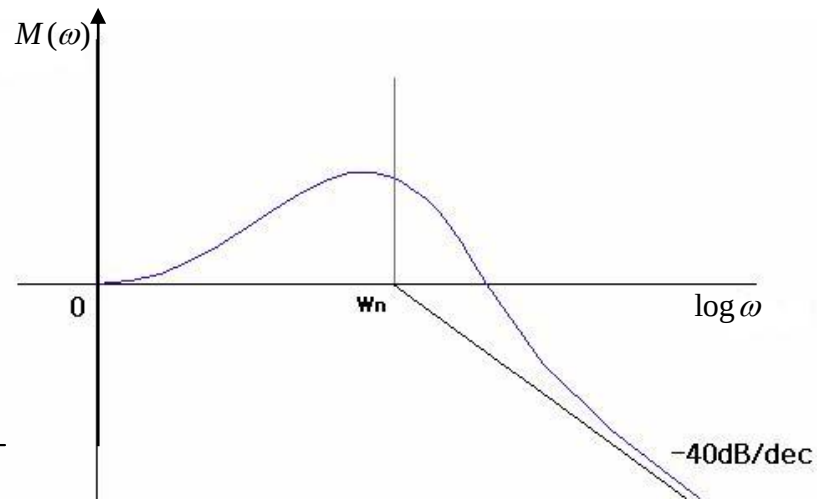
$$L_m\frac{1}{\left[1+\frac{2\zeta}{\omega_n}j\omega+\frac{1}{\omega_n^{\,2}}(j\omega)^2\right]}=20\log\frac{1}{\left[1+\frac{2\zeta}{\omega_n}j\omega+\frac{1}{\omega_n^{\,2}}(j\omega)^2\right]}=-20\log\left[\left(1-\frac{\omega^2}{\omega_n^{\,2}}\right)+\left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]^{1/2}$$

$$\frac{\omega}{\omega_n}<<1$$

$$\rightarrow\quad L_m\left[G(j\omega)\right]\cong-20\log 1=0dB$$

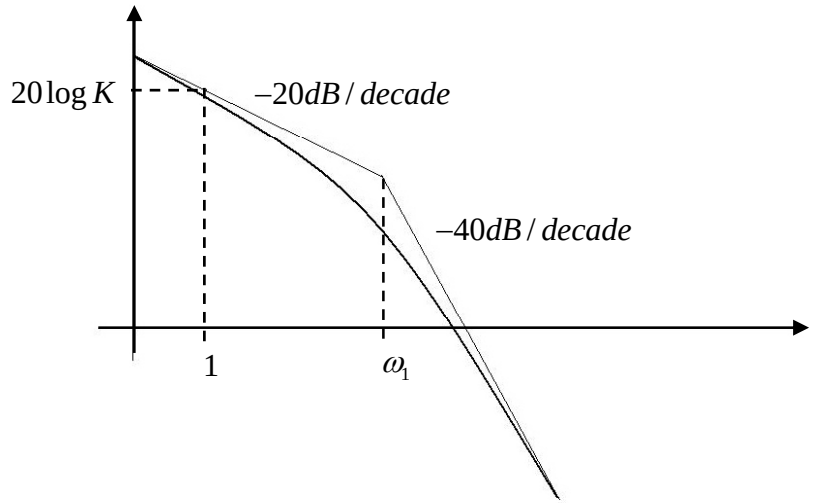
$$\frac{\omega}{\omega_n}>>1$$

$$\rightarrow\quad L_m\left[G(j\omega)\right]\cong-20\log\frac{\omega^2}{\omega_n^{\,2}}=-40\log\frac{\omega}{\omega_n}$$

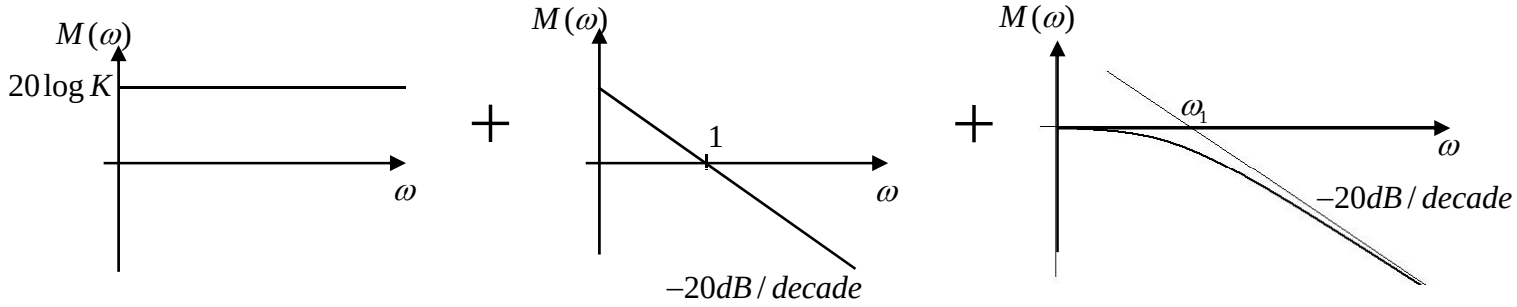


Transfer Functions and Frequency Magnitude Plot

ex)



$=$



$$= \frac{K}{(j\omega)\left(1+\frac{j\omega}{\omega_1}\right)} \Rightarrow \frac{K}{s\left(1+\frac{1}{\omega_1}s\right)}$$

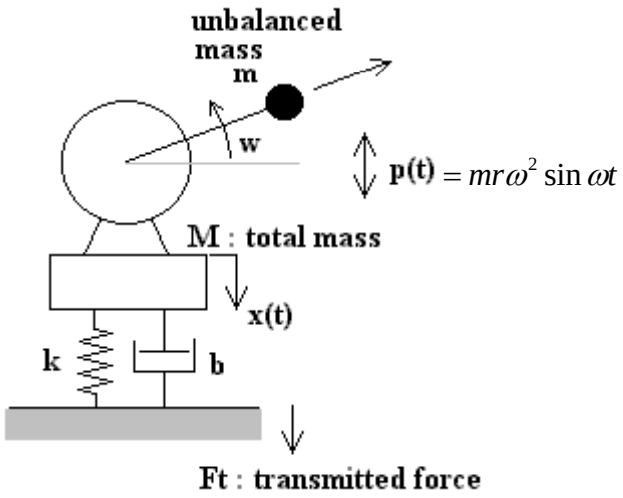
Vibration Isolation in Rotating Systems

Vibration due to rotating unbalance

$$(1) \quad M\ddot{x} + b\dot{x} + kx = p(t)$$

$$= mr\omega^2 \sin \omega t$$

$$\frac{X(s)}{P(s)} = \frac{1}{Ms^2 + bs + k} = G(s)$$



Vibration Isolation in Rotating Systems

(2) **Frequency response**

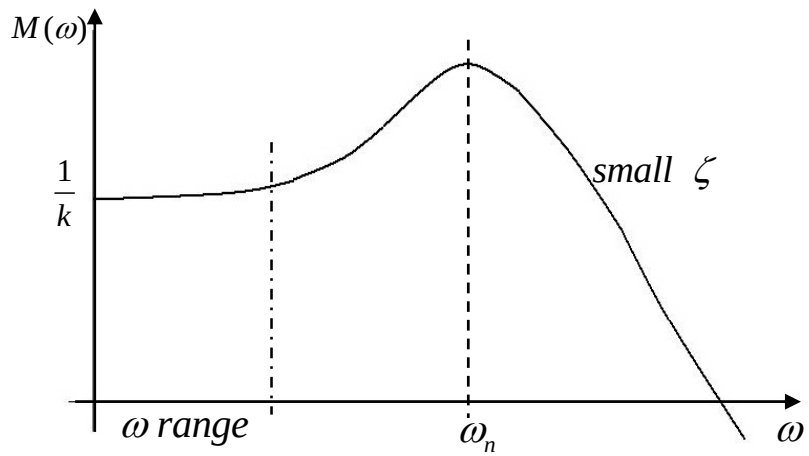
$$\frac{x(j\omega)}{p(j\omega)} = \frac{1}{-M\omega^2 + bj\omega + k} = G(j\omega)$$

$$x(t) = X\sin(\omega t + \phi) = \left|G(j\omega)\right|\sin(\omega t + \phi)\cdot mr\omega^2$$

$$\phi = -\tan^{-1}\frac{b\omega}{k - M\omega^2} = -\tan^{-1}\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n{}^2}$$

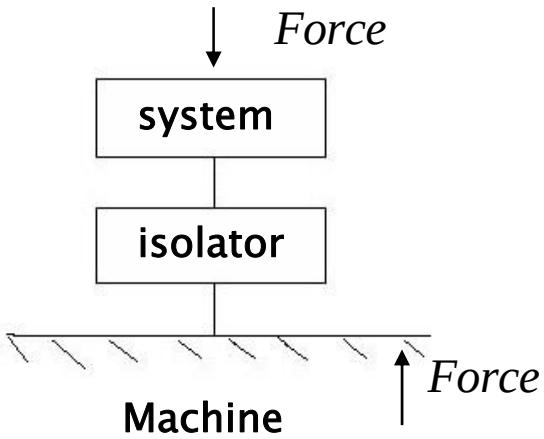
$$\left|G(j\omega)\right| = \frac{1}{\sqrt{\left(k - M\omega^2\right)^2 + b^2\omega^2}}$$

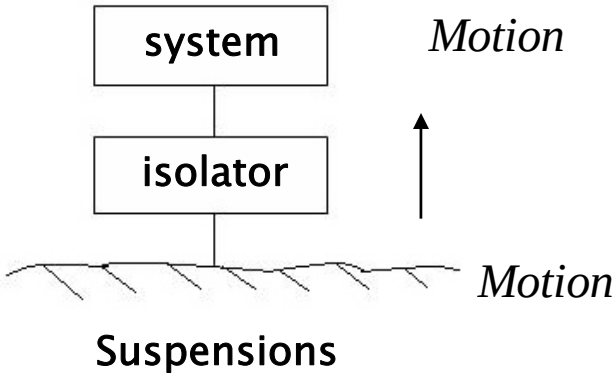
$$= \frac{1/k}{\sqrt{\left(1-\omega^2/\omega_n{}^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}}$$



Vibration Isolation in Rotating Systems

Vibration isolators





Reduce the magnitude of force transmitted from a machine to its foundation

Reduce the magnitude of motion transmitted from a vibratory foundation to a system

Isolator i) – load supporting elements, spring.

– energy-dissipating elements, damper.

ii) – Synthetic rubber : both load supporting and energy dissipating

Transmissibility

Transmissibility :

A measure of the reduction of transmitted force or motion afforded by an isolator.

$$TR(machine) = \frac{the\ \ force\ amplitude\ transmitted\ to\ the\ \ foundation}{the\ amplitude\ of\ \ the\ exciting\ \ force}$$

$$TR(suspensions) = \frac{the\ vibration\ amplitude\ of\ \ the\ system}{the\ vibration\ amplitude\ of\ \ the\ \ foundation}$$

Transmissibility for Force Excitation

$$p(t) = mr\omega^2 \sin \omega t = F_0 \sin \omega t$$

$$f(t) = b\dot{x} + kx = F_t \sin\left(\omega t + \phi\right)$$

$$X(s) = \frac{1}{Ms^2 + bs + k} P(s)$$

$$F_t(s) = (bs + k) X(s) = \frac{bs + k}{Ms^2 + bs + k} \cdot P(s)$$

Frequency response

$$\frac{F_t(j\omega)}{p(j\omega)} = \frac{bj\omega + k}{-M\omega^2 + bj\omega + k} = \frac{\left(b/M\right) j\omega + \left(k/M\right)}{-\omega^2 + \left(b/M\right) j\omega + k/M}$$

Transmissibility for Force Excitation

$$k/M = \omega_n^{\,2}, \qquad b/M = 2\zeta\omega_n \qquad \Rightarrow \frac{F_t(j\omega)}{p(j\omega)} = \frac{1 + j\left(2\zeta\omega/\omega_n\right)}{1 - \omega^2/\omega_n^{\,2} + j\left(2\zeta\omega/\omega_n\right)}$$

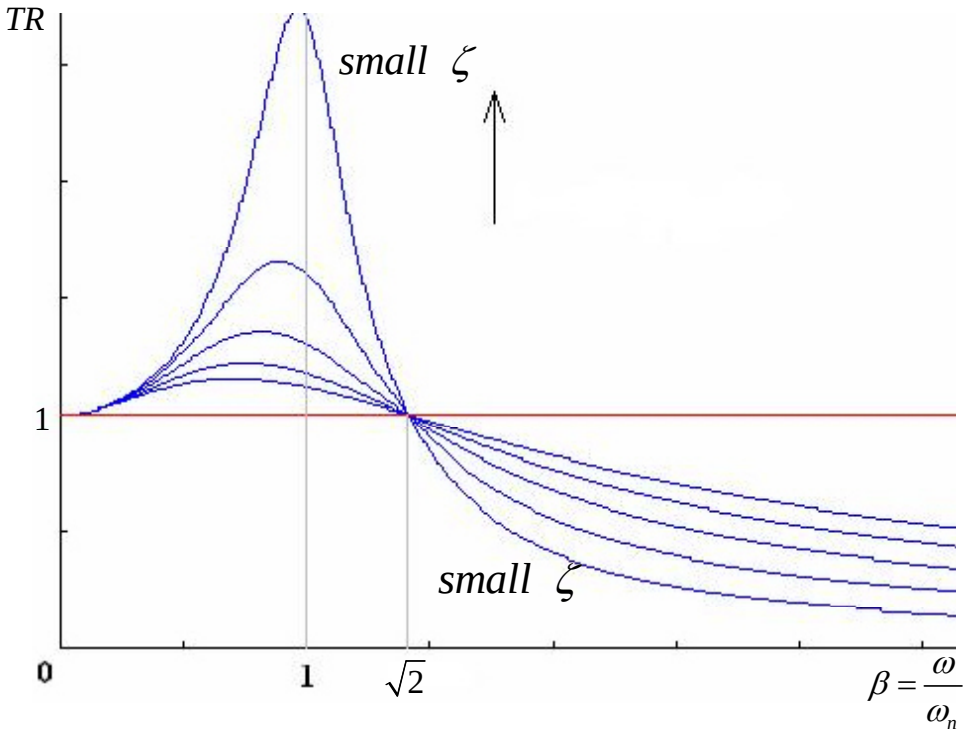
$$TR = \frac{F_t}{F_0} = \left|\frac{F_t(j\omega)}{p(j\omega)}\right|$$

$$= \frac{\sqrt{1 + \left(2\zeta\omega/\omega_n\right)^2}}{\sqrt{\left(1 - \omega^2/\omega_n^{\,2}\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}}$$

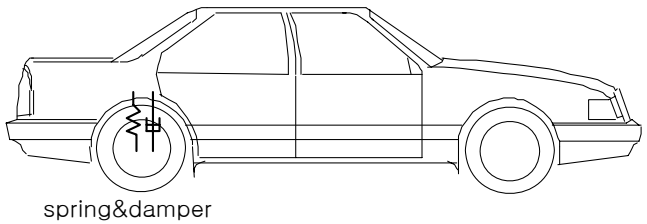
$$= \frac{\sqrt{1 + \left(2\zeta\beta\right)^2}}{\sqrt{\left(1 - \beta^2\right)^2 + \left(2\zeta\beta\right)^2}}$$

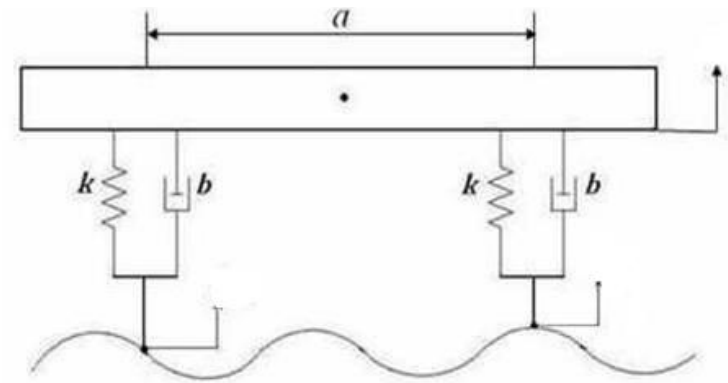
Note, $\beta = \sqrt{2}$, $TR = 1$

for any ζ

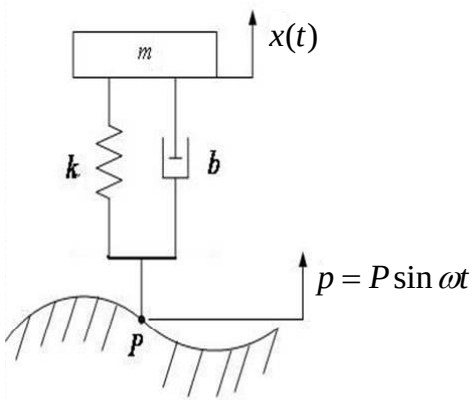


Automobile Suspension Systems





Simplified model



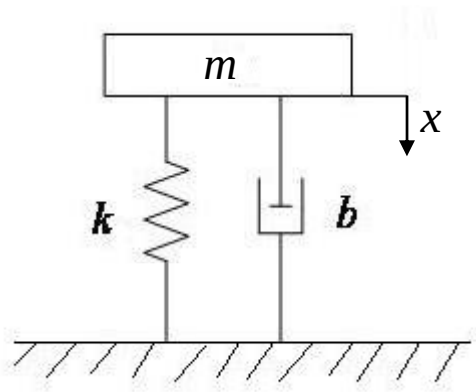
$$m\ddot{x} + b(\dot{x} - p) + k(x - p) = 0$$

$$\frac{X(s)}{P(s)} = \frac{bs + k}{ms^2 + bs + k}, \qquad \frac{x(j\omega)}{p(j\omega)} = \frac{bj\omega + k}{-m\omega^2 + bj\omega + k}$$

$$TR = \left|\frac{x(j\omega)}{p(j\omega)}\right| = \frac{\sqrt{b^2\omega^2 + k^2}}{\sqrt{\left(k - m\omega^2\right)^2 + b^2\omega^2}}$$

$$= \frac{\sqrt{1 + \left(2\zeta\beta\right)^2}}{\sqrt{\left(1 - \beta^2\right)^2 + \left(2\zeta\beta\right)^2}}$$

Dynamic Vibration Absorbers



$$P = mr\omega^2$$

$$TR = \frac{\left(mr\omega^2\right)\sqrt{b^2\omega^2 + k^2}}{\sqrt{\left(k - m\omega^2\right)^2 + b^2\omega^2}}$$

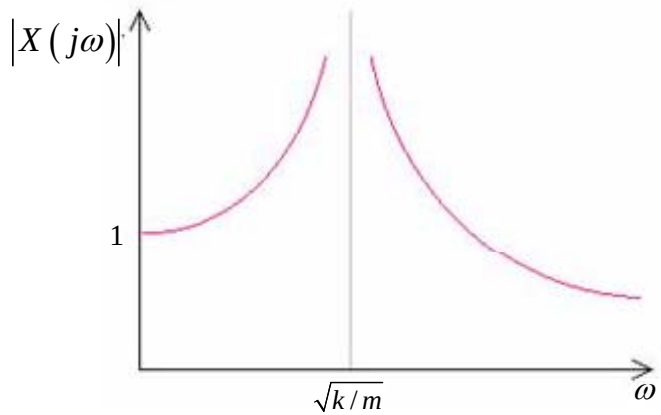
If b is small and $\omega = \omega_n$

Then resonance – excessive vibration

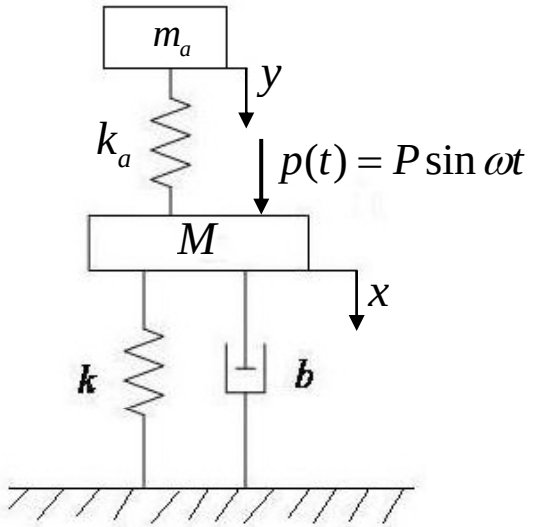
– extremely large force transmitted

ω is very close to ω_n , critical speed,

=> A dynamic vibration absorber



Dynamic Vibration Absorbers



$$M\ddot{x} = -kx - b\dot{x} - k_a(x-y) + p(t)$$

$$m_a\ddot{y} = -k_a(y-x)$$

Laplace Transform

$$\left(Ms^2 + bs + k + k_a\right)X(s) - k_aY(s) = P(s)$$

$$\left(m_as^2 + k_a\right)Y(s) - k_aX(s) = 0$$

$$\Rightarrow \quad \frac{X(s)}{P(s)} = \frac{m_as^2 + k_a}{\left(Ms^2 + bs + k + k_a\right)\left(m_as^2 + k_a\right) - k_a^{\,2}}$$

Frequency response

$$\frac{X(j\omega)}{P(j\omega)} = \frac{-m_a\omega^2 + k_a}{\left(-M\omega^2 + k + k_a\right)\left(-m_a\omega^2 + k_a\right) - k_a^{\,2}} \qquad \textit{for small } b$$

The transmitted force

$$f(t) = kx + b\dot{x} \cong kx$$

Dynamic Vibration Absorbers

$$|X(j\omega)| = \left| \frac{k_a - m_a\omega^2}{\left(-M\omega^2 + k + k_a\right)\left(k_a + m_a\omega^2\right) - k_a{}^2} \right| |P(j\omega)|$$

$$= \left| \frac{mr\omega^2\left(k_a - m_a\omega^2\right)}{\left(-M\omega^2 + k + k_a\right)\left(k_a + m_a\omega^2\right) - k_a{}^2} \right|$$

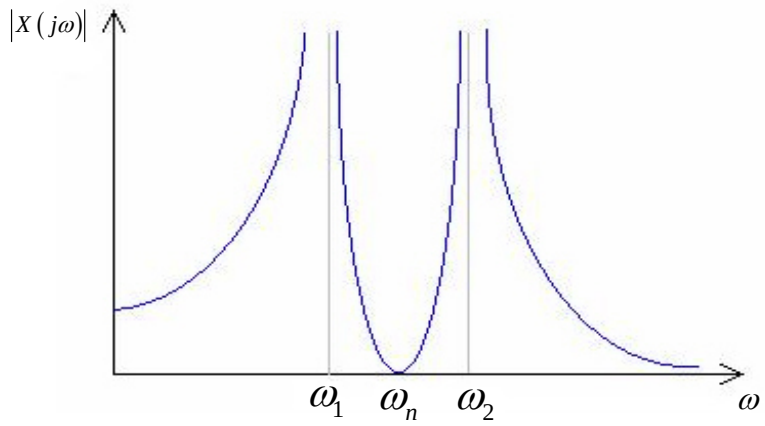
If we choose k_a, m_a so that,

$$k_a - m_a\omega^2 = 0$$

then, $|X(j\omega)| = 0$, $|f(t)| = 0$ at this frequency.

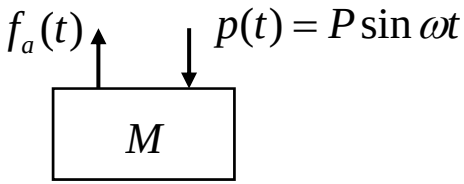
If $\omega_{exc} = \omega_n$

we use dynamic vibration absorbers.



Dynamic Vibration Absorbers

Physically,



$$f_a(t) = -k_a\left(x - y\right) = k_a y = p(t), \qquad \left(at \;\; \omega_{exc} = \omega_n, \quad x \cong 0\right)$$

Spring force $k_a y$ cancels $p(t)$.

$$\frac{Y(j\omega)}{P(j\omega)} = \frac{k_a}{\left(-M\omega^2 + k + k_a\right)\left(-m_a\omega^2 + k_a\right) - k_a{}^2}$$

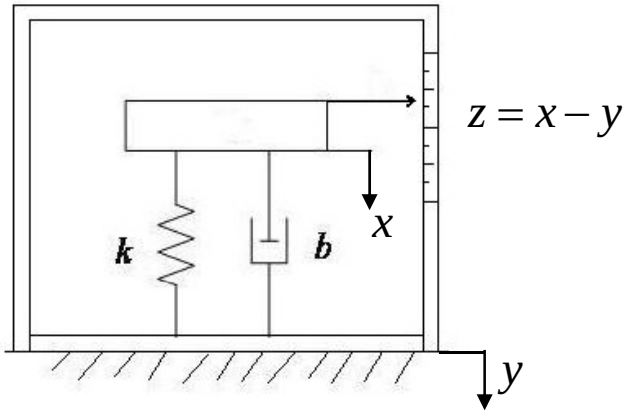
If we choose m_a, k_a so that, $k_a - m_a\omega^2 = 0$

then $\dfrac{Y(j\omega)}{P(j\omega)} = -\dfrac{1}{k_a}$ magnitude ratio $\left|\dfrac{1}{k_a}\right|$

$$p(t) = \sin\omega t$$

$$y(t) = \frac{1}{k_a} P\sin\left(\omega t - 180^\circ\right) = -\frac{P}{k_a}\sin\omega t \qquad \left(phase \;\; -180^\circ\right)$$

Seismograph



A device used to measure ground displacement during earthquakes

Equation of motion : $m\ddot{x} = k(y - x) + b(\dot{y} - \dot{x})$

measurement : $z = x - y$

$$\Rightarrow m(\ddot{y} + \ddot{z}) + b\dot{z} + kz = 0$$

$$m\ddot{z} + b\dot{z} + kz = -m\ddot{y}$$

$$\Rightarrow \textit{Transfer Function} : \frac{Z(s)}{Y(s)} = \frac{-ms^2}{ms^2 + bs + k} = \frac{-s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\left(\frac{k}{m} = \omega_n^2, \quad \frac{b}{m} = 2\zeta\omega_n \right)$$

Seismograph

$$\frac{Z(j\omega)}{Y(j\omega)} = \frac{\omega^2}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n{}^2} = \frac{\beta^2}{1-\beta^2 + 2\zeta\beta j} \qquad (\beta = \omega/\omega_n)$$

We hope, $z(t) \cong y(t)$

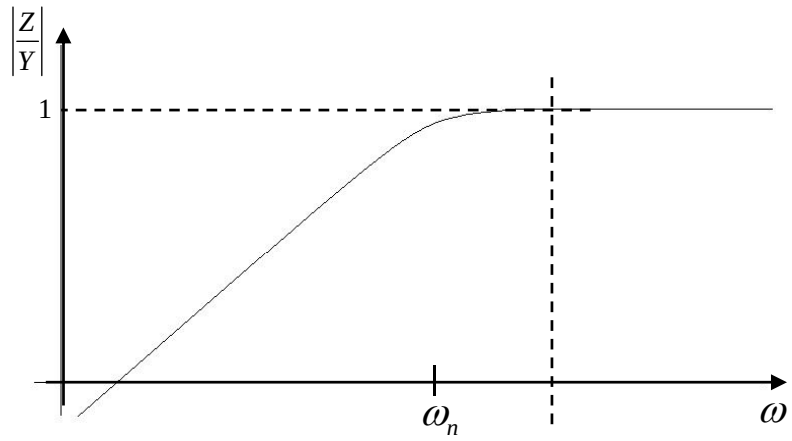
If, $\beta >> 1$, $\dfrac{Z(j\omega)}{Y(j\omega)} = -1$

$z(t) \cong y(t)$ *for* $\omega >> \omega_n$

Choose ω_n as small as possible.

(large mass, soft spring)

$$\left(\omega_n = \sqrt{\frac{k}{m}}\right)$$



Accelerometer

The system configuration is basically the same as the seismograph, but choice of undamped natural frequency is different.

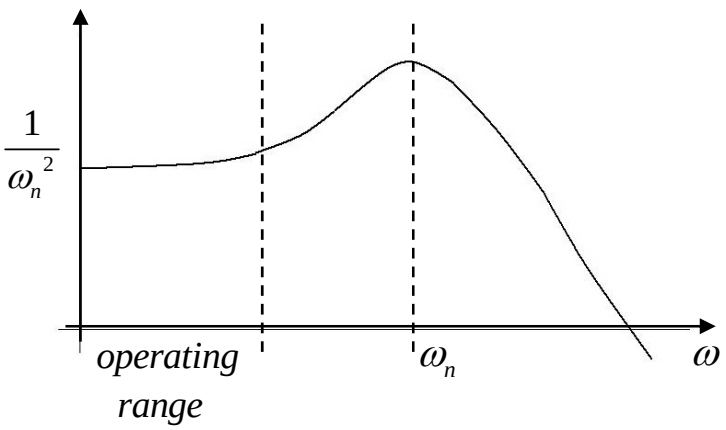
Equation of motion : the same as the seismograph, $m\ddot{z} + b\dot{z} + kz = -m\ddot{y}$

$$\Rightarrow \textit{Transfer Function} : \frac{Z(s)}{s^2 Y(s)} = \frac{-m}{ms^2 + bs + k} = \frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

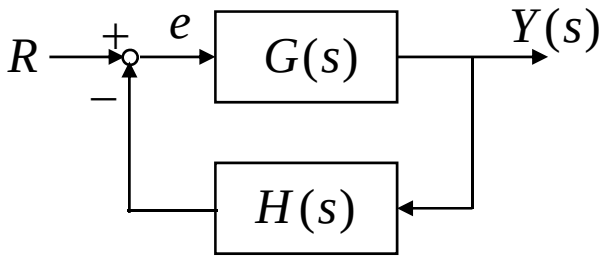
Frequency response

$$\omega_n >> \omega$$

$$\frac{Z(s)}{s^2 Y(s)} \cong -\frac{1}{\omega_n}$$



Performance Characteristics



$$Y = G(s)e$$

$$R - H(s)Y = e$$

$$\Rightarrow Y = G(s)\left(R - H(s)Y\right)$$

$$Y\left(1 + G(s)H(s)\right) = G(s)R$$

$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = G_0(s)$$

Step Input Response

Consider, 2nd order system

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2} = G_0(s)$$

Now, consider "time-response"-"pole" of the transfer function relations

$$Y(s) = \frac{1}{s}\frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (\textit{step input response})$$

$$= \frac{K/\omega_n{}^2}{s} + \frac{-\left(K/\omega_n{}^2\right)s - 2\zeta K/\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

pole $\quad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$$s = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

Step Input Response

$i)\quad 0<\zeta<1$

$$s=-\zeta\omega_n\pm\sqrt{1-\zeta^2}\,\omega_n j=-\zeta\omega_n\pm\omega_d j$$

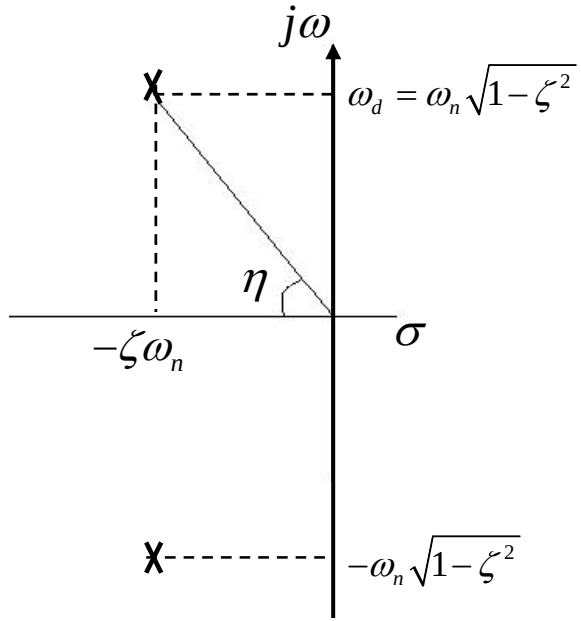
$$Y(s)=\frac{K/\omega_n^{\ 2}}{s}+\frac{-\left(K/\omega_n^{\ 2}\right)s-2\zeta K/\omega_n}{\left(s+\zeta\omega_n\right)^2+\left(1-\zeta^2\right)\omega_n^{\ 2}}$$

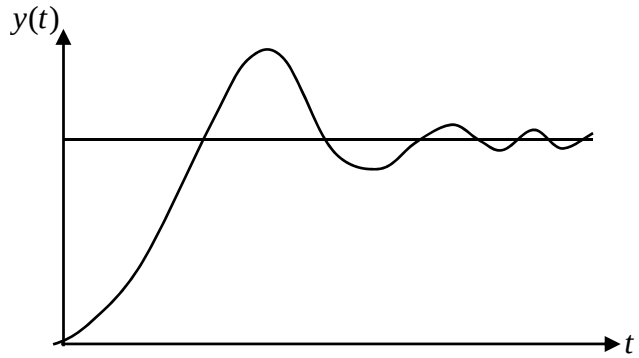
$$y(t)=\frac{K}{\omega_n^{\ 2}}+C_1e^{-\zeta\omega_n t}\cos\omega_d t+C_2e^{-\zeta\omega_n t}\sin\omega_d t$$

$$=\frac{K}{\omega_n^{\ 2}}+Ae^{-\zeta\omega_n t}\sin\left(\omega_d t+\phi\right),\qquad -\zeta\omega_n<0$$

$$\left(C_1=-\frac{K}{\omega_n^{\ 2}},\quad C_2=\frac{2\zeta K}{\omega_n^{\ 2}}\frac{1}{\sqrt{1-\zeta^2}}\right)$$

$$\cos\eta=\zeta$$





Step Input Response

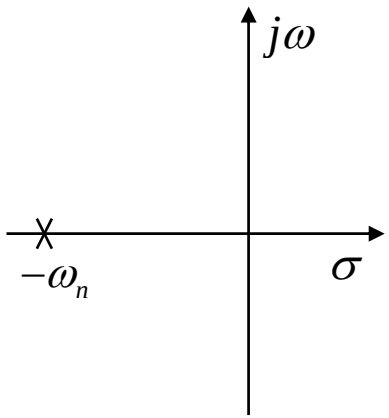
$ii) \quad \zeta = 1$

$$s = -\zeta\omega_n \pm 0j = -\omega_n$$

$$Y(s) = \frac{1}{s}\frac{K}{(s+\omega_n)^2}$$

$$= \frac{a}{s} + \frac{b}{s+\omega_n} + \frac{c}{(s+\omega_n)^2}$$

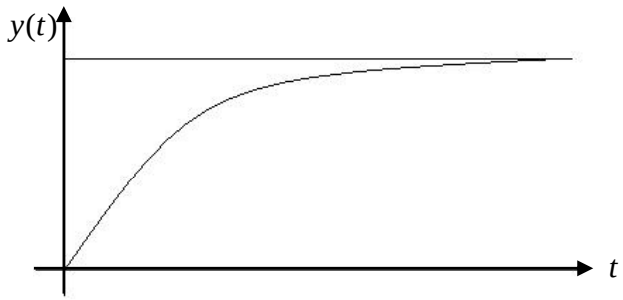
$$\left(a = \frac{K}{\omega_n^{\,2}}, \quad b = -\frac{K}{\omega_n^{\,2}}, \quad c = -\frac{K}{\omega_n} \right)$$



$$y(t) = a + b \cdot e^{-\omega_n t} + c \cdot te^{-\omega_n t}$$

$$\dot{y}(t) = -\omega_n b \cdot e^{-\omega_n t} + c \cdot e^{-\omega_n t} - \omega_n c \cdot te^{-\omega_n t}$$

$$= \left(\cancel{+\frac{1}{\omega_n} - \frac{1}{\omega_n}} \right) e^{-\omega_n t} + Kte^{-\omega_n t} > 0$$



Step Input Response

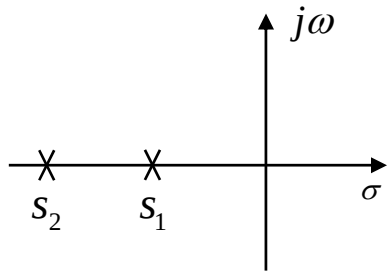
$iii)\quad \zeta > 1$

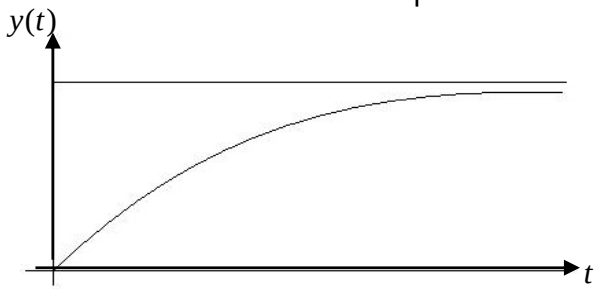
$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

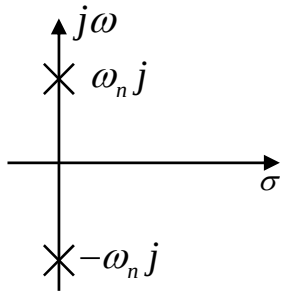
$$Y(s) = \frac{1}{s}\frac{K}{(s-s_1)(s-s_2)} = \frac{a}{s} + \frac{b}{s-s_1} + \frac{c}{s-s_2}$$

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \qquad s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$y(t) = a + b\cdot e^{s_1 t} + c\cdot e^{s_2 t}, \qquad s_2 < s_1 < 0$$





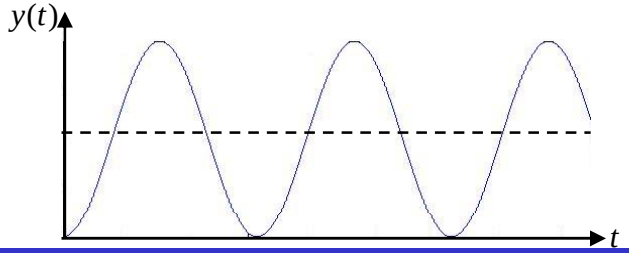


$iv)\quad \zeta = 0 \qquad s = \pm\omega_n j$

$$Y(s) = \frac{1}{s}\frac{K}{(s^2 + \omega_n{}^2)} = \frac{a}{s} + \frac{bs + c}{(s^2 + \omega_n{}^2)}$$

$$y(t) = a + \frac{c}{\omega_n}\cdot\sin\omega_n t + b\cdot\cos\omega_n t$$

$$= a + \sqrt{\left(c/\omega_n\right)^2 + b^2}\,\sin\left(\omega_n t + \phi\right)$$



Step Input Response

$v)\quad -1 < \zeta < 0$

$$s = -\zeta\omega_n \pm \sqrt{1-\zeta^2}\,\omega_n j = -\zeta\omega_n \pm \omega_d j$$

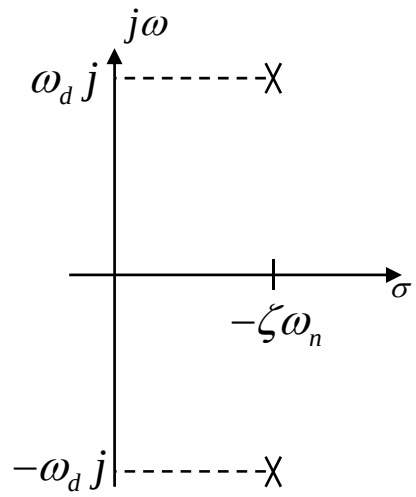
$$Y(s) = \frac{K/\omega_n^{\ 2}}{s} + \frac{-\left(K/\omega_n^{\ 2}\right)s - 2\zeta K/\omega_n}{\left(s+\zeta\omega_n\right)^2 + \left(1-\zeta^2\right)\omega_n^{\ 2}}$$

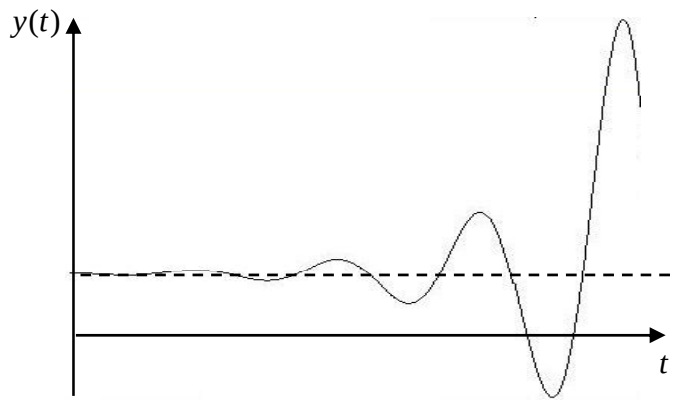
$$y(t) = \frac{K}{\omega_n^{\ 2}} + C_1 e^{-\zeta\omega_n t}\cos\omega_d t + C_2 e^{-\zeta\omega_n t}\sin\omega_d t$$

$$= \frac{K}{\omega_n^{\ 2}} + A e^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right)$$

$$-\zeta\omega_n > 0$$

$$\left(C_1 = -\frac{K}{\omega_n^{\ 2}}, \quad C_2 = \frac{2\zeta K}{\omega_n^{\ 2}}\frac{1}{\sqrt{1-\zeta^2}} \right)$$





Step Input Response

$vi)\quad \zeta < -1$

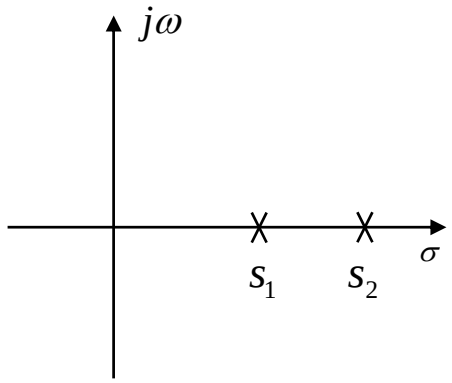
$$s = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n j = -\zeta\omega_n \pm \omega_d j$$

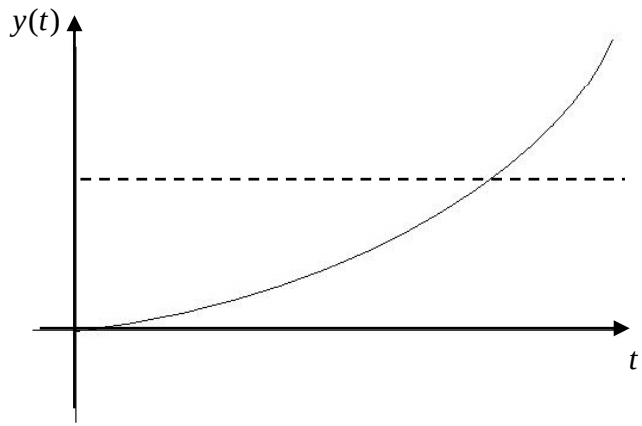
$$Y(s) = \frac{1}{s}\frac{K}{(s-s_1)(s-s_2)} = \frac{a}{s} + \frac{b}{s-s_1} + \frac{c}{s-s_2}$$

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \qquad s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$y(t) = a + be^{s_1 t} + ce^{s_2 t}$$

$$s_2 > s_1 > 0$$





Step Input Response

Poles : $s = \sigma \pm j\omega_d = -\zeta\omega_n \pm \sqrt{1-\zeta^2}\,\omega_n t$

$y(t) = Ce^{\sigma t}$