Chapter 11 Columns

11.1 Introduction

- Buckling Failures of Columns

1. Failures investigated so far in this course: failures caused by excessive strength or stiffness → strength and stiffness of members are important

2. Buckling failure of columns: long, slender members loaded axially in compression deflects → buckling may collapse eventually (instead of failures by direct compression of the material)

3. Example: compressing a plastic slender ruler, stepping on an aluminum can, think plate of a bridge under compression, etc.

4. Buckling is one of the major causes of failures in structures → should be considered in design process

11.2 Buckling and Stability

- Idealized Structure to Investigate Buckling and Stability ("Buckling Model")

1. Rigid bars $AB$ and $BC$ joined by a pin connection ~ rotational spring with stiffness $\beta_R$ is added at the pin → an idealized structure analogous to the column structure shown above (elasticity is concentrated vs distributed)

2. Hooke’s law for the rotational spring

$$M = \beta_R \theta$$

3. If the bars are perfectly aligned, the axial load $P$ acts through the longitudinal line → spring is uns________, and the bars are in direct compression

4. Suppose point $B$ moves a small distance laterally (by external disturbances, forces or imperfect geometry) → rigid bars rotate through small angles $\theta$
5. Axial forces and “restoring moment” $M_B$ developed in the spring show opposite effects $\rightarrow$ Axial force tends to __________ the lateral displacement, and $M_B$ tends to __________ it.

6. What happens after the disturbing force is removed?

1) Small $P \rightarrow \theta$ keeps __________ $\rightarrow$ returns to the original position: **Stable**

2) Large $P \rightarrow \theta$ keeps __________ $\rightarrow$ fails by lateral buckling: **Unstable**

“How large $P$ should be to make the system unstable?” $\rightarrow$ Critical load

○ Critical Load $P_{cr}$

1. Moment in the spring: $M_B = 2\beta_R\theta$

2. Under small angle $\theta$, the lateral displacement at point $B$: $\theta L/2$

3. Moment equilibrium for bar $BC$

$$M_B - P \cdot \left(\frac{\theta L}{2}\right) = 0$$

$$\left(2\beta_R - \frac{PL}{2}\right) \cdot \theta = 0$$

4. First solution of equilibrium equation: $\theta = 0$ $\rightarrow$ trivial solution representing the equilibrium at perfectly straight alignment regardless of the magnitude of the load

5. Second solution of equilibrium equation:

$$P_{cr} = \frac{4\beta_R}{L}$$

$\rightarrow$ The structure is in equilibrium regardless of the magnitude of the angle $\theta$

$\rightarrow$ Critical load is the **only load** for which the structure will be in equilibrium in the disturbed position, i.e. $\theta \neq 0$

6. What if $P \neq P_{cr}$, i.e. can’t sustain the equilibrium?

1) If $P < P_{cr}$, restoring moment is dominant $\rightarrow$ structure is __________

2) If $P > P_{cr}$, effect of the axial force is dominant $\rightarrow$ structure is __________

7. From the critical load derived above, it is seen that one can increase the stability by __________ing stiffness or __________ing length
Summary

1. $\theta = 0$: no disturbance $\rightarrow$ equilibrium for any $P$

2. Disturbance introduced to cause $\theta \neq 0$ and the source of the disturbance removed
   
   1) $P < P_{cr}$: goes back to the original equilibrium (stable equilibrium)
   
   2) $P = P_{cr}$: can sustain the equilibrium regardless of $\theta$ (neutral equilibrium) ~ “bifurcation” point
   
   3) $P > P_{cr}$: cannot sustain the equilibrium (unstable equilibrium)

3. These are analogous to a ball placed upon a smooth surface

Example 11-1: Consider two idealized columns. The first one consists of a single rigid bar $ABCD$ pinned at $D$ and laterally supported at $B$ by a spring with translational stiffness $\beta$. The second column consists of two rigid bars $ABC$ and $CD$ that are joined at $C$ by an elastic connection with rotational stiffness $\beta_R = \left(\frac{L}{C}\right) \beta L^2$. Find an expression for critical load $P_{cr}$ for each column.
### 11.3 Columns with Pinned Ends

- **Differential Equation for Deflection of an “Ideal Column” (i.e. perfectly straight) with Pinned Ends**

1. Bending-moment equation:
   \[ EI \ddot{v} = M \]

2. Moment equilibrium equation:
   \[ M + P \ddot{v} = 0 \]

3. Therefore, the deflection equation of the deflection curve is
   \[ EI \ddot{v} + P \ddot{v} = 0 \]

4. Homogeneous, linear, differential equation of second order with constant coefficients

- **Solution of Differential Equation**

1. For convenience, we introduce \( k^2 = P/EI \)

2. Rewrite the differential equation:
   \[ \ddot{v} + k^2 v = 0 \]

3. From mathematics, the general solution of the equation is
   \[ v = C_1 \sin kx + C_2 \cos kx \]

4. Boundary conditions to determine \( C_1 \) and \( C_2 \):
   \[ v(0) = \quad v(L) = \]

5. From the first condition, \( C_2 = \)

6. Thus the deflection of the column is \( v(x) = C_1 \sin kx \)

7. From the second condition, \( C_1 \sin kL = \)

8. **Case 1**: \( C_1 = \quad \rightarrow \quad v(x) = \), i.e. the column remains straight (for any \( kL \))

9. **Case 2**: \( \sin kL = \quad \rightarrow \) “Buckling equation”
The column sustains equilibrium if \( kL = n\pi, \ n = 1,2,3, \ldots \)

The corresponding axial (critical) loads are

\[
P = \frac{n^2 \pi^2 EI}{L^2}
\]

10. Deflection curves at neutral equilibrium at critical loads are

\[
v(x) = C_1 \sin kx = C_1 \sin \frac{n\pi x}{L}
\]

\( \bigcirc \) Critical Loads

1. The lowest critical load for a column with pinned ends:

\[
P_{cr} = \frac{\pi^2 EI}{L^2}
\]

2. The corresponding buckled shape (mode shape):

\[
v(x) = C_1 \sin \frac{\pi x}{L}
\]

3. Note: the amplitude \( C_1 \) of the buckled shape is un__________ (but small)

4. \( n = 1 \): “Fundamental” buckling mode

5. As \( n \) increases, “higher modes” appear

\( \rightarrow \) No practical interest because the fundamental load is reached first

\( \rightarrow \) To make higher modes occur, lateral supports should be provided at intermediate points
11.1 Columns with Pinned Ends (continued)

○ Critical Stress

1. **Critical stress**: the stress in the column when \( P = \)

\[
\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL^2}
\]

2. Using the **radius of gyration** \( r = \sqrt{I/A} \)

\[
\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}
\]

where \( L/r \) is called “slenderness ratio”

3. **Euler’s curve**: critical stress versus the slenderness ratio
   - Long and slender columns: buckle at \( \boxed{\text{stress}} \)
   - Short and stubby columns: buckle at \( \boxed{\text{stress}} \)
   - The curve is valid only for \( \sigma < \sigma_{pl} \) because we use \( \boxed{\text{law}} \)

○ Effects of Large Deflections, Imperfections, and Inelastic Behavior

1. Ideal elastic column with **small deflections** (Curve A): No deflection or undetermined deflection at \( P = P_{cr} \)

2. Ideal elastic column with **large deflection** (Curve B): Should use exact (nonlinear) expression for the curvature, i.e. instead of \( v'' \) \( \rightarrow \) Once the column begins buckling, an increasing load is required to cause an increase in the deflections

3. Elastic column with **imperfections** (Curve C): Imperfections such as initial curvature \( \rightarrow \) Imperfections produce deflections from the onset of loading; the larger the imperfections, the further curve C moves to the right

4. **Inelastic** column with **imperfections** (Curve D): As the material reaches the proportional limit, it becomes easier to increase deflections
Example 11-2: A long, slender column ABC is pin supported at the ends and compressed by an axial load \( P \). Lateral support is provided at the midpoint \( B \) (only in the direction within the plane). The column is constructed of a standard steel shape (IPN 220; Table E-2) having \( E = 200 \) GPa and proportional limit \( \sigma_{pl} = 300 \) MPa. The total length \( L = 8 \) m. Determine the allowable load \( P_{allow} \) using a factor of safety \( n = 2.5 \) with respect to Euler buckling of the column.
11.4 Columns with Other Support Conditions

- Column Fixed at the Base and Free at the Top
  1. Bending moment at distance $x$ from the base is $M = P(\delta - v)$
  2. Bending moment equation: $EIv'' = M =$
  3. Using the notation $k^2 = P/EI$ again, the equation becomes $v'' + k^2v = k^2\delta$
  4. Homogeneous solution (the same as the pinned-pinned case): $v_H = C_1 \sin kx + C_2 \cos kx$
  5. Particular solution: $v_p =$
  6. Consequently, the general solution is $v(x) = v_H + v_p = C_1 \sin kx + C_2 \cos kx +$
  7. Boundary conditions: $v(0) = , v'(0) = \text{ and } v(L) =$
  8. From the first condition, $C_2 =$
  9. From the second boundary condition, $C_1 =$
  10. Finally, the solution is $v(x) = \delta(1 - \cos kx) \rightarrow$ shape is identified but the amplitude is und
  11. From the third boundary condition, $\delta \cos kL =$
  12. The nontrivial solution (i.e. buckling equation) is $\cos kL = 0$
  13. Therefore, $kL = \frac{n\pi}{2}, \ n = 1,3,5,...$
  14. The critical loads are
      \[
      P_{cr} = \frac{n^2 \pi^2 EI}{4L^2}, \ n = 1,3,5,... \text{ and for } n = 1, P_{cr} = \frac{\pi^2 EI}{4L^2}\]
15. Buckled mode shapes are \( v(x) = \delta \left( 1 - \cos \frac{n \pi x}{2L} \right) \)

**Effective Lengths of Columns**

1. **Effective length** of a column: the length of the equivalent pinned-end column having a deflection curve matching the deflection of the given column.

2. As seen in the figure, the effective length of a column fixed at the base and free at the top is \( L_e = 2L \).

3. From the critical loads of the two column cases, we can derive a general formula for the critical load,

\[
P_{cr} = \frac{\pi^2 EI}{L_e^2}
\]

**Column with Both Ends Fixed against Rotation**

1. According to the deflected shape sketched based on the boundary conditions, it is noted that \( L_e = L \).

2. Therefore, the critical load is

\[
P_{cr} = \frac{4\pi^2 EI}{L^2}
\]

**Column Fixed at the Base and Pinned at the Top**

1. By solving the differential equation (details in the textbook), we find the buckling equation

\[
kL = \tan kL
\]

2. Solving this equation numerically, \( kL = 4.4934 \)

3. The corresponding critical load is

\[
P_{cr} = \frac{20.19 EI}{L^2} = \frac{2.046\pi^2 EI}{L^2}
\]
4. The effective length is $L_e = 0.699L \approx 0.7L$

**Summary of Results**

<table>
<thead>
<tr>
<th>(a) Pinned-pinned column</th>
<th>(b) Fixed-free column</th>
<th>(c) Fixed-fixed column</th>
<th>(d) Fixed-pinned column</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{cr} = \frac{\pi^2 EI}{L^2}$</td>
<td>$P_{cr} = \frac{\pi^2 EI}{4L^2}$</td>
<td>$P_{cr} = \frac{4\pi^2 EI}{L^2}$</td>
<td>$P_{cr} = \frac{2.046 \pi^2 EI}{L^2}$</td>
</tr>
<tr>
<td>$L_e = L$</td>
<td>$L_e = 2L$</td>
<td>$L_e = 0.5L$</td>
<td>$L_e = 0.699L$</td>
</tr>
<tr>
<td>$K = 1$</td>
<td>$K = 2$</td>
<td>$K = 0.5$</td>
<td>$K = 0.699$</td>
</tr>
</tbody>
</table>

**Example 11-3:** A viewing platform is supported by a row of aluminum pipe columns having length $L = 3.25 \text{ m}$ and outer diameter $d = 100 \text{ mm}$. Because of the manner in which the columns are constructed, we model each column as a fixed-pinned column. The columns are being designed to support compressive loads $P = 200 \text{ kN}$. Determine the minimum required thickness $t$ of the columns if a factor of safety $n = 3$ is required with respect to Euler buckling. The modulus of Elasticity of the aluminum is $E = 72 \text{ GPa}$ and the proportional stress limit is 480 MPa.
(Intended Blank for Notes)
11.5 Columns with Eccentric Axial Loads

- Differential Equation of Columns with Eccentricity

1. Consider a column with a small eccentricity $e$ under axial load $P$

2. This is equivalent to a column under centric load $P$ but with additional couples $M_0 = $

3. Bending moment in the column is obtained from a free-body-diagram from the moment equilibrium (around $A$)

\[ M = M_0 + P(-v) = Pe - Pv \]

4. Differential equation

\[ EI\ddot{v} = M = Pe - Pv \]

\[ \ddot{v} + k^2 v = \]

5. The general solution: $v = C_1 \sin kx + C_2 \cos kx + e$

6. Boundary conditions: $v(0) = v(L) = $

7. These conditions yield

\[ C_2 = \]

\[ C_1 = -e \frac{(1 - \cos kL)}{\sin kL} = -e \tan \frac{kL}{2} \]

8. The equation of the deflection curve is

\[ v(x) = -e \left( \tan \frac{kL}{2} \sin kx + \cos kx - 1 \right) \]

**Note:** the deflection for the centric load was $v(x) = C_1 \sin kx = C_1 \sin \frac{\pi x}{L}$

Undetermined (centric load) vs determined (eccentric load)

9. Critical load (both pinned ends)

\[ P_{cr} = \frac{\pi^2 EI}{L^2} \]
Maximum Deflection

1. Maximum deflection $\delta$ occurs at the midpoint, thus

$$\delta = -v \left( \frac{L}{2} \right)$$

$$= e \left( \tan \frac{kl}{2} \sin \frac{kl}{2} + \cos \frac{kl}{2} - 1 \right)$$

$$= e \left( \sec \frac{kl}{2} - 1 \right)$$

2. Consider an alternative expression for $k$

$$k = \frac{P}{\sqrt{EI}} = \frac{P\pi^2}{P_{cr}L^2} = \frac{\pi}{L} \sqrt{\frac{P}{P_{cr}}}$$

3. Using this, the maximum deflection is described in terms of the ratio $P/P_{cr}$

$$\delta = e \left[ \sec \left( \frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right) - 1 \right]$$

4. Load-deflection diagram ($\gamma$)

- The deflection increases as the load $P$ increases, but nonlinear even if linear elastic material is used $\rightarrow$ s________ rule does not work

- When the imperfection is increased from $e_1$ to $e_2$: the maximum deflection increases by

- As the load $P$ approaches the critical load $P_{cr}$ the deflection increases without limit

- An ideal column with a centrally applied load ($e = 0$) is the limiting case of a column with an eccentric load ($e > 0$)
Maximum Bending Moment

1. Maximum bending moment occurs when \( v = \)
   \[
   M_{\text{max}} = P(e + \ldots)
   \]

2. Thus the maximum bending moment is
   \[
   M_{\text{max}} = Pe \sec \frac{kL}{2} = Pe \sec \left( \frac{\pi}{2} \sqrt{\frac{P}{P_{\text{cr}}}} \right)
   \]

Example 11-4: A brass bar \( AB \) projecting from the side of a large machine is loaded at end \( B \) by a force \( P = 7 \) kN with an eccentricity \( e = 11 \) mm. The bar has a rectangular cross section with height \( h = 30 \) mm and width \( b = 15 \) mm. What is the longest permissible length \( L_{\text{max}} \) of the bar if the deflection at the end is limited to \( 3 \) mm? For the brass, use \( E = 100 \) GPa.
11.6 Secant Formula for Columns

⊙ Maximum Stress in a Column under an Eccentric Load

1. Maximum stress occurs at the (concave/convex) side of the column

\[ \sigma_{\text{max}} = \frac{P}{A} + \frac{M_{\text{max}}}{l} \]

2. Maximum moment

\[ M_{\text{max}} = Pe \sec \left( \frac{\pi}{2} \sqrt{\frac{P}{P_{\text{cr}}}} \right) \]

3. From \( P_{\text{cr}} = \pi^2 EI / L^2 \) and \( l = Ar^2 \) where \( r \) is the radius of gyration, the maximum moment is described as

\[ M_{\text{max}} = Pe \sec \left( \frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \]

4. Substituting this into the maximum stress formula above,

\[ \sigma_{\text{max}} = \frac{P}{A} + \frac{Pe}{l} \sec \left( \frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \]

\[ = \frac{P}{A} \left[ 1 + \frac{ec}{r^2} \sec \left( \frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right] \]

5. This so-called “secant formula” describes the maximum compressive stress in a column under eccentric load in terms of \( E, P/A \), \( L/r \) (slenderness ratio) and \( ec/r^2 \) (eccentricity ratio)

6. For given \( \sigma_{\text{max}} \) and \( E \), one can find the possible pairs of \( P/A \) and \( L/r \) for each eccentricity level \( (ec/r^2) \) and plot a graph of secant formula (↑)

7. For centric load, i.e. \( ec/r^2 = 0 \), the critical stress is

\[ \sigma_{\text{cr}} = \frac{P_{\text{cr}}}{A} = \frac{\pi^2 EI}{AL^2} = \frac{\pi^2 E}{(L/r)^2} \]

8. Secant formula and graph let us know the load-carrying capacity of a column in terms of slenderness and eccentricity (Trend?)
Example 11-5: A steel wide-flange column of HE 320A shape is pin-supported at the ends and has a length of 7.5 m. The column supports a centrally applied load $P_1 = 1800$ kN and an eccentrically applied load $P_2 = 200$ kN. Bending takes place about axis 1-1 of the cross section, and the eccentric load acts on axis 2-2 at a distance of 400 mm from the centroid $C$.

(a) Using the secant formula, and assuming $E = 210$ GPa, calculate the maximum compressive stress in the column.

(b) If the yield stress for the steel is $\sigma_Y = 300$ MPa, what is the factor of safety with respect to yielding?
Many thanks for your hard work in this semester to learn Mechanics of Materials. I wish you the very best on your remaining course work and future career and life.

Cheers,
Junho

2017