Thus the time rate of change of the *k*th generalized momentum is given by Eqn(2.60)

Finally, the equations of motion in terms of q_k :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \qquad k = 1, 2, \ldots, n \qquad \textbf{(2.61)}$$

: General form of Lagrange's Equations of Motion

There is one equation corresponding to each q_k.

The system of equations represents a coupled system of ordinary equations governing the evolution of the dynamical system in terms of the *n* generalized coordinates. ~ Finite D.O.F !

*** Continuous system (such as beam, plate and shell):**

PDE !

Alternatively, Lagrange's equations of motion may be

written in terms of the generalized momenta as

$$\frac{d}{dt}(p_k) - \frac{\partial T}{\partial q_k} = Q_k \, , \qquad\qquad k = 1, 2, \ldots, n$$

This means that Newton's Second Law (2.51) is transformed under a change of variables to generalized coordinates $q_1, q_2, \ldots q_n$ **.**

Hence <u>Newton's Second Law is *not* invariant under an</u>

arbitrary change of variables. The extra term represents inertial effects induced by the coordinate transformations.

Lagrange's equations allow the formulation of the equations of motion, independent of the physical significance of the variables.

Note that the dynamics of the system is thus

characterized by the kinetic energy and the virtual work done by generalized forces.

The hallmark of the Lagrangian formulation is that the energy contains the dynamic information.

The use of generalized coordinates, compatible with the constraints, results in the minimum number of variables needed to completely describe the motion.

Furthermore, for generalized coordinates adopted to the constraints, the forces of constraint do not contribute to the virtual work.

Hence the reactions do not appear in the resulting equations of motion.

Ex : a simple pendulum (Fig 2.11).

Assume that a particle of mass *m* is attached to a massless

rod that is free to rotate in a vertical plane

about a frictionless pin.

The motion of this single-degree-of-freedom system may

be described by the generalized coordinate θ.

The <u>Kinetic energy</u> of the system is given in terms of the

generalized velocity $\dot{\theta}$ as

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

From a previous example, the generalized force associated with the rotational coordinate of a pendulum was derived, based on virtual work, as

$$Q_\theta = -mgl\sin\theta$$

The equation of motion based on the Lagrangian

formulation is therefore represented by

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta}}) - \frac{\partial T}{\partial \theta} = Q_\theta$$

That is,

$$\frac{d}{dt}(ml^2\dot{\theta}) - 0 = -mgl\sin\theta$$

which can be set into the more familiar form

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

The systematic approach of the Lagrangian formulation is evident in this example.

The formulation is based on the Kinetic energy and the virtual work.

Since the variable θ is adopted to the constraint of circular motion, the equation of motion has been set up

without need to consider the force of constraint acting on the particle.

The constraint force is in fact the tension in the cable.

Ex :

Consider the two-degree-of-freedom system consisting of two carts coupled by linear elastic springs. (Fig. 2.12)

The generalized coordinates q_1 and q_2 represent the

displacements of the carts from the unstretched configurations of the springs. The kinetic energy is readily formulated as

$$T = \frac{1}{2} m_1 \dot{q}_1 + \frac{1}{2} m_2 \dot{q}_2$$

The generalized forces can be deduced by the method of virtual work.

Then

$$Q_1 = -k_1 q_1 + k_2 (q_2 - q_1)\textbf{,} \qquad Q_2 = -k_2 (q_2 - q_1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1\textbf{,} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2 \qquad \textbf{(2.62)}$$

The equations of motion (2.62) may be simplified and put in standard form as

$$m_1\ddot{q}_1 + (k_1 + k_2)q_1 - k_2 q_2 = 0$$

$$m_2\ddot{q}_2 - k_2 q_1 + k_2 q_2 = 0$$

In a Matrix Form ?

CONSERVATIVE SYSTEMS

Lagrange's equations of motion represent a unified approach to deriving the governing equations of a dynamical system.

Equations (2.61) are completely general, in that they apply generically to all mechanical systems.

The governing equations are based on the total Kinetic energy of a system and the generalized forces derived by the method of virtual work.

Only generalized forces directly affecting the generalized coordinates contribute to the virtual work.

Lagrange's equations of motion may also be expressed in several alternate forms, depending on the nature of the generalized forces.

For a *conservative system*, there exists a potential function in terms of the generalized coordinates

$$V = V(q_1, q_2, \ldots, q_n)$$

from which the generalized forces can be derived as

$$Q_k = -\frac{\partial V}{\partial q_k} \tag{2.63}$$

Substituting the generalized force (2.63) into

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \frac{\partial V}{\partial q_k} \tag{2.64}$$

Since the potential function only depends on the generalized coordinates,…

Thus

$$\frac{\partial T}{\partial \dot{q}_k} = \frac{\partial (T - V)}{\partial \dot{q}_k}$$

Rewriting Lagrange's equations (2.64) results in

$$\frac{d}{dt}\left[\frac{\partial (T - V)}{\partial \dot{q}_k}\right] - \frac{\partial (T - V)}{\partial q_k} = 0$$

This version of the equation has a particularly simple form. The scalar quantity in the parentheses is defined as

the ~

Lagrangian function:

$$L(q,\dot{q},t)=T(q,\dot{q},t)-V(q)$$

<u>It is a function of the generalized coordinates and velocities.</u>

The Lagrangian represents the difference between the total Kinetic energy and the total Potential energy of a

conservative system.

The equations of motion (2.61) can thus be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

which is the standard form of Lagrange's equations of motion for conservative systems.

A formulation based on the Lagrangian is convenience

that allows by-passing the determination of generalized forces from the method of virtual work.

It is interesting to note that for a conservative system all the dynamics are characterized by a single scalar function, the Lagrangian of the system.

The Lagrangian function simplifies the equations of motion and often aids in the understanding of the

dynamics of the system.

Practices !

1. **A particle of mass m is suspended by a massless wire of length $r = a + b\cos\omega t..(a > b > 0)$ to form a spherical pendulum. Find the equation of motion.**

Sol) **T ~ p.102,Eqn.(2.24), V =** $mgr\cos\theta$

- **2 DOF :** θ,ϕ **: governing eq.:**

  Paffian form of constraint ? Linearize :

  Incase : r **= constant ?**

2.A particle of mass m can slide without friction on the inside of a small tube which is bent in the form of a circle of radius *r*. The tube rotate about a vertical

diameter with a constant angular velocity ω.

<u>Write the equation of motion.</u>

$$T_\theta = \frac{1}{2} m (r \dot{\theta})^2$$

$$T_\omega = \frac{1}{2} m (\omega r \sin\theta)^2$$

V = $mgr\cos\theta$

Sol) T= $\frac{1}{2}mr^2(\dot{\theta}^2+\omega^2\sin^2\theta)$ **, V=** $mgr\cos\theta$ **, L=T-V**

3.A particle of mass m can slide on a smooth wire having the form $y=3x^2$ **, where the gravity acts in the direction of the negative y-axis.**

Obtain the equations of motion.

Sol) T= $\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$ **, V=** mgy **with** $y=3x^2$

Eliminate : y ~ Finally,

4.Text : p.120

Elevator !!